

Research Article

Exponential Stability for a Class of Stochastic Reaction-Diffusion Hopfield Neural Networks with Delays

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This paper studies the asymptotic behavior for a class of delayed reaction-diffusion Hopfield neural networks driven by finite-dimensional Wiener processes. Some new sufficient conditions are established to guarantee the mean square exponential stability of this system by using Poincaré's inequality and stochastic analysis technique. The proof of the almost surely exponential stability for this system is carried out by using the Burkholder-Davis-Gundy inequality, the Chebyshev inequality and the Borel-Cantelli lemma. Finally, an example is given to illustrate the effectiveness of the proposed approach, and the simulation is also given by using the Matlab.

1. Introduction

Recently, the dynamics of Hopfield neural networks with reaction-diffusion terms have been deeply investigated because their various generations have been widely used in some practical engineering problems such as pattern recognition, associate memory, and combinatorial optimization (see [1–3]). However, under closer scrutiny, that a more realistic model would include some of the past states of the system, and theory of functional differential equations systems has been extensively developed [4, 5], meanwhile many authors have considered the asymptotic behavior of the neural networks with delays [6–9]. In fact random perturbation is unavoidable in any situation [3, 10]; if we include some environment noise in these systems, we can obtain a more perfect model of this situation

[3, 11–16]. So, this paper is devoted to the exponential stability of the following delayed reaction-diffusion Hopfield neural networks driven by finite-dimensional Wiener processes:

$$\begin{aligned} du_i(t, x) &= \left(\sum_{j=1}^l \frac{\partial}{\partial x_j} \left(D_{ij}(x) \frac{\partial u_i}{\partial x_j} \right) - a_i u_i + \sum_{j=1}^n c_{ij} f_j(u_j(t-r, x)) \right) dt \\ &\quad + \sum_{j=1}^m g_{ij}(u_i(t-r, x)) dW_j, \\ \frac{\partial u_i}{\partial \nu} \Big|_{\partial \mathcal{O}} &= 0, \quad t \geq 0, \\ u_i(\theta, x) &= \phi_i(\theta, x), \quad x \in \mathcal{O} \in \mathbb{R}^l, \theta \in [-r, 0], i = 1, 2, \dots, n. \end{aligned} \quad (1.1)$$

There are n neural network units in this system and $u_i(t, x)$ denote the potential of the cell i at t and x . a_i are positive constants and denote the rate with which the i th unit will reset its potential to the resting state in isolation when it is disconnected from the network and external inputs at t , and c_{ij} are the output connection weights from the j th neuron to the i th neuron. f_j are the active functions of the neural network. r is the time delay of a neuron. \mathcal{O} denotes an open bounded and connected subset of \mathbb{R}^l with a sufficient regular boundary $\partial \mathcal{O}$, ν is the unit outward normal on $\partial \mathcal{O}$, $\partial u_i / \partial \nu = (\nabla u_i, \nu)_{\mathbb{R}^l}$, and g_{ij} are noise intensities. Initial data ϕ_i are \mathcal{F}_0 -measurable and bounded functions, almost surely.

We denote $(\Omega, \mathcal{F}, \mathbb{P})$ a complete probability space with filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions (see [10]). $W_i(t)$, $i = 1, 2, \dots, m$, are scale standard Brownian motions defined on $(\Omega, \mathcal{F}, \mathbb{P})$.

For convenience, we rewrite system (1.1) in the vector form:

$$\begin{aligned} du &= (\nabla \cdot (D(x) \circ \nabla u) - Au + Cf(u(t-r))) dt + G(u(t-r)) dW, \\ \frac{\partial u(t, x)}{\partial \nu} \Big|_{\partial \mathcal{O}} &= 0, \quad t \geq 0, \\ u(0, x) &= \phi(x), \end{aligned} \quad (1.2)$$

where $C = (c_{ij})_{n \times n}$, $u = (u_1, u_2, \dots, u_n)^T$, $\nabla u = (\nabla u_1, \dots, \nabla u_n)^T$, $W = (W_1, W_2, \dots, W_m)^T$, $f(u) = (f_1(u_1), f_2(u_2), \dots, f_n(u_n))^T$, $A = \text{Diag}(a_1, a_2, \dots, a_n)$, $\phi = (\phi_1, \phi_2, \dots, \phi_n)^T$, $G(u) = (g_{ij}(u_i))_{n \times m}$, $D = (D_{ij})_{n \times l}$, and $D \circ \nabla u = (D_{ij} \partial u_i / \partial x_j)_{n \times l}$ is the Hadamard product of matrix D and ∇u ; for the definition of divergence operator $\nabla \cdot u$, we refer to [2, 3].

2. Preliminaries and Notations

In this paper, we introduce the following Hilbert spaces $H \triangleq L^2(\mathcal{O})$, $V \triangleq H^1(\mathcal{O})$, according to [17–19], $V \subset H = H' \subset V'$, where H', V' denote the dual of the space H, V , respectively, the injection is continuous, and the embedding is compact. $\|\cdot\|, \|\|\cdot\|\|$ represent the norm in H, V , respectively.

$U \triangleq (L^2(\mathcal{O}))^n$ is the space of vector-valued Lebesgue measurable functions on \mathcal{O} , which is a Banach space under the norm $\|u\|_U = (\sum_{i=1}^n \|u_i(x)\|^2)^{1/2}$.

$C \triangleq C([-r, 0], U)$ is the Banach space of all continuous functions from $[-r, 0]$ to U , when equipped with the sup-norm $\|\phi\|_C = \sup_{-r \leq s \leq 0} \|\phi\|_U$.

With any continuous \mathcal{F}_t -adapted U -valued stochastic process $u(t) : \Omega \rightarrow U, t \geq -r$, we associate a continuous \mathcal{F}_t -adapted C -valued stochastic process $u_t : \Omega \rightarrow C, t > 0$, by setting $u_t(s, x)(\omega) = u(t + s, x)(\omega), s \in [-r, 0], x \in \mathcal{O}$.

$C_{\mathcal{F}_0}^b$ denote the space of all bounded continuous processes $\phi : [-r, 0] \times \Omega \rightarrow U$ such that $\phi(\theta, \cdot)$ is \mathcal{F}_0 -measurable for each $\theta \in [-r, 0]$ and $E\|\phi\|_C < \infty$.

$\mathcal{L}(K)$ is the set of all linear bounded operators from K into K ; when equipped with the operator norm, it becomes a Banach space.

In this paper, we assume the following.

H1 f_i and G_{ij} are Lipschitz continuous with positive Lipschitz constants k_1, k_2 such that $|f_i(u) - f_i(v)| \leq k_1|u - v|$ and $|G_{ij}(u) - G_{ij}(v)| \leq k_2|u - v|, \forall u, v \in \mathbb{R}$, and $f_i(0) = 0, g_{ij}(0) = 0$.

H2 There exists $\alpha > 0$ such that $D_{ij}(x) \geq \alpha/l$.

H3 Let $\eta = 2\alpha\beta^2 + 2k_3 - nk_1^2\sigma^2e^r - mk_2^2e^r - 2 > 0, k_3 = \min\{a_i\}, \sigma = \max\{|c_{ij}|\}$.

Remark 2.1. We can infer from H1 that system (1.1) has an equilibrium $u(t, x, \omega) = 0$.

Let us define the linear operator as follows:

$$\begin{aligned} \mathfrak{A} : \Pi(\mathfrak{A}) \in U &\longrightarrow U, \\ \mathfrak{A}u &= \nabla \cdot (D(x) \circ \nabla u), \end{aligned} \tag{2.1}$$

and $\Pi(\mathfrak{A}) = \{u \in H^2(\mathcal{O})^n, \partial u / \partial \nu|_{\partial \mathcal{O}} = 0\}$.

Lemma 2.2 (Poincaré's inequality). *Let \mathcal{O} be a bounded domain in \mathbb{R}^l and ϕ belong to a collection of twice differentiable functions defined on \mathcal{O} into \mathbb{R} ; then*

$$\|\phi\| \leq \beta^{-1} \|\phi\| \tag{2.2}$$

where the constant β depends on the size of \mathcal{O} .

Lemma 2.3. *Let us consider the equation*

$$\begin{aligned} \frac{du}{dt} &= \mathfrak{A}u, \quad t \geq 0, \\ u(0) &= \phi. \end{aligned} \tag{2.3}$$

For every $\phi \in U$, let $u(t) = S(t)\phi$ denote the solution of (2.3); then $S(t)$ is a contraction map in U .

Proof. Now we take the inner product of (2.3) with $u(t)$ in U ; by employing the Gaussian theorem and condition H2, we get that $(\mathfrak{A}u, u) \leq -\alpha\|u\|_{H^1(\mathcal{O})^n}^2$, (\cdot, \cdot) is the inner product in U , $\|u\|_{H^1(\mathcal{O})^n}^2$ denote the norm of $H^1(\mathcal{O})^n$ (see [3]), which means

$$\frac{1}{2} \frac{d}{dt} \|u(t)\|_U^2 + \alpha \|u(t)\|_{H^1(\mathcal{O})^n}^2 \leq 0. \tag{2.4}$$

Thanks to the Poincaré inequality, one obtains

$$\frac{d}{dt} \|u(t)\|_U^2 + 2\alpha\beta^2 \|u(t)\|_U^2 \leq 0. \quad (2.5)$$

Multiplying $e^{2\alpha\beta^2 t}$ in both sides of the inequality, we have

$$\frac{d}{dt} \left(e^{2\alpha\beta^2 t} \|u(t)\|_U^2 \right) \leq 0. \quad (2.6)$$

Integrating the above inequality from 0 to t , we obtain

$$\|u(t)\|_U^2 \leq e^{-2\alpha\beta^2 t} \|\phi\|_U^2. \quad (2.7)$$

By the definition of $\|T(t)\|_{\mathcal{L}(U)}$, we have $\|T(t)\|_{\mathcal{L}(U)} \leq 1$. \square

Definition 2.4 (see [20–22]). A stochastic process $u(t) : [-r, +\infty) \times \Omega \rightarrow U$ is called a global mild solution of (1.1) if

- (i) $u(t)$ is adapted to \mathcal{F}_t
- (ii) $u(t)$ is measurable with $\int_0^\infty \|u(t)\|_U^2 dt < \infty$ almost surely and

$$\begin{aligned} u(t) &= S(t)\phi - \int_0^t S(t-s)A ds + \int_0^t S(t-s)f(u(s-r))ds + \int_0^t S(t-s)G(u(s-r))dW, \\ u(t) &= \phi \in C_{\mathcal{F}_0}^b, \quad t \in [-r, 0], \end{aligned} \quad (2.8)$$

for all $t \in [-r, +\infty)$ with probability one.

Definition 2.5. Equation (1.1) is said to be almost surely exponentially stable if, for any solution $u(t, x, \omega)$ with initial data $\phi \in C_{\mathcal{F}_0}^b$, there exists a positive constant λ such that

$$\limsup_{t \rightarrow \infty} \ln \|u_t\|_C \leq -\lambda, \quad u_t \in C, \text{ almost surely.} \quad (2.9)$$

Definition 2.6. System (1.1) is said to be exponentially stable in the mean square sense if there exist positive constants κ and α such that, for any solution $u(t, x, \omega)$ with the initial condition $\phi \in C_{\mathcal{F}_0}^b$, one has

$$E\|u(t)\|_C^2 \leq \kappa e^{-\alpha(t-t_0)}, \quad t \geq t_0, u_t \in C. \quad (2.10)$$

3. Main Result

Theorem 3.1. *Suppose conditions H1–H3 hold; then (1.1) is exponentially stable in the mean square sense.*

Proof. Let u be the mild solution of (1.1); thanks to the Itô formula, we observe that

$$\begin{aligned} d\left(e^{\lambda t} u_i^2\right) &= \lambda e^{\lambda t} u_i^2 dt + e^{\lambda t} \left(2u_i \left(\sum_{j=1}^l \frac{\partial}{\partial x_j} \left(D_{ij} \frac{\partial u_i}{\partial x_j} \right) - a_i u_i + \sum_{j=1}^n c_{ij} f_j(u_j(t-r)) \right) \right) dt \\ &\quad + e^{\lambda t} \left(G_i G_i^T \right) dt + 2e^{\lambda t} u_i G_i dW, \quad G_i = (G_{i1}, G_{i2}, \dots, G_{im}), \end{aligned} \quad (3.1)$$

where λ is a positive constant that will be defined below. Then, by integration between 0 and t , we find that

$$\begin{aligned} e^{\lambda t} u_i^2(t) &= \phi_i(0)^2 + \int_0^t \lambda e^{\lambda s} u_i^2 ds + 2 \int_0^t e^{\lambda s} \left(u_i \sum_{j=1}^l \frac{\partial}{\partial x_j} \left(D_{ij} \frac{\partial u_i}{\partial x_j} \right) \right) ds - 2 \int_0^t e^{\lambda s} a_i u_i^2 ds \\ &\quad + 2 \int_0^t e^{\lambda s} u_i \sum_{j=1}^n c_{ij} f_j(u_j(s-r)) ds + \int_0^t e^{\lambda s} G_i G_i^T ds + 2 \int_0^t e^{\lambda s} u_i G_i dW. \end{aligned} \quad (3.2)$$

Integrating the above equation over \mathcal{O} , by virtue of Fubini's theorem, we prove that

$$\begin{aligned} e^{\lambda t} \left\| u_i^2 \right\|^2 &= \left\| \phi_i(0) \right\|^2 + \lambda \int_0^t e^{\lambda s} \int_{\mathcal{O}} u_i^2 dx ds + \int_0^t e^{\lambda s} \int_{\mathcal{O}} 2u_i \sum_{j=1}^l \frac{\partial}{\partial x_j} \left(D_{ij} \frac{\partial u_i}{\partial x_j} \right) dx ds \\ &\quad - 2 \int_0^t e^{\lambda s} \int_{\mathcal{O}} a_i u_i^2 dx ds + 2 \int_0^t e^{\lambda s} \int_{\mathcal{O}} u_i \sum_{j=1}^n c_{ij} f_j(u_j(s-r)) dx ds \\ &\quad + \int_0^t e^{\lambda s} \int_{\mathcal{O}} G_i G_i^T dx ds + 2 \int_0^t e^{\lambda s} \int_{\mathcal{O}} u_i G_i dx dW. \end{aligned} \quad (3.3)$$

Taking the expectation on both sides of the last equation, by means of [3, 10, 16]

$$2E \int_0^t \int_{\mathcal{O}} e^{\lambda s} u_i G_i dx dW = 0. \quad (3.4)$$

Then, by Fubini's theorem, we have

$$\begin{aligned} e^{\lambda t} E \left\| u_i^2 \right\|^2 &= E \left\| \phi_i(0) \right\|^2 + \lambda \int_0^t e^{\lambda s} \int_{\mathcal{O}} E u_i^2 dx ds + 2E \int_0^t e^{\lambda s} \int_{\mathcal{O}} u_i \sum_{j=1}^l \frac{\partial}{\partial x_j} \left(D_{ij} \frac{\partial u_i}{\partial x_j} \right) dx ds \\ &\quad - 2 \int_0^t e^{\lambda s} \int_{\mathcal{O}} a_i E u_i^2 dx ds + 2E \int_0^t e^{\lambda s} \int_{\mathcal{O}} u_i \sum_{j=1}^n c_{ij} f_j(u_j(s-r)) dx ds \\ &\quad + E \int_0^t e^{\lambda s} \int_{\mathcal{O}} G_i G_i^T dx ds \\ &\triangleq I_1 + I_2 + I_3 + I_4 + I_5 + I_6. \end{aligned} \quad (3.5)$$

We observe that

$$I_1 \triangleq E\|\phi_i(0)\|^2 \leq \sup_{\theta \in [-r, 0]} E\|\phi_i(\theta)\|^2, \quad (3.6)$$

$$I_2 \triangleq \lambda \int_0^t \int_{\mathcal{O}} e^{\lambda s} E u_i^2 dx ds = \lambda \int_0^t e^{\lambda s} E \|u_i\|^2 ds. \quad (3.7)$$

From the Neumann boundary condition, by means of Green's formula and H2 (see [3, 6, 7]), we know

$$\begin{aligned} I_3 &\triangleq 2E \int_0^t \int_{\mathcal{O}} e^{\lambda s} \left(u_i \sum_{j=1}^l \frac{\partial}{\partial x_j} \left(D_{ij} \frac{\partial u_i}{\partial x_j} \right) \right) dx ds \\ &= -2E \int_0^t \int_{\mathcal{O}} e^{\lambda s} \sum_{j=1}^l D_{ij} \left(\frac{\partial u_i}{\partial x_j} \right)^2 dx ds \\ &\leq -2\alpha \int_0^t e^{\lambda s} E \|u_i\|^2 ds \leq -2\alpha\beta^2 \int_0^t e^{\lambda s} E \|u_i\|^2 ds. \end{aligned} \quad (3.8)$$

Then, by using the positiveness of a_i , one gets the relation

$$I_4 \triangleq -2 \int_0^t \int_{\mathcal{O}} e^{\lambda s} a_i E u_i^2 dx ds \leq -2k_3 \int_0^t e^{\lambda s} E \|u_i\|^2 ds, \quad (3.9)$$

where $k_3 = \min\{a_1, a_2, \dots, a_n\} > 0$. By using the Young inequality as well as condition H1, we have that

$$\begin{aligned} I_5 &\triangleq 2E \int_0^t \int_{\mathcal{O}} e^{\lambda s} u_i \sum_{j=1}^n c_{ij} f_j dx ds \\ &\leq \int_0^t \int_{\mathcal{O}} e^{\lambda s} \left(E|u_i|^2 + E \left| \sum_{j=1}^n c_{ij} f_j \right|^2 \right) dx ds \\ &\leq \int_0^t \int_{\mathcal{O}} e^{\lambda s} \left(E|u_i|^2 + \sigma^2 \sum_{j=1}^n E |f_j(u_j(s-r))|^2 \right) dx ds \\ &\leq \int_0^t \int_{\mathcal{O}} e^{\lambda s} \left(E|u_i|^2 + \sigma^2 k_1^2 \sum_{j=1}^n E |u_j(s-r)|^2 \right) dx ds \\ &\leq \int_0^t e^{\lambda s} \left(E \|u_i\|^2 + \sigma^2 k_1^2 E \|u(s-r)\|_U^2 \right) ds, \end{aligned} \quad (3.10)$$

where $\sigma = \max |c_{ij}|$, and

$$I_6 \triangleq \int_0^t \int_{\mathcal{O}} e^{\lambda s} E G_i G_i^T dx ds \leq m k_2^2 \int_0^t e^{\lambda s} E \|u_i(s-r)\|^2 ds. \quad (3.11)$$

We infer from (3.6)–(3.11) that

$$\begin{aligned} e^{\lambda t} E \|u_i(t)\|^2 &\leq \sup_{\theta \in [-r, 0]} E \|\phi_i(\theta)\|^2 - (2\alpha\beta^2 + 2k_3 - 1 - \lambda) \int_0^t e^{\lambda s} E \|u_i\|^2 ds \\ &\quad + \sigma^2 k_1^2 \int_0^t e^{\lambda s} \|u(t-r)\|_{\bar{U}}^2 ds + mk_2^2 \int_0^t e^{\lambda s} E \|u_i(s-r)\|^2 ds. \end{aligned} \quad (3.12)$$

Adding (3.12) from $i = 1$ to $i = n$, we obtain

$$\begin{aligned} e^{\lambda t} E \|u\|_{\bar{U}}^2 &\leq E \|\phi\|_C^2 - (2\alpha\beta^2 + 2k_3 - 1 - \lambda) \int_0^t e^{\lambda s} E \|u\|_{\bar{U}}^2 ds \\ &\quad + (nk_1^2 \sigma^2 + mk_2^2) \int_0^t e^{\lambda s} E \|u(s-r)\|_{\bar{U}}^2 ds, \end{aligned} \quad (3.13)$$

due to

$$\begin{aligned} \int_0^t e^{\lambda s} E \|u(s-r)\|_{\bar{U}}^2 ds &\leq e^{\lambda r} \int_{-r}^t e^{\lambda s} E \|u(s)\|_{\bar{U}}^2 ds \\ &\leq e^{2\lambda r} \int_{-r}^0 E \|\phi(s)\|_{\bar{U}}^2 ds + e^{\lambda r} \int_0^t e^{\lambda s} E \|u(s)\|_{\bar{U}}^2 ds \\ &\leq r e^{2\lambda r} E \|\phi\|_C^2 + e^{\lambda r} \int_0^t e^{\lambda s} E \|u(s)\|_{\bar{U}}^2 ds; \end{aligned} \quad (3.14)$$

we induce from the previous equations that

$$e^{\lambda t} E \|u\|_{\bar{U}}^2 \leq -c_1 \int_0^t e^{\lambda s} E \|u\|_{\bar{U}}^2 ds + c_2, \quad (3.15)$$

where $c_1 = 2\alpha\beta^2 + 2k_3 - 1 - nk_1^2 \sigma^2 e^{\lambda r} - mk_2^2 e^{\lambda r} - \lambda$ and $c_2 = (1 + mk_2^2 r e^{2\lambda r} + nk_1^2 \sigma^2 r e^{2\lambda r}) E \|\phi\|_C^2$; so we choose $\lambda = 1$ such that $c_1 = \eta > 0$. By using the classical Gronwall inequality we see that

$$e^{\lambda t} E \|u\|_{\bar{U}}^2 \leq c_2 e^{-\eta t}; \quad (3.16)$$

in other words, we get

$$E \|u\|_{\bar{U}}^2 \leq c_2 e^{-(\eta+1)t}. \quad (3.17)$$

So, for $t + \theta \geq t/2 \geq 0$, we also have

$$\begin{aligned} E \|u(t + \theta)\|_{\bar{U}}^2 &\leq c_2 e^{-(\eta+1)(t+\theta)}, \\ &\leq c_2 e^{-\kappa t}, \quad \theta \in [-r, 0], \quad \kappa = \frac{(\eta+1)}{2}, \end{aligned} \quad (3.18)$$

and we can conclude that

$$E\|u_t\|_C^2 \leq c_2 e^{-\kappa t}. \quad (3.19)$$

□

Theorem 3.2. *If the system (1.1) satisfies hypotheses H1–H3, then it is almost surely exponentially stable.*

Proof. Let $u(t)$ be the mild solution of (1.1). By Definition 2.4 as well as the inequality $(\sum_{i=1}^n a_i)^2 \leq n \sum_{i=1}^n a_i^2, a_i \in \mathbb{R}$, we have

$$\begin{aligned} E \sup_{N \leq t \leq N+1} \|u(t)\|_U^2 &\leq 4 \sup_{N \leq t \leq N+1} \|S(t-N+1)u(N)\|_U^2 \\ &\quad + 4 \sup_{N \leq t \leq N+1} \left\| \int_{N-1}^t -AS(t-s)u \, ds \right\|_U^2 \\ &\quad + 4 \sup_{N \leq t \leq N+1} \left\| \int_{N-1}^t S(t-s)Cf(u(s-r)) \, ds \right\|_U^2 \\ &\quad + 4 \sup_{N \leq t \leq N+1} \left\| \int_{N-1}^t S(t-s)G(u(s-r)) \, dW \right\|_U^2 \\ &\triangleq I_1 + I_2 + I_3 + I_4. \end{aligned} \quad (3.20)$$

Using the contraction of the map $S(t)$ and the result of Theorem 3.1, we find

$$\begin{aligned} I_1 &\triangleq 4 \sup_{N \leq t \leq N+1} E\|(S(t-N+1)u(N-1))\|_U^2 \\ &\leq 4 \sup_{N \leq t \leq N+1} E\|u_{N-1}\|_C^2 \leq 4c_2 e^{-\kappa(N-1)}. \end{aligned} \quad (3.21)$$

By the Hölder inequality, we obtain

$$\begin{aligned} I_2 &\triangleq 4 \sup_{N \leq t \leq N+1} E \left\| \int_{N-1}^t -AS(t-s)u \, ds \right\|_U^2 \\ &\leq 4 \sup_{N \leq t \leq N+1} (t-N+1) \int_{N-1}^t E\|-AS(t-s)u\|_U^2 \, ds \\ &\leq 8 \sup_{N \leq t \leq N+1} \int_{N-1}^t E\|Au\|_U^2 \, ds \\ &\leq 8k_4^2 \int_{N-1}^{N+1} E\|u\|_U^2 \, ds \leq 8k_4^2 \int_{N-1}^{N+1} E\|u_s\|_C^2 \, ds \\ &\leq 8k_4^2 c_2 \int_{N-1}^{N+1} e^{-\kappa s} \, ds \leq 8k_4^2 \rho_1 e^{-\kappa(N-1)}, \end{aligned} \quad (3.22)$$

where $\rho_1 = c_2/\kappa, k_4 = \max\{a_1, a_2, \dots, a_n\}$.

By virtue of Theorem 3.1, Hölder inequality, and H1, we have

$$\begin{aligned}
I_3 &\triangleq 4 \sup_{N \leq t \leq N+1} \left\| E \int_{N-1}^t S(t-s) C f(u(s-r)) ds \right\|_U^2 \\
&\leq 4 \sup_{N \leq t \leq N+1} (t-N+1) E \int_{N-1}^t \|C f(u(s-r))\|_U^2 ds \\
&\leq 8\sigma^2 \sup_{N \leq t \leq N+1} E \int_{N-1}^t \|f(u(s-r))\|_U^2 ds \\
&\leq 8k_1^2 \sigma^2 \int_{N-1}^{N+1} E \|u(s-r)\|_U^2 ds \leq 8k_1^2 \sigma^2 \int_{N-1}^{N+1} E \|u_s\|_C^2 ds \\
&\leq 8k_1^2 c_2 \int_{N-1}^{N+1} e^{-\kappa s} ds \leq 8k_1^2 \rho_1 e^{-\kappa(N-1)}.
\end{aligned} \tag{3.23}$$

Then, by the Burkholder-Davis-Gundy inequality (see [18, 22]), there exists c_3 such that

$$\begin{aligned}
I_4 &\triangleq 4 \sup_{N \leq t \leq N+1} E \left\| \int_{N-1}^t S(t-s) G(u(s-r)) dW \right\|_U^2 \\
&\leq 4c_3 \sup_{N \leq t \leq N+1} E \int_{N-1}^t \|S(t-s) G(u(s-r)) I\|_U^2 ds \\
&\leq 4c_3 k_2^2 \sup_{N \leq t \leq N+1} \int_{N-1}^t E \|u(s-r)\|_U^2 ds \leq 4c_3 k_2^2 \int_{N-1}^{N+1} E \|u_s\|_C^2 ds \\
&\leq 4c_3 k_2^2 c_2 \int_{N-1}^{N+1} e^{-\kappa s} ds \leq 4c_3 k_2^2 \rho_1 e^{-\kappa(N-1)},
\end{aligned} \tag{3.24}$$

where $I = (1, 1, \dots, 1)^T$ is an m -dimensional vector.

We can deduce from (3.21)–(3.24) that

$$E \sup_{N \leq t \leq N+1} \|u(t)\|_U^2 \leq \rho_2 e^{-\kappa(N-1)}, \tag{3.25}$$

where $\rho_2 = 4c_2 + (8k_4^2 + 8k_1^2 + 4c_3 k_2^2) \rho_1$.

Thus, for any positive constants ε_N , thanks to the Chebyshev inequality we have that

$$\begin{aligned}
P \left(\sup_{N \leq t \leq N+1} \|u(t)\|_U > \varepsilon_N \right) &\leq \frac{1}{\varepsilon_N^2} \sup_{N \leq t \leq N+1} E \|u(t)\|_U^2 \\
&\leq \frac{1}{\varepsilon_N^2} \rho_2 e^{-\kappa(N-1)}.
\end{aligned} \tag{3.26}$$

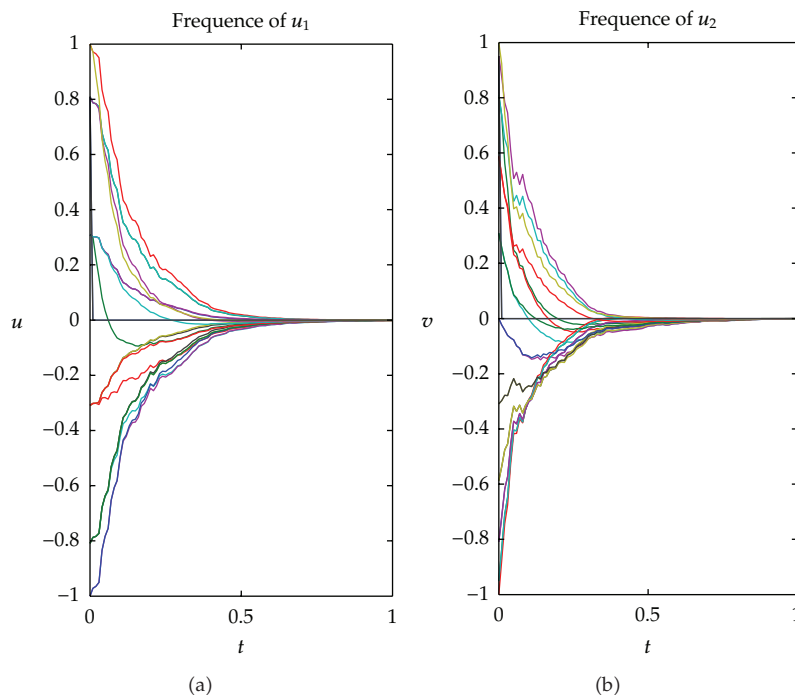


Figure 1

Due to the Borel-Cantelli lemma, we see that

$$\limsup_{t \rightarrow \infty} \frac{\ln \|u(t)\|_{\bar{U}}^2}{t} \leq -\kappa, \quad \text{almost surely.} \quad (3.27)$$

This completes the proof of the theorem. \square

4. Simulation

Consider two-dimensional stochastic reaction-diffusion recurrent neural networks with delay as follows:

$$\begin{aligned} du_1(t, x) &= (10\Delta u_1 - 7u_1 + 1.3 \tanh(u_1(t-1, x)))dt + u_1(t-1, x)dW, \\ du_2(t, x) &= (10\Delta u_2 - 7u_2 + \tanh(u_1(t-1, x)) - \tanh(u_2(t-1, x)))dt \\ &\quad + u_2(t-1, x)dW \\ \frac{\partial u_i(t, 0)}{\partial x} &= \frac{\partial u_i(t, 20)}{\partial x} = 0, \quad t \geq 0, \\ u_1(\theta, x) &= \cos(0.2\pi x), \quad u_2(\theta, x) = \cos(0.1\pi x), \quad x \in [0, 20], \theta \in [-1, 0]. \end{aligned} \quad (4.1)$$

Δ is the Laplace operator. We have $\beta \geq 1/20$, $\alpha \geq 10$, $k_1 = 1$, $k_2 = 1$, $k_3 = 7$, $\sigma = 1.3$, $n = 2$, and $\eta > 0$; by Theorems 3.1 and 3.2, this system is mean square exponentially stable as well as almost surely exponentially stable. The results can be shown in Figures 1, 2 and 3.

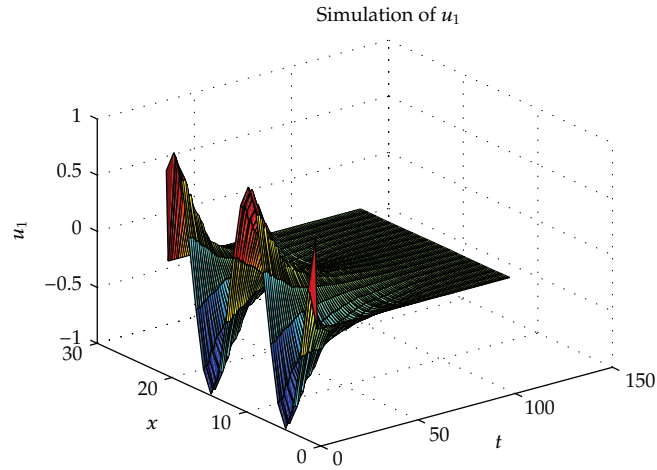


Figure 2

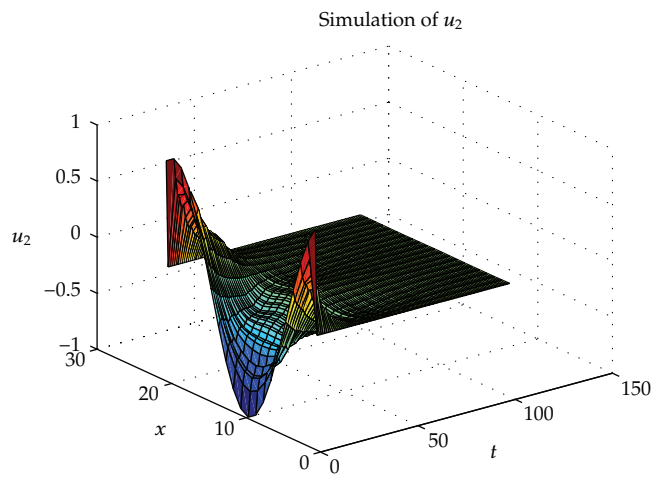


Figure 3

We use the forward Euler method to simulate this example [23–25]. We choose the time step $\Delta t = 0.01$ and space step $\Delta x = 1$, and $\delta = \Delta t / \Delta x^2 = 0.01$.

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