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*Research Article*

# Numerical Solution for Complex Systems of Fractional Order

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By using a complex transform, we impose a system of fractional order in the sense of Riemann-Liouville fractional operators. The analytic solution for this system is discussed. Here, we introduce a method of homotopy perturbation to obtain the approximate solutions. Moreover, applications are illustrated.

## 1. Introduction

Fractional models have been studied by many researchers to sufficiently describe the operation of variety of computational, physical, and biological processes and systems. Accordingly, considerable attention has been paid to the solution of fractional differential equations, integral equations, and fractional partial differential equations of physical phenomena. Most of these fractional differential equations have analytic solutions, approximation, and numerical techniques [1–3]. Numerical and analytical methods have included finite difference methods such as Adomian decomposition method, variational iteration method, homotopy perturbation method, and homotopy analysis method [4–7].

The idea of the fractional calculus (i.e., calculus of integrals and derivatives of any arbitrary real or complex order) was planted over 300 years ago. Abel in 1823 investigated the generalized tautochrone problem and for the first time applied fractional calculus techniques in a physical problem. Later Liouville applied fractional calculus to problems in potential theory. Since that time the fractional calculus has drawn the attention of many researchers in all areas of sciences (see [8–10]).

One of the most frequently used tools in the theory of fractional calculus is furnished by the Riemann-Liouville operators. It possesses advantages of fast convergence, higher stability, and higher accuracy to derive different types of numerical algorithms. In this paper, we will deal with scalar linear time-space fractional differential equations. The time is taken in sense of the Riemann-Liouville fractional operators. Also, This type of differential equation arises in many interesting applications. For example, the Fokker-Planck partial differential equation, bond pricing equations, and the Black-Scholes equations are in this class of differential equations (partial and fractional).

In [11], the author used complex transform to obtain a system of fractional order (nonhomogeneous) keeping the equivalency properties. By employing the homotopy perturbation method, the analytic solution is presented for coupled system of fractional order. Furthermore, applications are imposed such as wave equations of fractional order.

## 2. Fractional Calculus

This section concerns with some preliminaries and notations regarding the fractional calculus.

*Definition 2.1.* The fractional (arbitrary) order integral of the function $f$ of order $\alpha > 0$ is defined by

$$I_a^{\alpha} f(t) = \int_a^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau)\, d\tau. \tag{2.1}$$

When $a = 0$, we write $I_a^{\alpha} f(t) = f(t) * \phi_{\alpha}(t)$, where $(*)$ denoted the convolution product (see [12]), $\phi_{\alpha}(t) = t^{\alpha-1}/\Gamma(\alpha), t > 0$ and $\phi_{\alpha}(t) = 0, t \le 0$ and $\phi_{\alpha} \rightarrow \delta(t)$ as $\alpha \rightarrow 0$ where $\delta(t)$ is the delta function.

*Definition 2.2.* The fractional (arbitrary) order derivative of the function $f$ of order $0 \le \alpha < 1$ is defined by

$$D_a^{\alpha} f(t) = \frac{d}{dt} \int_a^t \frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)} f(\tau)\, d\tau = \frac{d}{dt} I_a^{1-\alpha} f(t). \tag{2.2}$$

*Remark 2.3.* From Definitions 2.1 and 2.2, $a = 0$, we have

$$\begin{aligned} D^{\alpha} t^{\mu} &= \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} t^{\mu-\alpha}, \quad \mu > -1;\ 0 < \alpha < 1, \\ I^{\alpha} t^{\mu} &= \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)} t^{\mu+\alpha}, \quad \mu > -1;\ \alpha > 0. \end{aligned} \tag{2.3}$$

The Leibniz rule is

$$D_a^{\alpha}[f(t)g(t)] = \sum_{k=0}^{\infty} \binom{\alpha}{k} D_a^{\alpha-k} f(t) D_a^k g(t) = \sum_{k=0}^{\infty} \binom{\alpha}{k} D_a^{\alpha-k} g(t) D_a^k f(t), \tag{2.4}$$

where

$$\binom{\alpha}{k} = \frac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha+1-k)}. \tag{2.5}$$

*Definition 2.4.* The Caputo fractional derivative of order $\mu > 0$ is defined, for a smooth function $f(t)$ by

$${}^{c}D^{\mu} f(t) := \frac{1}{\Gamma(n-\mu)} \int_0^t \frac{f^{(n)}(\zeta)}{(t-\zeta)^{\mu-n+1}}\, d\zeta, \tag{2.6}$$

where $n = [\mu] + 1$, (the notation $[\mu]$ stands for the largest integer not greater than $\mu$).

Note that there is a relationship between Riemann-Liouville differential operator and the Caputo operator

$$D_a^{\mu} f(t) = \frac{1}{\Gamma(1-\mu)} \frac{f(a)}{(t-a)^{\mu}} + {}^{c}D_a^{\mu} f(t), \tag{2.7}$$

and they are equivalent in a physical problem (i.e., a problem which specifies the initial conditions).

In this paper, we consider the following fractional differential equation:

$$D^{\alpha} u(t,z) = a(t,z) u_{zz} + b(t,z) u_z + c(t,z) u + f(t,z), \tag{2.8}$$

where $a \neq 0, b, c, u, f$ are complex valued functions, analytic in the domain $\mathfrak{D} := J \times U$; $J = [0,T], T \in (0,\infty)$ and $U := \{z \in \mathbb{C}, |z| \leq 1\}$.

The above equation involves well-known time fractional diffusion equations.

## 3. Complex Transforms

In this section, we will transform the fractional differential equation (2.8) into a coupled nonlinear system of fractional order has similar form. It was shown in [11] that the complex transform

$$u(t,z) = \sigma(z) \overline{u}(t,z), \tag{3.1}$$

where $\sigma \neq 0$ is a complex valued function of complex variable $z \in U$, reduces (2.8) into the system

$$\begin{aligned} D^{\alpha}\overline{v} &= \overline{a}_1 \overline{v}_{zz} - \overline{a}_2 \overline{w}_{zz} + \overline{b}_1 \overline{v}_z - \overline{b}_2 \overline{w}_z + \overline{c}_1 \overline{v} - \overline{c}_2 \overline{w} \\ D^{\alpha}\overline{w} &= \overline{a}_1 \overline{w}_{zz} + \overline{a}_2 \overline{v}_{zz} + \overline{b}_1 \overline{w}_z + \overline{b}_2 \overline{v}_z + \overline{c}_1 \overline{w} + \overline{c}_2 \overline{v}, \end{aligned} \tag{3.2}$$

where

$$\begin{aligned}
\overline{a}_1 &= a_1, \qquad \overline{a}_2 = a_2\\
\overline{b}_1 &= b_1 + \frac{2\sigma_{1z}(a_1\sigma_1 + a_2\sigma_2) + 2\sigma_{2z}(a_1\sigma_2 - a_2\sigma_1)}{\sigma_1^2 + \sigma_2^2}\\
\overline{b}_2 &= b_2 - \frac{2\sigma_{1z}(a_1\sigma_2 - a_2\sigma_1) + 2\sigma_{2z}(a_2\sigma_2 - a_1\sigma_1)}{\sigma_1^2 + \sigma_2^2}\\
\overline{c}_1 &= c_1 + \frac{\sigma_{1zz}(a_1\sigma_1 + a_2\sigma_2) + \sigma_{2zz}(a_1\sigma_2 - a_2\sigma_1) + \sigma_{1z}(b_1\sigma_1 + b_2\sigma_2) + \sigma_{2z}(b_1\sigma_2 - b_2\sigma_1)}{\sigma_1^2 + \sigma_2^2}\\
\overline{c}_2 &= c_2 - \frac{\sigma_{1zz}(a_1\sigma_2 - a_2\sigma_1) + \sigma_{2zz}(a_1\sigma_1 - a_2\sigma_2) + \sigma_{1z}(b_1\sigma_2 - b_2\sigma_1) - \sigma_{2z}(b_1\sigma_1 + b_2\sigma_2)}{\sigma_1^2 + \sigma_2^2}.\\
\sigma(z) &:= \sigma_1(z) + i\sigma_2(z), \qquad u(t,z) = v(t,z) + iw(t,z)\\
a(t,z) &= a_1(t,z) + ia_2(t,z), \qquad b(t,z) = b_1(t,z) + ib_2(t,z)\\
c(t,z) &= c_1(t,z) + ic_2(t,z), \qquad \overline{u}(t,z) = \overline{v}(t,z) + i\overline{w}(t,z).
\end{aligned} \tag{3.3}$$

Also, it was shown that the complex transform

$$u(t,z) = \rho(t,z)\overline{u}(t,z), \tag{3.4}$$

reduces the nonhomogenous equation

$$D^\alpha u(t,z) = a(t,z)u_{zz} + b(t,z)u_z + c(t,z)u + f(t,z), \tag{3.5}$$

into the system

$$\begin{aligned}
D^\alpha \overline{v} &= \overline{a}_1\overline{v}_{zz} - \overline{a}_2\overline{w}_{zz} + \overline{b}_1\overline{v}_z - \overline{b}_2\overline{w}_z + \overline{c}_1\overline{v} - \overline{c}_2\overline{w} + \overline{f}_1\\
D^\alpha \overline{w} &= \overline{a}_1\overline{w}_{zz} + \overline{a}_2\overline{v}_{zz} + \overline{b}_1\overline{w}_z + \overline{b}_2\overline{v}_z + \overline{c}_1\overline{w} + \overline{c}_2\overline{v} + \overline{f}_2,
\end{aligned} \tag{3.6}$$

where

$$\begin{aligned}
\overline{a}_1 &= a_1, \qquad \overline{a}_2 = a_2\\
\overline{b}_1 &= b_1 + \frac{2\rho_{1z}(a_1\rho_1 + a_2\rho_2) + 2\rho_{2z}(a_1\rho_2 - a_2\rho_1)}{\rho_1^2 + \rho_2^2}\\
\overline{b}_2 &= b_2 - \frac{2\rho_{1z}(a_1\rho_2 - a_2\rho_1) + 2\rho_{2z}(a_2\rho_2 - a_1\rho_1)}{\rho_1^2 + \rho_2^2}\\
\overline{c}_1 &= c_1 + \frac{\rho_{1zz}(a_1\rho_1 + a_2\rho_2) + \rho_{2zz}(a_1\rho_2 - a_2\rho_1) + \rho_{1z}(b_1\rho_1 + b_2\rho_2) + \rho_{2z}(b_1\rho_2 - b_2\rho_1)}{\rho_1^2 + \rho_2^2}
\end{aligned}$$

$$\overline{c}_2 = c_2 - \frac{\rho_{1zz}(a_1\rho_2 - a_2\rho_1) + \rho_{2zz}(a_1\rho_1 - a_2\rho_2) + \rho_{1z}(b_1\rho_2 - b_2\rho_1) - \rho_{2z}(b_1\rho_1 + b_2\rho_2)}{\rho_1^2 + \rho_2^2}$$

$$\overline{f}_1 = \frac{\rho_1(f_1 - h_1) + \rho_2(f_2 - h_2)}{\rho_1^2 + \rho_2^2}$$

$$\overline{f}_2 = \frac{\rho_2(f_1 - h_1) + \rho_1(f_2 - h_2)}{\rho_1^2 + \rho_2^2},$$

$$\overline{f} = \frac{f}{\rho} - \frac{\alpha\rho_t}{\rho} I^{1-\alpha}\overline{u} = \overline{f}_1 + i\overline{f}_2$$

$$h_1 = \rho_{1t} I^{1-\alpha}\overline{v} - \rho_{2t} I^{1-\alpha}\overline{w}$$

$$h_2 = \rho_{2t} I^{1-\alpha}\overline{v} + \rho_{1t} I^{1-\alpha}\overline{w},$$

$$\rho(t,z) := \rho_1(t,z) + i\rho_2(t,z) \neq 0. \tag{3.7}$$

## 4. Numerical Solution

Let us put

$$\begin{aligned} F_1(t,z,\overline{v},\overline{w}) &= \phi_1(t,z) - L_1(\overline{v},\overline{w}) - N_1(\overline{v},\overline{w}) \\ F_2(t,z,\overline{v},\overline{w}) &= \phi_2(t,z) - L_2(\overline{v},\overline{w}) - N_2(\overline{v},\overline{w}), \end{aligned} \tag{4.1}$$

where $\phi_1(t,z)$ and $\phi_2(t,z)$ are arbitrary functions;

$$\begin{aligned} L_1(\overline{v},\overline{w}) &= -\ell_1(\overline{v}) + \ell_1(\overline{w}) = -\left(\overline{a}_1\overline{v}_{zz} + \overline{b}_1\overline{v}_z + \overline{c}_1\overline{v}\right) + \left(\overline{a}_2\overline{w}_{zz} + \overline{b}_2\overline{w}_z + \overline{c}_2\overline{w}\right), \\ L_2(\overline{v},\overline{w}) &= -(\ell_2(\overline{v}) + \ell_2(\overline{w})) = -\left(\overline{a}_2\overline{v}_{zz} + \overline{b}_2\overline{v}_z + \overline{c}_2\overline{v} + \overline{a}_1\overline{w}_{zz} + \overline{b}_1\overline{w}_z + \overline{c}_1\overline{w}\right) \end{aligned} \tag{4.2}$$

are the linear parts of $F_1$ and $F_2$, respectively. While $N_1$ and $N_2$ are the nonlinear parts of $F_1$ and $F_2$, respectively. Moreover, let us set the homotopy system

$$\begin{aligned} (1-p)D^\alpha\overline{v}(t,z) + pD^\alpha\overline{v}(t,z) - \phi_1(t,z) + L_1(\overline{v},\overline{w}) + N_1(\overline{v},\overline{w}) &= 0, \quad p \in [0,1] \\ (1-p)D^\alpha\overline{w}(t,z) + pD^\alpha\overline{w}(t,z) - \phi_2(t,z) + L_2(\overline{v},\overline{w}) + N_2(\overline{v},\overline{w}) &= 0, \end{aligned} \tag{4.3}$$

where

$$\begin{aligned} \overline{v}(t,z) &= \sum_{n=0}^{\infty} v_n(t,z)p^n, \qquad \overline{w}(t,z) = \sum_{n=0}^{\infty} w_n(t,z)p^n, \\ N_1(\overline{v},\overline{w}) &= \sum_{k=0}^{\infty} N_k p^k, \qquad N_2(\overline{v},\overline{w}) = \sum_{k=0}^{\infty} \widetilde{N}_k p^k. \end{aligned} \tag{4.4}$$

Hence we obtain the following system:

$$
\begin{aligned}
D^{\alpha}\begin{pmatrix}\overline{v}_0(t,z)\\ \overline{v}_1(t,z)\\ \overline{v}_2(t,z)\\ \vdots\\ \overline{v}_n(t,z)\end{pmatrix} &= \ell_1\begin{pmatrix}0\\ \overline{v}_0(t,z)\\ \overline{v}_1(t,z)\\ \vdots\\ \overline{v}_{n-1}(t,z)\end{pmatrix} - \ell_1\begin{pmatrix}0\\ \overline{w}_0(t,z)\\ \overline{w}_1(t,z)\\ \vdots\\ \overline{w}_{n-1}(t,z)\end{pmatrix}\\
&\quad - \begin{pmatrix}0\\ N_0(\overline{v}_0(t,z))\\ N_1(\overline{v}_0(t,z),\overline{v}_1(t,z))\\ \vdots\\ N_{n-1}(\overline{v}_0(t,z),\overline{v}_1(t,z),\dots,\overline{v}_{n-1}(t,z))\end{pmatrix} + \begin{pmatrix}0\\ \phi_1(t,z)\\ 0\\ \vdots\\ 0\end{pmatrix},\\
D^{\alpha}\begin{pmatrix}\overline{w}_0(t,z)\\ \overline{w}_1(t,z)\\ \overline{w}_2(t,z)\\ \vdots\\ \overline{w}_n(t,z)\end{pmatrix} &= \ell_2\begin{pmatrix}0\\ \overline{v}_0(t,z)\\ \overline{v}_1(t,z)\\ \vdots\\ \overline{v}_{n-1}(t,z)\end{pmatrix} + \ell_2\begin{pmatrix}0\\ \overline{w}_0(t,z)\\ \overline{w}_1(t,z)\\ \vdots\\ \overline{w}_{n-1}(t,z)\end{pmatrix}\\
&\quad - \begin{pmatrix}0\\ \widetilde{N}_0(\overline{w}_0(t,z))\\ \widetilde{N}_1(\overline{w}_0(t,z),\overline{w}_1(t,z))\\ \vdots\\ \widetilde{N}_{n-1}(\overline{w}_0(t,z),\overline{w}_1(t,z),\dots,\overline{w}_{n-1}(t,z))\end{pmatrix} + \begin{pmatrix}0\\ \phi_2(t,z)\\ 0\\ \vdots\\ 0\end{pmatrix},
\end{aligned}
\tag{4.5}
$$

where

$$
\begin{aligned}
\overline{v}_0(t,z) &= \sum_{j=0}^{k-1}\nu(z)v_0^{(j)}\frac{t^j}{j!}, \quad \alpha\in(k-1,k),\ \nu(z)=\sum_{n=0}^{\infty}\nu_n z^n,\\
\overline{v}_1(t,z) &= -I^{\alpha}(L_1\overline{v}_0(t,z)) - I^{\alpha}N_0(\overline{v}_0(t,z)) + I^{\alpha}\phi_1(t,z),\\
&\ \ \vdots\\
\overline{v}_n(t,z) &= -I^{\alpha}(L_1\overline{v}_{n-1}(t,z)) - I^{\alpha}N_{n-1}(\overline{v}_0(t,z),\dots,\overline{v}_{n-1}(t,z)),\\
\overline{w}_0(t,z) &= \sum_{j=0}^{k-1}\varpi(z)w_0^{(j)}\frac{t^j}{j!}, \quad \alpha\in(k-1,k),\ \varpi(z)=\sum_{n=0}^{\infty}\varpi_n z^n,\\
\overline{w}_1(t,z) &= -I^{\alpha}(L_2\overline{w}_0(t,z)) - I^{\alpha}\widetilde{N}_0(\overline{w}_0(t,z)) + I^{\alpha}\phi_2(t,z),\\
&\ \ \vdots\\
\overline{w}_n(t,z) &= -I^{\alpha}(L_2\overline{w}_{n-1}(t,z)) - I^{\alpha}\widetilde{N}_{n-1}(\overline{w}_0(t,z),\dots,\overline{w}_{n-1}(t,z)).
\end{aligned}
\tag{4.6}
$$

Consequently, we have the approximate solution

$$\overline{v}(t,z) = \sum_{j=0}^{\infty} \nu(z) v_0^{(j)} \frac{t^j}{j!} - I^{\alpha}\left( \sum_{j=0}^{\infty} L_1 \overline{v}_j(t,z) + \sum_{j=0}^{\infty} N_j - \phi_1(t,z) \right)$$
$$\overline{w}(t,z) = \sum_{j=0}^{\infty} \varpi(z) w_0^{(j)} \frac{t^j}{j!} - I^{\alpha}\left( \sum_{j=0}^{\infty} L_2 \overline{w}_j(t,z) + \sum_{j=0}^{\infty} \widetilde{N}_j - \phi_2(t,z) \right). \tag{4.7}$$

Thus, we impose a nonlinear integral equation in the following formula:

$$\overline{v}(t,z) = \sum_{j=0}^{\infty} \nu(z) v_0^{(j)} \frac{t^j}{j!} + \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} F_1(\tau,\zeta,u)\, d\tau$$
$$\overline{w}(t,z) = \sum_{j=0}^{\infty} \varpi(z) w_0^{(j)} \frac{t^j}{j!} + \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} F_2(\tau,\zeta,u)\, d\tau. \tag{4.8}$$

Now we can sake the main result of this section.

**Theorem 4.1.** *Consider the fractional differential system* (3.6) *subject to the initial conditions*

$$\left(\overline{v}^{(m)}(0,z) = \overline{v}_0^{(m)}(z), \overline{w}^{(m)}(0,z) = \overline{w}_0^{(m)}(z),\ m = 0,1,2,\ldots,k-1\right). \tag{4.9}$$

*The homotopy perturbation technique implies that the initial value problem (*(3.6)–(4.9)*) can be expressed as a nonlinear integral equation of the form* (4.8).

We proceed to prove the analytical convergence of our solution.

**Theorem 4.2.** *Suppose the sequence* $u_n(t,z) = \binom{\overline{v}_n(t,z)}{\overline{w}_n(t,z)}$ *of the homotopy series* $\overline{v}(t,z) = \sum_{n=0}^{\infty} \overline{v}_n(t,z)p^n$ *and* $\overline{w}(t,z) = \sum_{n=0}^{\infty} \overline{w}_n(t,z)p^n$ *is defined for* $p \in [0,1]$. *Assume the initial approximation* $u_0(t,z) = \binom{\overline{v}_0(t,z)}{\overline{w}_0(t,z)}$ *inside the domain of the solution* $u(t,z) = \binom{\overline{v}(t,z)}{\overline{w}(t,z)}$. *If* $\|u_{n+1}\| \le \rho\|u_n\|$ *for all* $n$, *where* $0 < \rho < 1$, *then the solution is absolutely convergent when* $p = 1$.

*Proof.* Let $C_n(t,z)$ be the sequence of partial sum of the homotopy series. Our aim is to show that $C_n(t,z)$ is a Cauchy sequence. Consider

$$\begin{aligned}\|C_{n+1}(t,z) - C_n(t,z)\| &= \|u_{n+1}(t,z)\| \\ &\le \rho\|u_n(t,z)\| \le \rho^2\|u_{n-1}(t,z)\| \\ &\le \cdots \le \rho^{n+1}\|u_0(t,z)\|.\end{aligned} \tag{4.10}$$

For $n \geq m, n \in \mathbb{N}$, we have

$$\begin{aligned}
\|C_n(t,z) - C_m(t,z)\| &= \|C_n(t,z) - C_{n-1}(t,z) + C_{n-1}(t,z) - C_{n-2}(t,z) + \cdots + C_{m+1}(t,z) - C_m(t,z)\| \\
&\leq \|C_n(t,z) - C_{n-1}(t,z)\| + \|C_{n-1}(t,z) - C_{n-2}(t,z)\| \\
&\quad + \cdots + \|C_{m+1}(t,z) - C_m(t,z)\| \\
&\leq \frac{1-\rho^{n-m}}{1-\rho}\rho^{m+1}\|u_0(t,z)\|.
\end{aligned} \tag{4.11}$$

Hence

$$\lim_{n,m\to\infty}\|C_n(t,z) - C_m(t,z)\| = 0; \tag{4.12}$$

therefore, $C_n(t,z)$ is a Cauchy sequence in the complex Banach space and consequently yields that the series solution is convergent. This completes the proof. □

Recently the homotopy methods are used to obtain approximate analytic solutions of the time-fractional nonlinear equation and time-space-fractional nonlinear equation (see [12–17]).

## 5. Applications

In this section, we will consider the pump wave equations along the fiber (Schrödinger equations). These types of equations are the fundamental equations for describing non-relativistic quantum mechanical behavior taking the form

$$iD^{\alpha}u(t,z) = -\frac{1}{2}u_{zz}(t,z) - |u|^2 u(t,z). \tag{5.1}$$

Under the transform $u = \overline{u} = \overline{v} + i\overline{w}$ such that either $|u|^2 = |\overline{v}|^2$ or $|u|^2 = |\overline{w}|^2$, we have the uncoupled system

$$\begin{aligned}
iD^{\alpha}\overline{v}(t,z) &= -\frac{1}{2}\overline{v}_{zz}(t,z) - |\overline{v}|^2\overline{v}(t,z) \\
iD^{\alpha}\overline{w}(t,z) &= -\frac{1}{2}\overline{w}_{zz}(t,z) - |\overline{w}|^2\overline{w}(t,z),
\end{aligned} \tag{5.2}$$

where $0 < \alpha \leq 1$. Subject to the initial conditions

$$\overline{v}_0(0,z) = e^{iz}, \qquad \overline{w}_0(0,z) = 1. \tag{5.3}$$

Operating (5.2) by $I^{\alpha}$, we have

$$\begin{aligned} i\overline{v}(t,z) &= \overline{v}_0(0,z) + I^{\alpha}\left[-\frac{1}{2}\overline{v}_{zz}(t,z) - |\overline{v}|^2\overline{v}(t,z)\right] \\ i\overline{w}(t,z) &= \overline{w}_0(0,z) + I^{\alpha}\left[-\frac{1}{2}\overline{w}_{zz}(t,z) - |\overline{w}|^2\overline{w}(t,z)\right]. \end{aligned} \tag{5.4}$$

By the same computation as in Section 5, we receive

$$\begin{aligned} \overline{v}_0 &= e^{iz}, & \overline{w}_0 &= 1 \\ \overline{v}_1 &= \frac{it^{\alpha}}{2\Gamma(\alpha+1)}e^{iz}, & \overline{w}_1 &= \frac{it^{\alpha}}{\Gamma(\alpha+1)} \\ \overline{v}_2 &= \frac{(it^{\alpha})^2}{2^2\Gamma(2\alpha+1)}e^{iz}, & \overline{w}_2 &= \frac{(it^{\alpha})^2}{\Gamma(2\alpha+1)} \\ &\vdots \\ \overline{v}_n &= \frac{(it^{\alpha})^n}{2^n\Gamma(n\alpha+1)}e^{iz}, & \overline{w}_n &= \frac{(it^{\alpha})^n}{\Gamma(n\alpha+1)}. \end{aligned} \tag{5.5}$$

Thus the solution $\overline{u}$ is given by

$$\overline{u}(t,z) = \left(\sum_{n=0}^{\infty}\frac{(it^{\alpha})^n}{2^n\Gamma(n\alpha+1)}e^{iz}, \sum_{n=0}^{\infty}\frac{(it^{\alpha})^n}{\Gamma(n\alpha+1)}\right)^T. \tag{5.6}$$

Moreover, under the same transform, (5.1) reduces to coupled system

$$\begin{aligned} iD^{\alpha}\overline{v}(t,z) &= -\frac{1}{2}\overline{v}_{zz}(t,z) - \left(|\overline{v}|^2 + |\overline{w}|^2\right)\overline{v}(t,z) \\ iD^{\alpha}\overline{w}(t,z) &= -\frac{1}{2}\overline{w}_{zz}(t,z) - \left(|\overline{v}|^2 + |\overline{w}|^2\right)\overline{w}(t,z), \end{aligned} \tag{5.7}$$

Operating (5.7) by $I^{\alpha}$, we have

$$\begin{aligned} i\overline{v}(t,z) &= \overline{v}_0(0,z) + I^{\alpha}\left[-\frac{1}{2}\overline{v}_{zz}(t,z) - \left(|\overline{v}|^2 + |\overline{w}|^2\right)\overline{v}(t,z)\right] \\ i\overline{w}(t,z) &= \overline{w}_0(0,z) + I^{\alpha}\left[-\frac{1}{2}\overline{w}_{zz}(t,z) - \left(|\overline{v}|^2 + |\overline{w}|^2\right)\overline{w}(t,z)\right]. \end{aligned} \tag{5.8}$$

Therefore,

$$\overline{u}(t,z) = \left(\sum_{n=0}^{\infty}\frac{(it^{\alpha})^n}{2^n\Gamma(n\alpha+1)}e^{iz}, \sum_{n=0}^{\infty}\frac{(it^{\alpha})^n}{\Gamma(n\alpha+1)}\right)^T, \quad \left|e^{iz}\right| = 1. \tag{5.9}$$

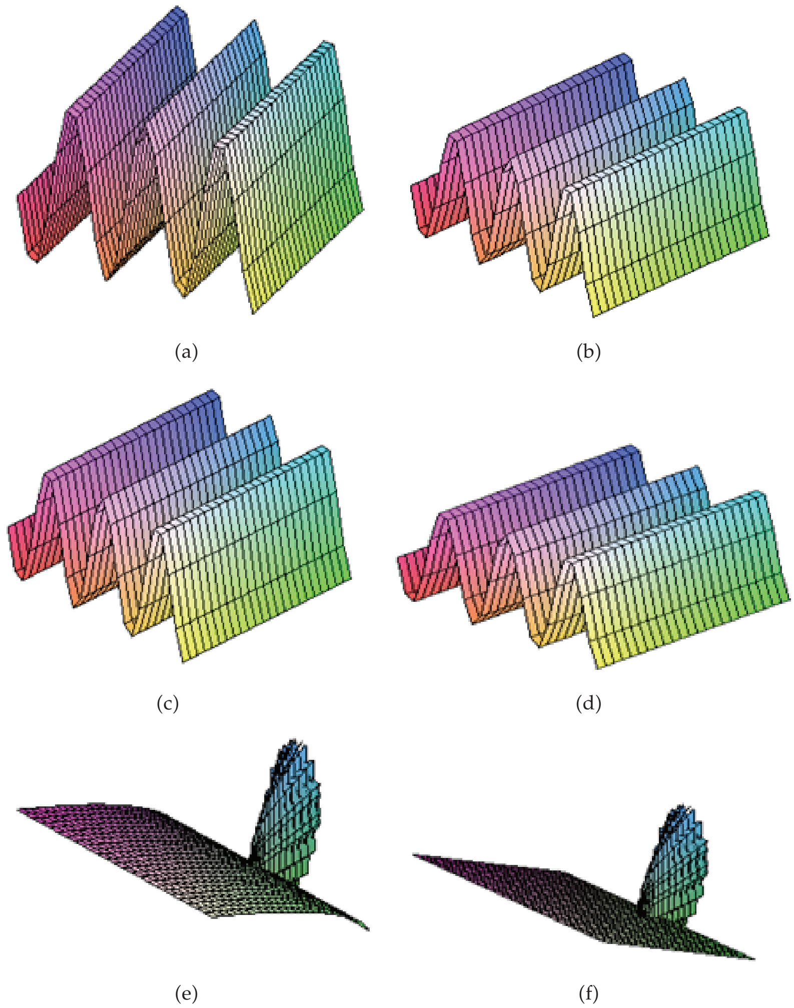

Figure 1: ((a)–(d)) The solution $\overline{v}$ when $\alpha = 0.5, \alpha = 0.75, \alpha = 0.9$, and $\alpha = 1$, respectively. ((e),(f)) the solution $(\overline{u}, \overline{v})$ when $\alpha = 0.5$ and $\alpha = 1$.

## 6. Conclusion

We suggested two types of complex transforms for systems of fractional differential equations. We concluded that the complex fractional differential equations can be transformed into coupled and uncoupled system of homogeneous and nonhomogeneous types. Moreover, we employed the homotopy perturbation scheme for solving the nonlinear complex fractional differential systems. The convergence of the method is discussed in a domain that contains the initial solution. The Schrödinger equation is illustrated as an application. This type of equation is used in the quantum mechanics, which describes how the quantum state of a physical system changes with time. In the standard quantum mechanics, the wave function is the most complete explanation that can be specified to a physical system. Solutions of the Schrdinger′s equation describe not only molecular, atomic, and subatomic systems, but also macroscopic systems (see Figure 1).

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