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Research Article

Numerical Solutions of Stochastic Differential Equations with Piecewise Continuous Arguments under Khasminskii-Type Conditions

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The main purpose of this paper is to investigate the convergence of the Euler method to stochastic differential equations with piecewise continuous arguments (SEPCAs). The classical Khasminskii-type theorem gives a powerful tool to examine the global existence of solutions for stochastic differential equations (SDEs) without the linear growth condition by the use of the Lyapunov functions. However, there is no such result for SEPCAs. Firstly, this paper shows SEPCAs which have nonexplosion global solutions under local Lipschitz condition without the linear growth condition. Then the convergence in probability of numerical solutions to SEPCAs under the same conditions is established. Finally, an example is provided to illustrate our theory.

1. Introduction

Stochastic modeling has come to play an important role in many branches of science and industry. Such models have been used with great success in a variety of application areas, including biology, epidemiology, mechanics, economics, and finance. Most stochastic differential equations are nonlinear and cannot be solved explicitly, but it is very important to research the existence and uniqueness of solution of stochastic differential equations. Many authors have studied the problem of SDEs. The classical existence-and-uniqueness theorem requires the coefficients $f(x(t))$ and $g(x(t))$ to satisfy the local Lipschitz condition and the linear growth condition (see [1]). However, there are many SDEs that do not satisfy the linear growth condition, so more general conditions have been introduced to replace theirs. Khasminskii [2] has studied Khasminskii's test for SDEs which are the most powerful conditions. Similarly, the classical existence-and-uniqueness theorem for stochastic differential delay equations (SDDEs) requires the coefficients $f(x(t), x(t-\tau))$ and $g(x(t), x(t-\tau))$ to satisfy the local Lipschitz condition and the linear growth condition (see [3–6]). Mao [7] has proved Khasminskii-type theorem, and this is a natural generalization of the classical Khasminskii test.

In recent years, differential equations with piecewise continuous arguments (EPCAs) had attracted much attention, and many useful conclusions were obtained. These systems have applications in certain biomedical models, control systems with feedback delay in the work of L. Cooke and J. Wiener [8]. The general theory and basic results for EPCAs have by now been thoroughly investigated in the book of Wiener [9]. A typical EPCA contains arguments that are constant on certain intervals. The solutions are determined by a finite set of initial data, rather than by an initial function, as in the case of general functional differential equation. A solution is defined as a continuous, sectionally smooth function that satisfies the equation within these intervals. Continuity of a solution at a point joining any two consecutive intervals leads to recursion relations for the solution at such points. Hence, EPCAs represent a hybrid of continuous and discrete dynamical systems and combine the properties of both differential and difference equations.

However, up to now, there are few people who have considered the influence of noise to EPCAs. Actually, the environment, and accidental events may greatly influence the systems. Thus, analyzing SEPCAs is an interesting topic both in theory and applications. In this paper, we give the Khasminskii-type theorems for SEPCAs, which shows that SEPCAs have nonexplosion global solutions under local Lipschitz condition without the linear growth condition.

On the other hand, there is in general no explicit solution to an SEPCA, whence numerical solutions are required in practice. Numerical solutions to SDEs have been discussed under the local Lipschitz condition and the linear growth condition by many authors (see [5]). Mao [10] gives the convergence in probability of numerical solutions to SDDEs under Khasminskii-Type conditions. Dai and Liu [11] give the mean-square stability of the numerical solutions of linear stochastic differential equations with piecewise continuous arguments. However, SEPCAs do not have the convergence results. The other main aim of this paper is to establish convergence of numerical solution for SEPCAs under the differential conditions.

The paper is organized as follows. In Section 2, we introduce necessary notations and Euler method. In Section 3, we obtain the existence and uniqueness of solution to stochastic differential equations with piecewise continuous arguments under Khasminskii-type conditions. Then the convergence in probability of numerical solutions to stochastic differential equations with piecewise continuous arguments under the same conditions is established. Finally, an example is provided to illustrate our theory.

2. Preliminary Notation and Euler Method

In this paper, unless otherwise specified, let $|x|$ be the Euclidean norm in $x \in R^n$. If A is a vector or matrix, its transpose is defined by A^T. If A is a matrix, its trace norm is defined by $|A| = \sqrt{\operatorname{trace}(A^T A)}$. For simplicity, we also have to denote by $a \wedge b = \min\{a, b\}$, $a \vee b = \max\{a, b\}$.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$, satisfying the usual conditions. $\mathcal{L}^1([0, \infty), R^n)$ and $\mathcal{L}^2([0, \infty), R^n)$ denote the family of all real-valued $\mathcal{F}_t$-adapted process $f(t)_{t \geq 0}$, such that for every $T > 0$, $\int_0^T |f(t)| dt < \infty$ almost surely and

$\int_0^T |f(t)|^2 dt < \infty$ almost surely, respectively. For any $a, b \in R$ with $a < b$, denote $C([a,b]; R^n)$ as the family of continuous functions ϕ from $[a,b]$ to R^n with the norm $\|\phi\| = \sup_{a \le \theta \le b} |\phi(\theta)|$. Denote $C^b_{\mathcal{F}_t}([a,b]; R^n)$ as the family of all bounded $\mathcal{F}_t$-measurable $C([a,b]; R^n)$-valued random variables. Let $B(t) = (B_1(t), \dots, B_d(t))^T$ be a d-dimensional Brownian motion defined on the probability space. Let $C^2(R^n; R_+)$ denote the family of all continuous nonnegative functions $V(x)$ defined on R^n such that they are continuously twice differentiable in x. Given $V \in C^2(R^n; R_+)$, we define the operator $LV : R^n \times R^n \to R$ by

$$LV(x,y) = V_x(x) f(x,y) + \frac{1}{2} \text{trace}\left[g^T(x,y) V_{xx}(x) g(x,y) \right], \tag{2.1}$$

where

$$V_x(x) = \left(\frac{\partial V(x)}{\partial x_1}, \dots, \frac{\partial V(x)}{\partial x_n} \right), \qquad V_{xx}(x) = \left(\frac{\partial^2 V(x)}{\partial x_i \partial x_j} \right)_{n \times n}. \tag{2.2}$$

Let us emphasize that LV is defined on $R^n \times R^n$, while V is only defined on R^n.

Throughout this paper, we consider stochastic differential equations with piecewise continuous arguments

$$dx(t) = f(x(t), x([t]))dt + g(x(t), x([t]))dB(t) \quad \forall t \ge 0, \tag{2.3}$$

with initial data $x(0) = c_0$, where $f : R^n \times R^n \to R^n$, $g : R^n \times R^n \to R^{n \times d}$, c_0 is a vector, and $[\cdot]$ denotes the greatest-integer function. By the definition of stochastic differential, this equation is equivalent to the following stochastic integral equation:

$$x(t) = x(0) + \int_0^t f(x(s), x([s]))ds + \int_0^t g(x(s), x([s]))dB(s) \quad \forall t \ge 0. \tag{2.4}$$

Moreover, we also require the coefficients $f(x(t), x([t]))$ and $g(x(t), x([t]))$ to be sufficiently smooth.

To be precise, let us state the following conditions.

(H1) (The local Lipschitz condition) For every integer $i \ge 1$, there exists a positive constant L_i such that

$$\left| f(x,y) - f(\overline{x}, \overline{y}) \right|^2 \vee \left| g(x,y) - g(\overline{x}, \overline{y}) \right|^2 \le L_i \left(|x - \overline{x}|^2 + \left| y - \overline{y} \right|^2 \right), \tag{2.5}$$

for those $x, \overline{x}, y, \overline{y} \in R^n$ with $|x| \vee |\overline{x}| \vee |y| \vee |\overline{y}| \le i$.

(H2) (Linear growth condition) There exists a positive constant K such that

$$\left| f(x,y) \right|^2 \vee \left| g(x,y) \right|^2 \le K \left(1 + |x|^2 + |y|^2 \right), \tag{2.6}$$

for all $(x,y) \in R^n \times R^n$.

(H3) There are a function $V \in C^2(R^n; R_+)$ and a positive constant α such that

$$\liminf_{|x|\to\infty} V(x) = \infty, \tag{2.7}$$

$$LV(x,y) \le \alpha\big(1 + V(x) + V(y)\big), \tag{2.8}$$

for all $(x, y) \in R^n \times R^n$.

Let us first give the definition of the solution.

Definition 2.1 (see [11]). An R^n-valued stochastic process $\{x(t)\}$ is called a solution of (2.3) on $[0,\infty)$ if it has the following properties:

(1) $\{x(t)\}$ is continuous on $[0,\infty)$ and $\mathcal{F}_t$-adapted,

(2) $\{f(x(t), x([t]))\} \in \mathcal{L}^1([0,\infty), R^n)$ and $\{g(x(t), x([t]))\} \in \mathcal{L}^2([0,\infty), R^{n\times d})$,

(3) equation (2.4) is satisfied on each interval $[n, n+1) \subset [0,\infty)$ with integral end-points almost surely. A solution $\{x(t)\}$ is said to be unique if any other solution $\{\overline{x}(t)\}$ is indistinguishable from $\{x(t)\}$, that is,

$$P\{x(t) = \overline{x}(t)\ \forall t \in [0,\infty)\} = 1. \tag{2.9}$$

Let $h = 1/m$ be a given stepsize with integer $m \ge 1$, and let the gridpoints t_n be defined by $t_n = nh (n = 0, 1, 2, \ldots)$. For simplicity, we assume that $T = Nh$. We consider the Euler-Maruyama method to (2.3),

$$y_{n+1} = y_n + f\Big(y_n, y^h([nh])\Big)h + g\Big(y_n, y^h([nh])\Big)\Delta B_n, \tag{2.10}$$

for $n = 0, 1, 2, \ldots$, where $\Delta B_n = B(t_n) - B(t_{n-1})$, $y^h([nh])$ is approximation to the exact solution $x([nh])$. Let $n = km+l$ $(k = 0, 1, 2, \ldots,\ l = 0, 1, 2, \ldots, m-1)$. The adaptation of the Euler method to (2.3) leads to a numerical process of the following type:

$$y_{km+l+1} = y_{km+l} + f(y_{km+l}, y_{km})h + g(y_{km+l}, y_{km})\Delta B_{km+l}, \tag{2.11}$$

where $\Delta B_{km+l} = B(t_{km+l}) - B(t_{km+l-1})$, y_{km+l} and y_{km} are approximations to the exact solution $x(t_{km+l})$ and $x([t_{km+l}])$, respectively. The continuous Euler-Maruyama approximate solution is defined by

$$y(t) = y(0) + \int_0^t f(z(s), z([s]))ds + \int_0^t g(z(s), z([s]))dB(s), \tag{2.12}$$

where $z(t) = y_{km+l}$ and $z([t]) = y_{km}$ for $t \in [t_{km+l}, t_{km+l+1})$. It is not difficult to see that $y(t_{km+l}) = z(t_{km+l}) = y_{km+l}$ for $k = 0, 1, 2, \ldots,\ l = 0, 1, 2, \ldots, m-1$. For sufficiently large integer i, define the stopping times $\eta_i = \inf\{t \ge 0 : |x(t)| \ge i\}$, $\theta_i = \inf\{t \ge 0 : |y(t)| \ge i\}$.

3. Convergence in Probability of the Euler-Maruyama Method

In this section, we concentrate on (2.3) under the local Lipschitz condition (H1) without the linear growth condition (H2) to establish the generalized existence and uniqueness theorem for stochastic differential equations with piecewise continuous arguments. We then give the convergence in probability of the EM method to (2.3) under the local Lipschitz condition (H1) and some additional conditions (H3).

Theorem 3.1. *Under the conditions (H1) and (H3), there is a unique global solution $x(t)$ to* (2.3) *with initial data $x(0) = c_0$ on $t \in [0, \infty)$. Moreover, the solution has the property that*

$$EV(x(t)) < \infty, \quad \text{for any } t \geq 0. \tag{3.1}$$

Proof. Applying the standard truncation technique to (2.3), we obtain the unique maximal local solution $x(t)$ exists on $[0, \eta_e)$ under the local Lipschitz condition in a similar way as the proof of [10, Theorem 3.15, page 91], where η_e is the explosion time. For each integer $i \geq |c_0|$, define the stopping time

$$\eta_i = \inf\{t \in [0, \eta_e) : |x(t)| \geq i\}. \tag{3.2}$$

Clearly, η_i is increasing as $i \to \infty$. We denote that $\eta_\infty = \lim_{i\to\infty}\eta_i$ and $\inf\emptyset = \infty$. Hence, $\eta_\infty \leq \eta_e$ almost surely. If we can obtain $\eta_\infty = \infty$ almost surely, then $\eta_e = \infty$ almost surely.

In what follows, we will prove $\eta_\infty = \infty$ almost surely and assertion (3.1). By the Itô formula and condition (2.8), we derive that

$$\begin{aligned} dV(x(t)) &= LV(x(t), x([t]))dt + V_x(x(t))g(x(t), x([t]))dB(t) \\ &\leq \alpha[1 + V(x(t)) + V(x([t]))]dt + V_x(x(t))g(x(t), x([t]))dB(t), \end{aligned} \tag{3.3}$$

for $0 \leq t < \eta_\infty$. Now, for $t_1 \in [0, 1)$, we can integrate both sides of (3.3) from 0 to $\eta_i \wedge t_1$,

$$\begin{aligned} V(x(\eta_i \wedge t_1)) \leq{}& V(x(0)) + \alpha[1 + V(x(0))] \\ &+ \alpha\int_0^{\eta_i\wedge t_1} V(x(t))dt + \int_0^{\eta_i\wedge t_1} V_x(x(t))g(x(t), x([t]))dB(t). \end{aligned} \tag{3.4}$$

We take the expectations in both sides of (3.4),

$$\begin{aligned} EV(x(\eta_i \wedge t_1)) &\leq V(x(0)) + \alpha[1 + V(x(0))] + \alpha E\int_0^{\eta_i\wedge t_1} V(x(t))dt \\ &\leq \beta_1 + \alpha E\int_0^{\eta_i\wedge t_1} V(x(t))dt, \end{aligned} \tag{3.5}$$

where

$$\beta_1 = V(x(0)) + \alpha[1 + V(x(0))] = (1 + \alpha)V(c_0) + \alpha < \infty. \tag{3.6}$$

It is easy to compute

$$EV(x(\eta_i \wedge t_1)) \le \beta_1 + \alpha E\int_0^{\eta_i \wedge t_1} V(x(t))dt$$
$$= \beta_1 + \alpha \int_0^{t_1} EV(x(\eta_i \wedge t))dt. \tag{3.7}$$

Now the Gronwall inequality yields that

$$EV(x(\eta_i \wedge t_1)) \le \beta_1 e^{\alpha t_1} \le \beta_1 e^{\alpha}, \quad 0 \le t_1 < 1. \tag{3.8}$$

So we have

$$EV(x(\eta_i \wedge 1)) = \lim_{t_1 \to 1} EV(x(\eta_i \wedge t_1)) \le \beta_1 e^{\alpha}. \tag{3.9}$$

Defining

$$\gamma_i = \inf_{|x| \ge i} V(x), \quad \forall i \ge |c_0|, \tag{3.10}$$

denoting I_A as the indicator function of a set A, we compute

$$\beta_1 e^{\alpha} \ge EV(x(\eta_i \wedge 1)) \ge E(V(x(\eta_i))I_{\{\eta_i \le 1\}}) \ge \gamma_i P(\eta_i \le 1). \tag{3.11}$$

Letting $i \to \infty$, we have that $P(\eta_\infty \le 1) = 0$, namely,

$$P(\eta_\infty > 1) = 1. \tag{3.12}$$

By (3.8) and (3.12),

$$EV(x(t_1)) \le \beta_1 e^{\alpha}, \quad 0 \le t_1 \le 1. \tag{3.13}$$

Now let us prove $\eta_\infty > 2$, for $t_2 \in [1, 2)$, and we can integrate both sides of (3.3) from 1 to $\eta_i \wedge t_2$ and take the expectations

$$EV(x(\eta_i \wedge t_2)) \le EV(x(1)) + \alpha[1 + EV(x(1))] + \alpha E\int_1^{\eta_i \wedge t_2} V(x(t))dt$$
$$\le \beta_2 + \alpha \int_1^{t_2} EV(x(\eta_i \wedge t))dt, \tag{3.14}$$

where

$$\beta_2 \le \beta_1 e^{\alpha} + \alpha(1 + \beta_1 e^{\alpha}) < \infty. \tag{3.15}$$

Now the Gronwall inequality yields that

$$EV(x(\eta_i \wedge t_2)) \le \beta_2 e^{\alpha(t_2-1)} \le \beta_2 e^{\alpha}, \quad 1 \le t_2 < 2. \tag{3.16}$$

Hence, we have

$$EV(x(\eta_i \wedge 2)) = \lim_{t_1 \to 2} EV(x(\eta_i \wedge t_2)) \le \beta_2 e^{\alpha}. \tag{3.17}$$

By (3.10) and (3.17), we compute

$$\beta_2 e^{\alpha} \ge EV(x(\eta_i \wedge 2)) \ge E(V(x(\eta_i)) I_{\{\eta_i \le 2\}}) \ge \gamma_i P(\eta_i \le 2). \tag{3.18}$$

Letting $i \to \infty$, we have that $P(\eta_\infty \le 2) = 0$, namely,

$$P(\eta_\infty > 2) = 1. \tag{3.19}$$

From (3.16) and (3.19), we yield

$$EV(x(t_2)) \le \beta_2 e^{\alpha}, \quad 1 \le t_2 \le 2. \tag{3.20}$$

Repeating this procedure, we can show that, for any integer $j \ge 1$, $\eta_\infty > j$ almost surely,

$$EV(x(t_j)) \le \beta_j e^{\alpha}, \quad j-1 \le t_j \le j, \tag{3.21}$$

where

$$\beta_j \le \beta_{j-1} e^{\alpha} + \alpha(1 + \beta_{j-1} e^{\alpha}) < \infty. \tag{3.22}$$

By (3.13), (3.20), and (3.21), we obtain

$$EV(x(t)) \le \beta_j e^{\alpha} < \infty, \quad 0 \le t \le j. \tag{3.23}$$

Therefore, we must have $\eta_\infty = \infty$ almost surely as well as the required assertion (3.1). The proof is completed. □

Theorem 3.2. *Under the conditions (H1) and (H3), if $\varepsilon \in (0,1)$ and $T > 0$, then there exists a sufficiently large integer $\hat{i}$, dependent on ε and T such that*

$$P(\eta_i \le T) \le \varepsilon, \quad \forall i \ge \hat{i}. \tag{3.24}$$

Proof. By Theorem 3.1, we have

$$EV(x(\eta_i \wedge t)) \le \beta_j e^{\alpha} < \infty, \quad 0 \le t \le j. \tag{3.25}$$

Choose j large enough for $j > T$. From (3.25), we get

$$EV(x(\eta_i \wedge ([T]+1))) \leq \beta_{[T]+1} e^{\alpha} < \infty. \tag{3.26}$$

It follows from (3.10) and (3.26) that

$$\begin{aligned} \beta_{[T]+1} e^{\alpha} &\geq EV(x(\eta_i \wedge ([T]+1))) \\ &\geq E\left(V(x(\eta_i)) I_{\{\eta_i \leq [T]+1\}}\right) \\ &\geq \gamma_i P(\eta_i \leq [T]+1), \end{aligned} \tag{3.27}$$

while by (H3), $\gamma_i \to \infty$ as $i \to \infty$. Thus, there is a sufficiently large integer $\hat{i}$ such that

$$\gamma_i \geq \frac{\beta_{[T]+1} e^{\alpha}}{\varepsilon}, \quad \forall i \geq \hat{i}. \tag{3.28}$$

Therefore, we get that

$$P(\eta_i \leq T) \leq P(\eta_i \leq [T]+1) \leq \frac{\beta_{[T]+1} e^{\alpha}}{\gamma_i} < \varepsilon. \tag{3.29}$$

The proof is completed. □

The following lemma shows that both $y(t)$ and $z(t)$ are close to each other.

Lemma 3.3. *Under the condition (H1), let $T > 0$ be arbitrary. Then*

$$E\left(\sup_{0 \leq t \leq T \wedge \theta_i} |y(t) - z(t)|^2\right) \leq C_1(i) h^{1/2}, \tag{3.30}$$

where $C_1(i) = 4(2i^2 L_i + |f(0,0)|^2 \vee |g(0,0)|^2)(1 + (16\sqrt{3}/3) d (T+1)^{1/2})$.

Proof. For $t \in [0, T \wedge \theta_i)$, there are two integers k and l such that $t \in [t_{km+l}, t_{km+l+1})$. So we compute

$$\begin{aligned} |y(t) - z(t)|^2 &= \left| \int_{t_{km+l}}^{t} f(z(s), z([s])) ds + \int_{t_{km+l}}^{t} g(z(s), z([s])) dB(s) \right|^2 \\ &= \left| \int_{t_{km+l}}^{t} f(y_{km+l}, y_{km}) ds + \int_{t_{km+l}}^{t} g(y_{km+l}, y_{km}) dB(s) \right|^2 \\ &= \left| f(y_{km+l}, y_{km})(t - t_{km+l}) + g(y_{km+l}, y_{km})(B(t) - B(t_{km+l})) \right|^2 \\ &\leq 2 \left| f(y_{km+l}, y_{km}) \right|^2 h^2 + 2 \left| g(y_{km+l}, y_{km}) \right|^2 |B(t) - B(t_{km+l})|^2, \end{aligned} \tag{3.31}$$

since

$$\begin{aligned}\left|f\left(y_{km+l}, y_{km}\right)\right|^2 &\leq 2\left|f\left(y_{km+l}, y_{km}\right) - f(0,0)\right|^2 + 2\left|f(0,0)\right|^2 \\ &\leq 2L_i\left(\left|y_{km+l}\right|^2 + \left|y_{km}\right|^2\right) + 2\left|f(0,0)\right|^2 \\ &\leq 2\left(2i^2 L_i + \left|f(0,0)\right|^2\right).\end{aligned} \tag{3.32}$$

Similarly, we obtain that

$$\left|g\left(y_{km+l}, y_{km}\right)\right|^2 \leq 2\left(2i^2 L_i + \left|g(0,0)\right|^2\right). \tag{3.33}$$

Substituting (3.32) and (3.33) into (3.31) gives

$$\left|y(t) - z(t)\right|^2 \leq C\left(h^2 + |B(t) - B(t_{km+l})|^2\right), \tag{3.34}$$

where $C = 4(2i^2 L_i + |f(0,0)|^2 \vee |g(0,0)|^2)$. Let $n_t = km + l$ for $t \in [t_{km+l}, t_{km+l+1})$, then we have that

$$\begin{aligned}E\left(\sup_{0\leq t\leq T\wedge\theta_i} |B(t) - B(t_{n_t})|^2\right) &\leq \sum_{i=1}^{d} E\left(\sup_{0\leq t\leq T\wedge\theta_i} |B_i(t) - B_i(t_{n_t})|^2\right) \\ &\leq \sum_{i=1}^{d} E\left(\sup_{u=0,1,2,\ldots,N}\ \sup_{t_u\leq t\leq t_{u+1}\wedge T} |B_i(t) - B_i(t_u)|^2\right) \\ &\leq \sum_{i=1}^{d} \left[E\left(\sup_{u=0,1,2,\ldots,N}\ \sup_{t_u\leq t\leq t_{u+1}\wedge T} |B_i(t) - B_i(t_u)|^4\right)\right]^{1/2},\end{aligned} \tag{3.35}$$

while by the Doob martingale inequality, we have

$$\begin{aligned}E\left(\sup_{u=0,1,2,\ldots,N}\ \sup_{t_u\leq t\leq t_{u+1}\wedge T} |B_i(t) - B_i(t_u)|^4\right) &\leq \sum_{u=0}^{N} E\left(\sup_{t_u\leq t\leq t_{u+1}\wedge T} |B_i(t) - B_i(t_u)|^4\right) \\ &\leq \left(\frac{4}{3}\right)^4 \sum_{u=0}^{N} E|B_i(t_{u+1}\wedge T) - B_i(t_u)|^4 \\ &\leq \frac{256}{27}\sum_{u=0}^{N} h^2 \\ &\leq \frac{256}{27}(T+1)h.\end{aligned} \tag{3.36}$$

Substituting (3.36) into (3.35) yields

$$E\left(\sup_{0\le t\le T\wedge\theta_i}|B(t)-B(t_{n_t})|^2\right)\le\sum_{i=1}^{d}\left(\frac{256}{27}(T+1)h\right)^{1/2}=\frac{16\sqrt{3}}{9}d(T+1)^{1/2}h^{1/2}. \tag{3.37}$$

Thus, we obtain

$$\begin{aligned}E\left(\sup_{0\le t\le T\wedge\theta_i}\left|y(t)-z(t)\right|^2\right)&\le Ch^2+C\frac{16\sqrt{3}}{9}d(T+1)^{1/2}h^{1/2}\\&\le C\left(1+\frac{16\sqrt{3}}{3}d(T+1)^{1/2}\right)h^{1/2}\\&\le C_1(i)h^{1/2},\end{aligned} \tag{3.38}$$

where $C_1(i)=4(2i^2L_i+|f(0,0)|^2\vee|g(0,0)|^2)(1+(16\sqrt{3}/3)d(T+1)^{1/2})$. The proof is completed. □

Lemma 3.4. *Under the condition (H1), for any $T>0$, there exists a positive constant $C_2(i)$ dependent on i and independent of h such that*

$$E\left(\sup_{0\le t\le T}\left|x(t\wedge\eta_i\wedge\theta_i)-y(t\wedge\eta_i\wedge\theta_i)\right|^2\right)\le C_2(i)h^{1/2}, \tag{3.39}$$

where $C_2(i)=8T(T+4)L_iC_1(i)e^{8T(T+4)L_i}$.

Proof. It follows from (2.4) and (2.12) that

$$\begin{aligned}&\left|x(t\wedge\eta_i\wedge\theta_i)-y(t\wedge\eta_i\wedge\theta_i)\right|^2\\&\quad\le2\left|\int_0^{t\wedge\eta_i\wedge\theta_i}f(x(s),x([s]))-f(z(s),z([s]))ds\right|^2\\&\qquad+2\left|\int_0^{t\wedge\eta_i\wedge\theta_i}g(x(s),x([s]))-g(z(s),z([s]))dB(s)\right|^2.\end{aligned} \tag{3.40}$$

By the Hölder inequality, we obtain

$$\begin{aligned}
&\left|x(t\wedge\eta_i\wedge\theta_i)-y(t\wedge\eta_i\wedge\theta_i)\right|^2\\
&\quad\le 2T\int_0^t\left|f(x(s\wedge\eta_i\wedge\theta_i),x([s\wedge\eta_i\wedge\theta_i]))-f(z(s\wedge\eta_i\wedge\theta_i),z([s\wedge\eta_i\wedge\theta_i]))\right|^2ds\\
&\qquad+2\left|\int_0^t g(x(s\wedge\eta_i\wedge\theta_i),x([s\wedge\eta_i\wedge\theta_i]))-g(z(s\wedge\eta_i\wedge\theta_i),z([s\wedge\eta_i\wedge\theta_i]))dB(s)\right|^2.
\end{aligned} \tag{3.41}$$

This implies that for any $0\le t_1\le T$,

$$\begin{aligned}
&E\sup_{0\le t\le t_1}\left(\left|x(t\wedge\eta_i\wedge\theta_i)-y(t\wedge\eta_i\wedge\theta_i)\right|^2\right)\\
&\quad\le 2TE\sup_{0\le t\le t_1}\int_0^t\left|f(x(s\wedge\eta_i\wedge\theta_i),x([s\wedge\eta_i\wedge\theta_i]))\right.\\
&\qquad\qquad\left.-f(z(s\wedge\eta_i\wedge\theta_i),z([s\wedge\eta_i\wedge\theta_i]))\right|^2ds\\
&\qquad+2E\left[\sup_{0\le t\le t_1}\left|\int_0^t g(x(s\wedge\eta_i\wedge\theta_i),x([s\wedge\eta_i\wedge\theta_i]))\right.\right.\\
&\qquad\qquad\left.\left.-g(z(s\wedge\eta_i\wedge\theta_i),z([s\wedge\eta_i\wedge\theta_i]))dB(s)\right|^2\right].
\end{aligned} \tag{3.42}$$

By Doob martingale inequality, it is not difficult to show that

$$\begin{aligned}
&E\sup_{0\le t\le t_1}\left(\left|x(t\wedge\eta_i\wedge\theta_i)-y(t\wedge\eta_i\wedge\theta_i)\right|^2\right)\\
&\quad\le 2TE\int_0^{t_1}\left|f(x(s\wedge\eta_i\wedge\theta_i),x([s\wedge\eta_i\wedge\theta_i]))-f(z(s\wedge\eta_i\wedge\theta_i),z([s\wedge\eta_i\wedge\theta_i]))\right|^2ds\\
&\qquad+8E\int_0^{t_1}\left|g(x(s\wedge\eta_i\wedge\theta_i),x([s\wedge\eta_i\wedge\theta_i]))-g(z(s\wedge\eta_i\wedge\theta_i),z([s\wedge\eta_i\wedge\theta_i]))\right|^2ds.
\end{aligned} \tag{3.43}$$

Note from (H1) and Lemma 3.3 that

$$\begin{aligned}
&E\int_0^{t_1} \left|f\left(x(s\wedge\eta_i\wedge\theta_i),x([s\wedge\eta_i\wedge\theta_i])\right)-f\left(z(s\wedge\eta_i\wedge\theta_i),z([s\wedge\eta_i\wedge\theta_i])\right)\right|^2 ds\\
&\quad\le L_iE\int_0^{t_1} \left|x(s\wedge\eta_i\wedge\theta_i)-z(s\wedge\eta_i\wedge\theta_i)\right|^2+\left|x([s\wedge\eta_i\wedge\theta_i])-z([s\wedge\eta_i\wedge\theta_i])\right|^2 ds\\
&\quad\le 2L_iE\int_0^{t_1} \left|x(s\wedge\eta_i\wedge\theta_i)-y(s\wedge\eta_i\wedge\theta_i)\right|^2 ds\\
&\qquad+2L_iE\int_0^{t_1} \left|y(s\wedge\eta_i\wedge\theta_i)-z(s\wedge\eta_i\wedge\theta_i)\right|^2 ds\\
&\qquad+2L_iE\int_0^{t_1} \left|x([s\wedge\eta_i\wedge\theta_i])-y([s\wedge\eta_i\wedge\theta_i])\right|^2 ds\\
&\qquad+2L_iE\int_0^{t_1} \left|y([s\wedge\eta_i\wedge\theta_i])-z([s\wedge\eta_i\wedge\theta_i])\right|^2 ds\\
&\quad\le 4L_i\int_0^{t_1} E\sup_{0\le t\le s\wedge\eta_i\wedge\theta_i}\left|x(t)-y(t)\right|^2 ds+4L_iTC_1(i)h^{1/2}.
\end{aligned}\tag{3.44}$$

Similarly, we obtain that

$$\begin{aligned}
&E\int_0^{t_1} \left|g\left(x(s\wedge\eta_i\wedge\theta_i),x([s\wedge\eta_i\wedge\theta_i])\right)-g\left(z(s\wedge\eta_i\wedge\theta_i),z([s\wedge\eta_i\wedge\theta_i])\right)\right|^2 ds\\
&\quad\le 4L_i\int_0^{t_1} E\sup_{0\le t\le s\wedge\eta_i\wedge\theta_i}\left|x(t)-y(t)\right|^2 ds+4L_iTC_1(i)h^{1/2}.
\end{aligned}\tag{3.45}$$

Substituting (3.44) and (3.45) into (3.43) gives

$$\begin{aligned}
&E\sup_{0\le t\le t_1}\left(\left|x(t\wedge\eta_i\wedge\theta_i)-y(t\wedge\eta_i\wedge\theta_i)\right|^2\right)\\
&\quad\le 8(T+4)L_i\int_0^{t_1} E\sup_{0\le t\le s\wedge\eta_i\wedge\theta_i}\left|x(t)-y(t)\right|^2 ds+8(T+4)L_iTC_1(i)h^{1/2}.
\end{aligned}\tag{3.46}$$

By the Gronwall inequality, we must get

$$E\sup_{0\le t\le T}\left(\left|x(t\wedge\eta_i\wedge\theta_i)-y(t\wedge\eta_i\wedge\theta_i)\right|^2\right)\le C_2(i)h^{1/2},\tag{3.47}$$

where $C_2(i)=8T(T+4)L_iC(1+4d)e^{8T(T+4)L_i}$. □

Lemma 3.5. *Under the conditions (H1) and (H3) if $\varepsilon \in (0,1)$ and $T > 0$, then there exists a sufficiently large integer $\widehat{i}$ (dependent on ε and T) and sufficiently small $\widehat{h}$ such that*

$$P(\theta_{\widehat{i}} \leq T) \leq \varepsilon \quad \forall h \leq \widehat{h}. \tag{3.48}$$

Proof. By Itô formula, we have

$$\begin{aligned} dV(y(t)) &= \left(V_y(y(t))f(z(t),z([t])) + \frac{1}{2}\text{trace}\left[g^T(z(t),z([t]))V_{yy}(y(t))g(z(t),z([t]))\right]\right)dt \\ &\quad + V_y(y(t))g(z(t),z([t]))dB(t) \\ &= \Big(LV(y(t),y([t])) + V_y(y(t))\left[f(z(t),z([t])) - f(y(t),y([t]))\right] \\ &\quad + \frac{1}{2}\text{trace}\Big[g^T(z(t),z([t]))V_{yy}(y(t))g(z(t),z([t])) \\ &\qquad - g^T(y(t),y([t]))V_{yy}(y(t))g(y(t),y([t]))\Big]\Big)dt \\ &\quad + V_y(y(t))g(z(t),z([t]))dB(t). \end{aligned} \tag{3.49}$$

By condition (H1),

$$\begin{aligned} &V_y(y(t))\left[f(z(t),z([t])) - f(y(t),y([t]))\right] \\ &\quad + \frac{1}{2}\text{trace}\left[g^T(z(t),z([t]))V_{yy}(y(t))g(z(t),z([t])) - g^T(y(t),y([t]))V_{yy}(y(t))g(y(t),y([t]))\right] \\ &\quad = (V_y(y(s))\left[f(z(s),z([s])) - f(y(s),y([s]))\right] \\ &\qquad + \frac{1}{2}\text{trace}\Big(\left[g^T(z(s),z([s])) - g^T(y(s),y([s]))\right]V_{yy}(y(s))g(z(s),z([s])) \\ &\qquad + \frac{1}{2}\text{trace}\Big(g^T(y(s),y([s]))V_{yy}(y(s))\left[g(z(s),z([s])) - g(y(s),y([s]))\right] \\ &\quad \leq c_i(|y(t) - z(t)| + |y([t]) - z([t])|), \end{aligned} \tag{3.50}$$

where c_i denotes a positive constant independent of h. Substituting (3.50) into (3.49), we obtain that, for $0 \leq t \leq \theta_i$,

$$\begin{aligned} dV(y(t)) &\leq (LV(y(t),y([t])) + c_i(|y(t) - z(t)| + |y([t]) - z([t])|))dt \\ &\quad + V_y(y(t))g(z(t),z([t]))dB(t). \end{aligned} \tag{3.51}$$

Hence, for $t \in [n, n+1)$, we can integrate both sides of (3.51) from n to $t \wedge \theta_i$ and take the expectations

$$\begin{aligned} EV\left(y(t\wedge\theta_i)\right) &\leq EV\left(y(n)\right) + E\int_n^{t\wedge\theta_i} LV\left(y(s), y([s])\right)ds \\ &\quad + E\int_n^{t\wedge\theta_i} c_i\left(\left|y(t)-z(t)\right| + \left|y([t])-z([t])\right|\right)ds, \end{aligned} \tag{3.52}$$

while

$$\begin{aligned} & c_iE\int_n^t \left(\left|y(s\wedge\theta_i)-z(s\wedge\theta_i)\right| + \left|y([s\wedge\theta_i])-z([s\wedge\theta_i])\right|\right)ds \\ &\quad = c_i\int_n^t E\left|y(s\wedge\theta_i)-z(s\wedge\theta_i)\right|ds + c_i\int_n^t E\left|y([s\wedge\theta_i])-z([s\wedge\theta_i])\right|ds \\ &\quad \leq c_i\int_n^t \left(E\left|y(s\wedge\theta_i)-z(s\wedge\theta_i)\right|^2\right)^{1/2}ds + c_i\int_n^t \left(E\left|y([s\wedge\theta_i])-z([s\wedge\theta_i])\right|^2\right)^{1/2}ds \\ &\quad \leq 2c_i\int_n^t \left(E\sup_{0\leq u\leq s\wedge\theta_i}\left|y(u)-z(u)\right|^2\right)^{1/2}ds \\ &\quad \leq 2c_i\int_n^t \left[C_1(i)h^{1/2}\right]^{1/2}ds \\ &\quad \leq 2c_iT\left[C_1(i)h^{1/2}\right]^{1/2} \\ &\quad \leq C_3(i)h^{1/4}, \end{aligned} \tag{3.53}$$

where $C_3(i) = 2c_iT(C_1(i))^{1/2}$. Substituting this into (3.52) yields that

$$EV\left(y(t\wedge\theta_i)\right) \leq EV\left(y(n)\right) + \widetilde{\beta} + E\int_n^{t\wedge\theta_i}\left(LV\left(y(s), y([s])\right)\right)ds, \tag{3.54}$$

where $\widetilde{\beta} = C_3(i)h^{1/4}$. For $t \in [0, 1)$, by condition (H3), we obtain that

$$\begin{aligned} EV\left(y(t\wedge\theta_i)\right) &\leq V\left(y(0)\right) + \alpha E\int_0^{t\wedge\theta_i}\left[1 + V\left(y(s)\right) + V\left(y([s])\right)\right]ds + \widetilde{\beta} \\ &\leq V\left(y(0)\right) + \alpha\left[1 + V\left(y(0)\right)\right] + \widetilde{\beta} + \alpha E\int_0^{t\wedge\theta_i} V\left(y(s)\right)ds \\ &\leq \widetilde{\beta} + \widehat{\beta}_1 + \alpha\int_0^t EV\left(y(s\wedge\theta_i)\right)ds, \end{aligned} \tag{3.55}$$

where

$$\widehat{\beta}_1 = V(y(0)) + \alpha[1 + V(y(0))] = (1+\alpha)V(c_0) + \alpha < \infty. \tag{3.56}$$

Hence, by the Gronwall inequality,

$$\begin{aligned} EV(y(t \wedge \theta_i)) &\leq \widetilde{\beta} + \widehat{\beta}_1 + \alpha \int_0^t EV(y(s \wedge \theta_i))ds \\ &\leq \left(\widetilde{\beta} + \widehat{\beta}_1\right)e^{\alpha t} \\ &\leq \left(\widetilde{\beta} + \widehat{\beta}_1\right)e^{\alpha} \\ &< \infty, \end{aligned} \tag{3.57}$$

for $0 \leq t < 1$. Consequently,

$$EV(y(1 \wedge \theta_i)) = \lim_{t \to 1} EV(y(t \wedge \theta_i)) \leq \left(\widetilde{\beta} + \widehat{\beta}_1\right)e^{\alpha} < \infty. \tag{3.58}$$

Define

$$\gamma_i = \inf_{|y| \geq i} V(y), \quad \forall i \geq |c_0|, \tag{3.59}$$

and denote I_A as the indicator function of a set A, then we have

$$\left(\widetilde{\beta} + \widehat{\beta}_1\right)e^{\alpha} \geq EV(y(\theta_i \wedge 1)) \geq E\left(V(y(\theta_i))I_{\{\theta_i \leq 1\}}\right) \geq \gamma_i P(\theta_i \leq 1). \tag{3.60}$$

Letting $i \to \infty$, we have that $P(\theta_\infty \leq 1) = 0$, namely,

$$P(\theta_\infty > 1) = 1. \tag{3.61}$$

By (3.57) and (3.61),

$$EV(y(t)) \leq \left(\widetilde{\beta} + \widehat{\beta}_1\right)e^{\alpha} < \infty, \quad 0 \leq t \leq 1. \tag{3.62}$$

For $t \in [1, 2)$, by (3.54), we have

$$\begin{aligned} EV(y(t \wedge \theta_i)) &\leq EV(y(1)) + \alpha[1 + EV(y(1))] + \widetilde{\beta} + \alpha \int_1^t EV(y(s \wedge \theta_i))ds \\ &\leq \widetilde{\beta} + \widehat{\beta}_2 + \alpha \int_1^t EV(y(s \wedge \theta_i))ds, \end{aligned} \tag{3.63}$$

where

$$\widehat{\beta}_2 \leq (1+\alpha)\left(\widetilde{\beta}+\widehat{\beta}_1\right)e^{\alpha}+\alpha<\infty. \tag{3.64}$$

Hence, by the Gronwall inequality,

$$\begin{aligned}EV\left(y(t\wedge\theta_i)\right) &\leq \widetilde{\beta}+\widehat{\beta}_2+\alpha\int_1^t EV\left(y(s\wedge\theta_i)\right)ds\\ &\leq \left(\widetilde{\beta}+\widehat{\beta}_2\right)e^{\alpha(t-1)}\\ &\leq \left(\widetilde{\beta}+\widehat{\beta}_2\right)e^{\alpha}\\ &<\infty,\end{aligned} \tag{3.65}$$

for $1 \leq t < 2$. Consequently, we can obtain that

$$EV\left(y(2\wedge\theta_i)\right)=\lim_{t\to 2}EV\left(y(t\wedge\theta_i)\right)\leq\left(\widetilde{\beta}+\widehat{\beta}_2\right)e^{\alpha}<\infty. \tag{3.66}$$

In the same way, we have

$$EV\left(y(t)\right)\leq\left(\widetilde{\beta}+\widehat{\beta}_2\right)e^{\alpha}<\infty,\quad 1\leq t\leq 2. \tag{3.67}$$

Repeating this procedure, for $t \in [N-1, T)$, we can show that

$$EV\left(y(t\wedge\theta_i)\right)\leq\left(\widetilde{\beta}+\widehat{\beta}_T\right)e^{\alpha}<\infty, \tag{3.68}$$

where

$$\widehat{\beta}_T \leq (1+\alpha)\left(\widetilde{\beta}+\widehat{\beta}_{N-1}\right)e^{\alpha}+\alpha<\infty. \tag{3.69}$$

Consequently, we can obtain that

$$EV\left(y(T\wedge\theta_i)\right)=\lim_{t\to T}EV\left(y(t\wedge\theta_i)\right)\leq\left(\widetilde{\beta}+\widehat{\beta}_T\right)e^{\alpha}. \tag{3.70}$$

We compute

$$\left(\widetilde{\beta}+\widehat{\beta}_T\right)e^{\alpha}\geq EV\left(y(T\wedge\theta_i)\right)\geq E\left(V\left(y(\theta_i)\right)I_{\{\theta_i\leq T\}}\right)\geq\gamma_i P(\theta_i\leq T). \tag{3.71}$$

Then we have

$$P(\theta_i \le T) \le \frac{\left(\tilde{\beta} + \hat{\beta}_T\right)e^{\alpha}}{\gamma_i} = \frac{\left(C_3(i)h^{1/4} + \hat{\beta}_T\right)e^{\alpha}}{\gamma_i}. \tag{3.72}$$

Now, for any $\varepsilon \in (0,1)$, choose $i = \hat{i}$ sufficiently large for

$$\frac{\hat{\beta}_T e^{\alpha}}{\gamma_{\hat{i}}} \le \frac{\varepsilon}{2}, \tag{3.73}$$

and then choose $\hat{h}$ sufficiently small for

$$\frac{C_3\left(\hat{i}\right)\hat{h}^{1/4}e^{\alpha}}{\gamma_{\hat{i}}} \le \frac{\varepsilon}{2}. \tag{3.74}$$

Hence,

$$P\left(\theta_{\hat{i}} \le T\right) \le \varepsilon \quad \forall h \le \hat{h}. \tag{3.75}$$

□

The following theorems describe the convergence in probability of the EM method to (2.3) under the local Lipschitz condition (H1) and some additional conditions (H3).

Theorem 3.6. *Under the conditions (H1) and (H3), for arbitrarily small $\sigma \in (0,1)$,*

$$\lim_{h \to 0} P\left(\omega : \sup_{0 \le t \le T}\left|x(t) - y(t)\right| > \sigma\right) = 0, \tag{3.76}$$

for any $T > 0$.

Proof. For arbitrarily small $\sigma, \varepsilon \in (0,1)$. We set

$$\overline{\Omega} = \left\{\omega : \sup_{0 \le t \le T}\left|x(t) - y(t)\right| > \sigma\right\}. \tag{3.77}$$

By Theorem 3.2 and Lemma 3.5, there exists a pair of $\hat{i}$ and $\hat{h}$ such that

$$P\left(\eta_{\hat{i}} \le T\right) \le \frac{\varepsilon}{3},$$
$$P\left(\theta_{\hat{i}} \le T\right) \le \frac{\varepsilon}{3}, \quad \forall h \le \hat{h}. \tag{3.78}$$

For $h \le \hat{h}$,

$$\begin{aligned} P\left(\overline{\Omega}\right) &\le P\left(\overline{\Omega} \cap \{\eta_{\hat{i}} \wedge \theta_{\hat{i}} > T\}\right) + P(\eta_{\hat{i}} \wedge \theta_{\hat{i}} \le T) \\ &\le P\left(\overline{\Omega} \cap \{\eta_{\hat{i}} \wedge \theta_{\hat{i}} > T\}\right) + P(\eta_{\hat{i}} \le T) + P(\theta_{\hat{i}} \le T) \\ &\le P\left(\overline{\Omega} \cap \{\eta_{\hat{i}} \wedge \theta_{\hat{i}} > T\}\right) + \frac{2\varepsilon}{3}. \end{aligned} \tag{3.79}$$

By Lemma 3.4, we get

$$\begin{aligned} \sigma^2 P\left(\overline{\Omega} \cap \{\eta_{\hat{i}} \wedge \theta_{\hat{i}} > T\}\right) &\le E\left[\sup_{0\le t\le T} \left|x(t \wedge \eta_{\hat{i}} \wedge \theta_{\hat{i}}) - y(t \wedge \eta_{\hat{i}} \wedge \theta_{\hat{i}})\right|^2 I_{\{\eta_{\hat{i}} \wedge \theta_{\hat{i}} > T\}}\right] \\ &\le E \sup_{0\le t\le T} \left|x(t \wedge \eta_{\hat{i}} \wedge \theta_{\hat{i}}) - y(t \wedge \eta_{\hat{i}} \wedge \theta_{\hat{i}})\right|^2 \\ &\le C_2\left(\hat{i}\right) h^{1/2}. \end{aligned} \tag{3.80}$$

Hence,

$$P\left(\overline{\Omega} \cap \{\eta_{\hat{i}} \wedge \theta_{\hat{i}} > T\}\right) \le \frac{C_2\left(\hat{i}\right) h^{1/2}}{\sigma^2}. \tag{3.81}$$

For all sufficiently small h, we obtain

$$P\left(\overline{\Omega} \cap \{\eta_{\hat{i}} \wedge \theta_{\hat{i}} > T\}\right) \le \frac{\varepsilon}{3}. \tag{3.82}$$

From (3.79) and (3.82), we see that for all sufficiently small h,

$$P\left(\overline{\Omega}\right) \le \varepsilon, \tag{3.83}$$

which proves the theorem. □

Of course, $z(t)$ is computable but $y(t)$ is not, so the following theorem is much more useful in practice.

Theorem 3.7. *Under the conditions (H1) and (H3), for arbitrarily small* $\sigma \in (0,1)$,

$$\lim_{h\to 0} P\left(\omega : \sup_{0\le t\le T} |x(t) - z(t)| > \sigma\right) = 0, \tag{3.84}$$

for any $T > 0$.

Proof. For arbitrarily small $\sigma, \varepsilon \in (0,1)$. We denote

$$\widehat{\Omega} = \left\{\omega : \sup_{0\le t\le T} |x(t) - z(t)| > \sigma\right\}. \tag{3.85}$$

In the same way as Theorem 3.6, we can see that

$$P\left(\widehat{\Omega}\right) \le P\left(\widehat{\Omega} \cap \{\eta_{\hat{i}} \wedge \theta_{\hat{i}} > T\}\right) + \frac{2\varepsilon}{3}. \tag{3.86}$$

But by Lemma 3.3, we get

$$\begin{aligned}
\sigma^2 P\left(\widehat{\Omega} \cap \{\eta_{\hat{i}} \wedge \theta_{\hat{i}} > T\}\right) &\le E\left[\sup_{0\le t\le T} \left|x\left(t \wedge \eta_{\hat{i}} \wedge \theta_{\hat{i}}\right) - z\left(t \wedge \eta_{\hat{i}} \wedge \theta_{\hat{i}}\right)\right|^2 I_{\{\eta_{\hat{i}} \wedge \theta_{\hat{i}} > T\}}\right] \\
&\le E \sup_{0\le t\le T} \left|x\left(t \wedge \eta_{\hat{i}} \wedge \theta_{\hat{i}}\right) - z\left(t \wedge \eta_{\hat{i}} \wedge \theta_{\hat{i}}\right)\right|^2 \\
&\le 2E \sup_{0\le t\le T} \left|x\left(t \wedge \eta_{\hat{i}} \wedge \theta_{\hat{i}}\right) - y\left(t \wedge \eta_{\hat{i}} \wedge \theta_{\hat{i}}\right)\right|^2 \\
&\quad + 2E \sup_{0\le t\le T} \left|y\left(t \wedge \eta_{\hat{i}} \wedge \theta_{\hat{i}}\right) - z\left(t \wedge \eta_{\hat{i}} \wedge \theta_{\hat{i}}\right)\right|^2 \\
&\le 2E \sup_{0\le t\le T} \left|x\left(t \wedge \eta_{\hat{i}} \wedge \theta_{\hat{i}}\right) - y\left(t \wedge \eta_{\hat{i}} \wedge \theta_{\hat{i}}\right)\right|^2 \\
&\quad + 2E \sup_{0\le t\le T} \left|y\left(t \wedge \theta_{\hat{i}}\right) - z\left(t \wedge \theta_{\hat{i}}\right)\right|^2 \\
&\le 2\left(C_2\left(\hat{i}\right) + C_1\left(\hat{i}\right)\right) h^{1/2}.
\end{aligned} \tag{3.87}$$

therefore,

$$P\left(\widehat{\Omega} \cap \{\eta_{\hat{i}} \wedge \theta_{\hat{i}} > T\}\right) \le \frac{2\left(C_2\left(\hat{i}\right) + C_1\left(\hat{i}\right)\right) h^{1/2}}{\sigma^2}. \tag{3.88}$$

For all sufficiently small h, we obtain

$$P\left(\widehat{\Omega} \cap \{\eta_{\hat{i}} \wedge \theta_{\hat{i}} > T\}\right) \le \frac{\varepsilon}{3}. \tag{3.89}$$

From (3.86) and (3.89), we see that for all sufficiently small h,

$$P\left(\widehat{\Omega}\right) \le \varepsilon, \tag{3.90}$$

which proves the assertion (3.84). □

4. Numerical Example

Let us now discuss a numerical example to demonstrate the results which we obtain.

Example 4.1. Let us consider the stochastic differential equations with piecewise continuous arguments

$$dx(t) = \left[-x^3(t) + x([t])\right]dt + \left[\sin x^2(t) + x([t])\right]dB(t) \quad \forall t \geq 0. \tag{4.1}$$

Defining $V(x) = x^2$, we have

$$LV(x,y) = 2x\left(-x^3 + y\right) + \left(\sin x^2 + y\right)^2 \leq -2x^4 + 2xy + 2\left(\sin x^2\right)^2 + 2y^2 \leq 3\left(1 + x^2 + y^2\right), \tag{4.2}$$

where $\alpha = 3$. In other words, the equation satisfies condition (H3). By Theorem 3.1, we can conclude that the SEPCA (4.1) has a unique global solution $x(t)$ on $t \in [0,\infty)$. Moreover, the EM method can be applied to approximate the solution of the SEPCA (4.1). Given the stepsize $h = 1/m$, by (2.10), (2.11), and (2.12), the Euler method to (4.1) leads to a numerical process of the following type:

$$y_{km+l+1} = y_{km+l} + \left(-y_{km+l}^3 + y_{km}\right)h + \left(\sin y_{km+l}^2 + y_{km}\right)\Delta B_{km+l}. \tag{4.3}$$

The continuous Euler-Maruyama approximate solution is defined by

$$y(t) = y(0) + \int_0^t \left(-z^3(s) + z([s])\right)ds + \int_0^t \left(\sin z^2(s) + z([s])\right)dB(s), \tag{4.4}$$

where $z(t) = y_{km+l}$ and $z([t]) = y_{km}$ for $t \in [t_{km+l}, t_{km+l+1})$. By Theorems 3.6 and 3.7, we also have the convergence in probability of the EM method to (4.1) under the local Lipschitz condition (H1) and some additional conditions (H3).

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