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Research Article

On Decomposable Measures Induced by Metrics

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We prove that for a given normalized compact metric space it can induce a σ-max-superdecomposable measure, by constructing a Hausdorff pseudometric on its power set. We also prove that the restriction of this set function to the algebra of all measurable sets is a σ-max-decomposable measure. Finally we conclude this paper with two open problems.

1. Introduction

The classical measure theory is one of the most important theories in mathematics, and it was extended, generalized, and deeply examined in many directions [1]. Nonadditive measure [2, 3] is an extension of the measure in the sense that the additivity of the measure is replaced with a weaker condition, the monotonicity. There are many kinds of nonadditive measures [1, 4]: the Choquet capacity, the decomposable measure [5, 6], the λ-additive measure, the belief measure, the plausibility measure, and so fourth. Many important types of nonadditive measures occur in various branches of mathematics, such as potential theory [7], harmonic analysis, fractal geometry [8], functional analysis [9], the theory of nonlinear differential equations, and in optimization [1, 4, 10]. The Hausdorff distance introduced by Felix Hausdorff in the early 20th century as a way to measure the distance has many applications [8, 11–13]. In this paper, we will give a method for inducing a σ-max-superdecomposable measure from a given normalized compact metric space, by defining a Hausdorff pseudometric on the power set. Furthermore, we will prove that the restriction of the σ-max-superdecomposable measure to the algebra of all measurable sets is a σ-max-decomposable measure.

2. Preliminaries

Most notations and results on metric space and measure theory which are used in this paper can be found in [4, 14]. For simplicity, we consider only the normalized metric spaces (X, d), that is, $\operatorname{diam} X = \sup\{d(x, y) : x, y \in X\} = 1$. But it is not difficult to generalize the results obtained in this paper to the bounded metric spaces. Let $P(X)$ be the space of all subsets of X. A distance function, called the Hausdorff distance, on $P(X)$ is defined as follows.

Definition 2.1 (see [14]). Let (X, d) be a normalized metric space, and let A and B be elements in $P(X)$.

(i) If $x \in X$, the "distance" from x to B is

$$d(x, B) = d(B, x) = \inf_{y \in B}\{d(x, y)\} \tag{2.1}$$

with the convention $(x, \emptyset) = 1$.

(ii) The "distance" from A to B is

$$d(A, B) = \sup_{x \in A}\{d(x, B)\} \tag{2.2}$$

with the convention $d(\emptyset, B) = 0$.

(iii) The Hausdorff distance, $h(A, B)$, between A and B is

$$h(A, B) = \max\{d(A, B), d(B, A)\}. \tag{2.3}$$

A nonempty subset R of $P(X)$ is called an algebra if for every $E, F \in R$, $E \cup F \in R$ and $E^C \in R$, where E^C is the complement of E. A σ-algebra is an algebra which is closed under the formation of countable unions [4].

Definition 2.2. Let R be an algebra. A set function $\mu : R \to [0, 1]$ with $\mu(\emptyset) = 0$ and $\mu(X) = 1$ is:

(1) a max-decomposable measure, if and only if $\mu(A \cup B) = \max\{\mu(A), \mu(B)\}$, for each pair (A, B) of disjoint elements of R (see [6]);

(2) a σ-max-decomposable measure, if and only if

$$\mu\left(\bigcup_{i \in \mathbb{N}} A_i\right) = \sup\{\mu(A_i) : i \in \mathbb{N}\}, \tag{2.4}$$

for each sequence $(A_i)_{i \in \mathbb{N}}$ of disjoint elements of R (see [6]);

(3) a max-superdecomposable measure if and only if $\mu(A \cup B) \geq \max\{\mu(A), \mu(B)\}$;

(4) a σ-max-superdecomposable measure if and only if

$$\mu\left(\bigcup_{i \in \mathbb{N}} A_i\right) \geq \sup\{\mu(A_i) : i \in \mathbb{N}\}. \tag{2.5}$$

3. Main Results

Theorem 3.1. *Let (X, d) be a normalized metric space. Then $(P(X), h)$ is a normalized pseudometric space.*

Proof. It follows from Definition 2.1 that $h(\emptyset, \emptyset) = 0$ and $h(\emptyset, A) = 1$ for all nonempty subset $A \in P(X)$. Then it is clear that $h(A, A) = 0$ and $h(A, B) = h(B, A) \leq 1$ for all $A, B \in P(X)$.

Let $A, B, C \in P(X)$. If at least one of the three sets is empty, then one can easily prove the triangle inequality. Thus, without loss of generality, suppose that the three sets are not empty. For any three points $x_0 \in A$, $y_0 \in B$, and $z_0 \in C$, we have that

$$d(x_0, y_0) + d(y_0, z_0) \geq d(x_0, z_0), \tag{3.1}$$

which implies that

$$d(A, y_0) + d(y_0, z_0) = \inf_{x \in A} d(x, y_0) + d(y_0, z_0) \geq \inf_{x \in A} d(x, z_0) = d(A, z_0). \tag{3.2}$$

Consequently, we get that

$$\sup_{y \in B} d(A, y) + d(y_0, z_0) \geq d(A, y_0) + d(y_0, z_0) \geq d(A, z_0). \tag{3.3}$$

By the arbitrariness of y_0, we have that

$$\sup_{y \in B} d(A, y) + d(B, z_0) = \sup_{y \in B} d(A, y) + \inf_{y_0 \in B} d(y_0, z_0) \geq d(A, z_0). \tag{3.4}$$

Then we have that

$$\sup_{y \in B} d(A, y) + \sup_{z \in C} d(B, z) \geq \sup_{y \in B} d(A, y) + d(B, z_0) \geq d(A, z_0), \tag{3.5}$$

which implies that

$$d(B, A) + d(C, B) \geq d(C, A). \tag{3.6}$$

Similarly, we can get that

$$d(B, C) + d(A, B) \geq d(A, C). \tag{3.7}$$

It follows that

$$\begin{aligned} h(A,B)+h(B,C) &= \max\{d(A,B),d(B,A)\}+\max\{d(B,C),d(C,B)\} \\ &\geq \max\{d(A,C),d(C,A)\}=h(A,C). \end{aligned} \tag{3.8}$$

We conclude that $(P(X),h)$ is a normalized pseudometric space. □

Let μ be a normalized measure on an algebra $R \subseteq P(X)$ and μ^* be the outer measure induced by μ. Let $\rho : P(X)\times P(X) \to \mathbb{R}^+$ be defined by the equation $\rho(A,B)=\mu^*(A\Delta B)$, where the symmetric difference of A and B is defined by $A\Delta B=(A\cap B^C)\cup(A^C\cap B)$. Then $(P(X),\rho)$ is a normalized pseudometric space and $\mu^*(A)=\rho(A,\emptyset)$ for all $A\in P(X)$ [15]. Now, we consider the converse of this process for the normalized pseudometric space $(P(X),h)$. Since $h(A,\emptyset)=1$ for all nonempty subset $A\in P(X)$, it would not get any nontrivial results if the set function μ is defined by $\mu(A)=h(A,\emptyset)$. Thus, we give the following definition.

Definition 3.2. Let (X,d) be a normalized metric space. Now, we define a set function μ on $P(X)$ by

$$\mu(A)=1-h(X,A), \tag{3.9}$$

for all $A\in P(X)$.

Theorem 3.3. *Let (X,d) be a normalized metric space. Then the set function μ is a max-superdecomposable measure on $P(X)$.*

Proof. It is easy to see $\mu(\emptyset)=0$ and $\mu(X)=1$. Let $A,B\in P(X)$ with $A\subseteq B$. By the definition of μ, we have that

$$\begin{aligned} \mu(A) &= 1-\max\left\{\sup_{x\in X} d(x,A),\sup_{y\in A} d(X,y)\right\} \\ &= 1-\sup_{x\in X}\left(\inf_{y\in A} d(x,y)\right) \\ &\leq 1-\sup_{x\in X}\left(\inf_{y\in B} d(x,y)\right) \\ &= 1-\max\left\{\sup_{x\in X} d(x,B),\sup_{y\in B} d(X,y)\right\} \\ &= \mu(B), \end{aligned} \tag{3.10}$$

which shows the set function μ is monotonous. Thus, for any two sets $A,B\in P(X)$, we have

$$\mu(A\cup B)\geq \max\{\mu(A),\mu(B)\}. \tag{3.11}$$

□

Theorem 3.4. *Let (X, d) be a normalized metric space. Then the set function μ is a σ-max-superdecomposable measure on $P(X)$.*

Proof. Due to the monotonicity of μ, for each sequence $(A_i)_{i\in\mathbb{N}}$ of elements of $P(X)$ and every positive integer n, by mathematical induction we have that

$$\mu\left(\bigcup_{i\in\mathbb{N}} A_i\right) \geq \max\{\mu(A_1), \mu(A_2), \ldots, \mu(A_n)\}, \tag{3.12}$$

which implies that

$$\mu\left(\bigcup_{i\in\mathbb{N}} A_i\right) \geq \sup\{\mu(A_i) : i \in \mathbb{N}\}. \tag{3.13}$$

□

Lemma 3.5. *Let (X, d) be a normalized metric space. If $(A_i)_{i\in\mathbb{N}}$ is an increasing sequence in $P(X)$ such that $\bigcup_{i=1}^{\infty} A_i = A$, then $\lim_{i\to\infty} d(x, A_i) = d(x, A)$ for any point $x \in X$.*

Proof. Since $A_i \subseteq A$, it follows from Definition 2.1 that $d(x, A_i) = \inf_{y\in A_i} d(x, y) \geq d(x, A)$. If $\lim_{i\to\infty} d(x, A_i) = a > b = d(x, A)$, then for the decreasing sequence $(d(x, A_i))_{i\in\mathbb{N}}$, we have $d(x, y) \geq a$ for all $y \in A_i$, $i \in \mathbb{N}$.

On the other hand, from $d(x, A) = \inf_{y\in A} d(x, y) = b$, it follows that there exists a point $y_0 \in A$ such that $d(x, y_0) \leq (a + b)/2$. Since $\bigcup_{i=1}^{\infty} A_i = A$, there exists a positive integer i_0 such that $y_0 \in A_{i_0}$. Thus we get that $d(x, y_0) \geq a$ which contradicts $d(x, y_0) \leq (a + b)/2$. We conclude that $\lim_{i\to\infty} d(x, A_i) = d(x, A)$ for any point $x \in X$. □

Lemma 3.6. *Let (X, d) be a normalized compact metric space. If $(A_i)_{i\in\mathbb{N}}$ is an increasing sequence in $P(X)$ such that $\bigcup_{i=1}^{\infty} A_i = A$, then $\lim_{i\to\infty} h(A_i, A) = 0$.*

Proof. Since $A_i \subseteq A$, it follows from Definition 2.1 that

$$h(A_i, A) = \max\left\{\sup_{x\in A_i} d(x, A), \sup_{x\in A} d(x, A_i)\right\} = \sup_{x\in A} d(x, A_i). \tag{3.14}$$

If $\lim_{i\to\infty} h(A_i, A) = a > 0$, then for the decreasing sequence $(h(A_i, A))_{i\in\mathbb{N}}$, we have $h(A_i, A) \geq a$ for all $i \in \mathbb{N}$. Consequently there exists a point $x_i \in A$ for each A_i such that $d(x_i, A_i) > a/2$. Since X is a compact metric space, passing to subsequence if necessary, we may assume that the sequence $(x_i)_{i\in\mathbb{N}}$ converges to a point x in the closure of A and $\lim_{i\to\infty} d(x_i, A_i) = b \geq a/2$. However since

$$\begin{aligned} |d(x_i, A_i) - d(x, A)| &\leq |d(x_i, A_i) - d(x, A_i)| + |d(x, A_i) - d(x, A)| \\ &\leq d(x_i, x) + |d(x, A_i) - d(x, A)|, \end{aligned} \tag{3.15}$$

it follows from Lemma 3.5 and $\lim_{i\to\infty} d(x_i, x) = 0$ that

$$\lim_{i\to\infty} d(x_i, A_i) = d(x, A) = 0. \tag{3.16}$$

This is a contradiction. Thus we have $\lim_{i\to\infty} h(A_i, A) = 0$. □

Lemma 3.7. *Let (X, d) be a normalized compact metric space. If $(A_i)_{i\in\mathbb{N}}$ is an increasing sequence in $P(X)$ such that $\bigcup_{i=1}^{\infty} A_i = A$, then μ is continuous from below, that is, $\lim_{i\to\infty}\mu(A_i) = \mu(A)$.*

Proof. By the definition of μ, we have that

$$\left|\mu(A_i) - \mu(A)\right| = |h(A_i, X) - h(A, X)| \le h(A_i, A), \tag{3.17}$$

for all $i \in \mathbb{N}$. By Lemma 3.6, we have that $\lim_{i\to\infty}\mu(A_i) = \mu(A)$. □

Definition 3.8. A set E in $P(X)$ is μ-measurable if, for every set A in $P(X)$,

$$\mu(A) = \max\left\{\mu(A\cap E), \mu\left(A\cap E^C\right)\right\}. \tag{3.18}$$

Theorem 3.9. *If $\mathbb{S}$ is the class of all μ-measurable sets, then $\mathbb{S}$ is an algebra.*

Proof. It is easy to see that $\emptyset, X \in \mathbb{S}$, and that if $E \in \mathbb{S}$ then $E^C \in \mathbb{S}$. Let $E, F \in \mathbb{S}$ and $A \in P(X)$. It follows that

$$\begin{aligned}\mu(A\cap(E\cup F)) &= \max\left\{\mu(A\cap(E\cup F)\cap F), \mu\left(A\cap(E\cup F)\cap F^C\right)\right\}\\ &= \max\left\{\mu(A\cap F), \mu\left(A\cap E\cap F^C\right)\right\},\end{aligned} \tag{3.19}$$

which implies that

$$\begin{aligned}\max\left\{\mu(A\cap(E\cup F)), \mu\left(A\cap(E\cup F)^C\right)\right\} &= \max\left\{\mu(A\cap F), \mu\left(A\cap E\cap F^C\right), \mu\left(A\cap(E\cup F)^C\right)\right\}\\ &= \max\left\{\mu(A\cap F), \mu\left(A\cap F^C\cap E\right), \mu\left(A\cap F^C\cap E^C\right)\right\}\\ &= \max\left\{\mu(A\cap F), \mu\left(A\cap F^C\right)\right\} = \mu(A).\end{aligned} \tag{3.20}$$

Thus, $\mathbb{S}$ is closed under the formation of union. □

Theorem 3.10. *The restriction of set function μ to $\mathbb{S}$, $\mu|_{\mathbb{S}}$, is a σ-max-decomposable measure.*

Proof. Let E_1, E_2 be two disjoint sets in $\mathbb{S}$. It follows that

$$\mu(E_1\cup E_2) = \max\left\{\mu((E_1\cup E_2)\cap E_1), \mu\left((E_1\cup E_2)\cap E_1^C\right)\right\} = \max\{\mu(E_1), \mu(E_2)\}. \tag{3.21}$$

Let $\{E_i\}_{i=1}^{\infty}$ be a disjoint sequence set in $\mathbb{S}$ with $\bigcup_{i=1}^{\infty} E_i = E \in \mathbb{S}$. By mathematical induction, we can get that

$$\mu\left(\bigcup_{i=1}^{n} E_i\right) = \max\{\mu(A_1), \mu(A_2), \ldots, \mu(A_n)\} \tag{3.22}$$

for every positive integer n. Since μ is continuous from below and $\lim_{n\to\infty} \bigcup_{i=1}^{n} E_i = E$, we have

$$\mu(E) = \lim_{n\to\infty} \mu\left(\bigcup_{i=1}^{n} E_i\right) = \lim_{n\to\infty} \max\{\mu(E_1), \mu(E_2), \ldots, \mu(E_n)\} = \sup\{\mu(E_i) : i \in \mathbb{N}\}, \tag{3.23}$$

which implies that $\mu|_{\mathbb{S}}$ is a σ-max-decomposable measure. □

4. Concluding Remarks

For any given normalized compact metric space, we have proved that it can induce a σ-max-superdecomposable measure, by constructing a Hausdorff pseudometric on its power set. We have also proved that the restriction of the set function to the algebra of all measurable sets is a σ-max-decomposable measure. However, the following problems remain open.

Problem 1. Is μ a σ-subadditive measure on $P(X)$?

Problem 2. Is the class of all μ-measurable sets a σ-algebra?

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