Research Article

Optimality Conditions for Infinite Order Distributed Parabolic Systems with Multiple Time Delays Given in Integral Form

Bahaa G. M.[1, 2]

[1] *Department of Mathematics, Faculty of Science, Taibah University,*
  Al-Madinah Al-Munawwarah, Saudi Arabia
[2] *Department of Mathematics, Faculty of Science, Beni-Suef University, Beni-Suef, Egypt*

Correspondence should be addressed to G. M. Bahaa, bahaa_gm@hotmail.com

The optimal boundary control problem for $(n \times n)$ infinite order distributed parabolic systems with multiple time delays given in the integral form both in the state equations and in the Neumann boundary conditions is considered. Constraints on controls are imposed. Necessary and suffacient optimality conditions for the Neumann problem with the quadratic performance functional are derived.

1. Introduction

Distributed parameters systems with delays can be used to describe many phenomena in the real world. As is well known, heat conduction, properties of elastic-plastic material, fluid dynamics, diffusion-reaction processes, the transmission of the signals at a certain distance by using electric long lines, and so forth, all lie within this area. The object that we are studying (temperature, displacement, concentration, velocity, etc.) is usually referred to as the state.

During the last twenty years, equations with deviating argument have been applied not only in applied mathematics, physics, and automatic control, but also in some problems of economy and biology. Currently, the theory of equations with deviating arguments constitutes a very important subfield of mathematical control theory.

Consequently, equations with deviating arguments are widely applied in optimal control problems of distributed parameter system with time delays [1].

The optimal control problems of distributed parabolic systems with time-delayed boundary conditions have been widely discussed in many papers and monographs. A fundamental study of such problems is given by [2] and was next developed by [3, 4]. It was

also intensively investigated by [1, 5–16] in which linear quadratic problem for parabolic systems with time delays given in the different form (constant, time delays, time-varying delays, time delays given in the integral form, etc.) was presented.

The necessary and sufficient conditions of optimality for systems consist of only one equation and for $(n \times n)$ systems governed by different types of partial differential equations defined on spaces of functions of infinitely many variables and also for infinite order systems are discussed for example in [9, 11, 15–18] in which the argument of [19, 20] was used.

Making use of the Dubovitskii-Milyutin Theorem in [13, 21–28] the necessary and sufficient conditions of optimality for similar systems governed by second order operator with an infinite number of variables and also for infinite order systems were investigated. The interest in the study of this class of operators is stimulated by problems in quantum field theory.

In particular, the papers of [1, 8] present necessary and sufficient optimality conditions for the Neumann problem with quadratic performance functionals, applied to a single one equation of second-order parabolic system with fixed time delay and with multiple time delays given in the integral form both in the state equations and in the Neumann boundary conditions, respectively. Such systems constitute a more complex case of distributed parameter systems with time delays given in the integral form.

Also in [9, 11] time-optimal boundary control for a single one equation distributed infinite order parabolic and hyperbolic systems in which constant time lags appear in the integral form both in the state equation and in the Neumann boundary condition is present. Some specific properties of the optimal control are discussed.

In this paper we recall the problem in a more general formulation. A distributed parameter for infinite order parabolic $(n \times n)$ systems with multiple time delays given in the integral form both in the state equations and in the Neumann boundary conditions is considered. Such an infinite order parabolic system can be treated as a generalization of the mathematical model for a plasma control process. The quadratic performance functionals defined over a fixed time horizon are taken and some constraints are imposed on the initial state and the boundary control. Such a system may be viewed as a linear representation of many diffusion processes, in which time-delayed signals are introduced at a spatial boundary, and there is a freedom in choosing the controlled process initial state. Following a line of the Lions scheme, necessary and sufficient optimality conditions for the Neumann problem applied to the above system were derived. The optimal control is characterized by the adjoint equations.

This paper is organized as follows. In Section 1, we introduce spaces of functions of infinite order. In Section 2, we formulate the mixed Neumann problem for infinite order parabolic operator with multiple time delays given in the integral form. In Section 3, the boundary optimal control problem for this case is formulated, then we give the necessary and sufficient conditions for the control to be an optimal. In Section 4, we generalized the discussion to two cases, the first case: the optimal control for (2×2) coupled infinite order parabolic systems is studied. The second case: the optimal control for $(n \times n)$ coupled infinite order parabolic systems was to be formulated.

2. Sobolev Spaces with Infinite Order

The object of this section is to give the definition of some function spaces of infinite order and the chains of the constructed spaces which will be used later.

Let Ω be a bounded open set of $\mathbb{R}^n$ with a smooth boundary Γ, which is a C^∞ manifold of dimension $(n-1)$. Locally, Ω is totally on one side of Γ. We define the infinite order Sobolev space $W^\infty\{a_\alpha, 2\}(\Omega)$ of infinite order of periodic functions $\phi(x)$ defined on Ω [29–31] as follows:

$$W^\infty\{a_\alpha, 2\}(\Omega) = \left\{ \phi(x) \in C^\infty(\Omega) : \sum_{|\alpha|=0}^{\infty} a_\alpha \|D^\alpha \phi\|_2^2 < \infty \right\}, \tag{2.1}$$

where $C^\infty(\Omega)$ is the space of infinitely differentiable functions, $a_\alpha \geq 0$ is a numerical sequence, and $\|\cdot\|_2$ is the canonical norm in the space $L^2(\Omega)$, and

$$D^\alpha = \frac{\partial^{|\alpha|}}{(\partial x_1)^{\alpha_1} \cdots (\partial x_n)^{\alpha_n}}, \tag{2.2}$$

$\alpha = (\alpha_1, \ldots, \alpha_n)$ being a multi-index for differentiation, $|\alpha| = \sum_{i=1}^{n} \alpha_i$.

The space $W^{-\infty}\{a_\alpha, 2\}(\Omega)$ is defined as the formal conjugate space to the space $W^\infty\{a_\alpha, 2\}(\Omega)$, namely:

$$W^{-\infty}\{a_\alpha, 2\}(\Omega) = \left\{ \psi(x) : \psi(x) = \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} a_\alpha D^\alpha \psi_\alpha(x) \right\}, \tag{2.3}$$

where $\psi_\alpha \in L^2(\Omega)$ and $\sum_{|\alpha|=0}^{\infty} a_\alpha \|\psi_\alpha\|_2^2 < \infty$.

The duality pairing of the spaces $W^\infty\{a_\alpha, 2\}(\Omega)$ and $W^{-\infty}\{a_\alpha, 2\}(\Omega)$ is postulated by the formula:

$$(\phi, \psi) = \sum_{|\alpha|=0}^{\infty} a_\alpha \int_\Omega \psi_\alpha(x) D^\alpha \phi(x) dx, \tag{2.4}$$

where

$$\phi \in W^\infty\{a_\alpha, 2\}(\Omega), \qquad \psi \in W^{-\infty}\{a_\alpha, 2\}(\Omega). \tag{2.5}$$

From above, $W^\infty\{a_\alpha, 2\}(\Omega)$ is everywhere dense in $L^2(\Omega)$ with topological inclusions and $W^{-\infty}\{a_\alpha, 2\}(\Omega)$ denotes the topological dual space with respect to $L^2(\Omega)$, so we have the following chain of inclusions:

$$W^\infty\{a_\alpha, 2\}(\Omega) \subseteq L^2(\Omega) \subseteq W^{-\infty}\{a_\alpha, 2\}(\Omega). \tag{2.6}$$

We now introduce $L^2(0,T;L^2(\Omega))$ which we will denoted by $L^2(Q)$, where $Q = \Omega\times]0,T[$ denotes the space of measurable functions $t \rightarrow \phi(t)$ such that

$$\|\phi\|_{L^2(Q)} = \left(\int_0^T \|\phi(t)\|_2^2 dt\right)^{1/2} < \infty, \qquad (2.7)$$

endowed with the scalar product $(f,g) = \int_0^T (f(t),g(t))_{L^2(\Omega)} dt$, $L^2(Q)$ is a Hilbert space.

In the same manner we define the spaces $L^2(0,T;W^\infty\{a_\alpha,2\}(\Omega))$, and $L^2(0,T; W^{-\infty}\{a_\alpha,2\}(\Omega))$, as its formal conjugate.

Also, we have the following chain of inclusions:

$$L^2(0,T;W^\infty\{a_\alpha,2\}(\Omega)) \subseteq L^2(Q) \subseteq L^2(0,T;W^{-\infty}\{a_\alpha,2\}(\Omega)). \qquad (2.8)$$

The construction of the Cartesian product of n-times to the above Hilbert spaces can be constructed, for example

$$(W^\infty\{a_\alpha,2\}(\Omega))^n = \underbrace{W^\infty\{a_\alpha,2\}(\Omega) \times W^\infty\{a_\alpha,2\}(\Omega) \times \cdots \times W^\infty\{a_\alpha,2\}(\Omega)}_{n\text{-times}}$$

$$= \prod_{i=1}^n (W^\infty\{a_\alpha,2\}(\Omega))^i, \qquad (2.9)$$

with norm defined by:

$$\|\phi\|_{(W^\infty\{a_\alpha,2\}(\Omega))^n} = \sum_{i=1}^n \|\phi_i\|_{W^\infty\{a_\alpha,2\}(\Omega)}, \qquad (2.10)$$

where $\phi = (\phi_1,\phi_2,\ldots,\phi_n) = (\phi_i)_{i=1}^n$ is a vector function and $\phi_i \in W^\infty\{a_\alpha,2\}(\Omega)$.

Finally, we have the following chain of inclusions:

$$\left(L^2(0,T;W^\infty\{a_\alpha,2\}(\Omega))\right)^n \subseteq \left(L^2(Q)\right)^n \subseteq \left(L^2(0,T;W^{-\infty}\{a_\alpha,2\}(\Omega))\right)^n, \qquad (2.11)$$

where $(L^2(0,T;W^{-\infty}\{a_\alpha,2\}(\Omega)))^n$ are the dual spaces of $(L^2(0,T;W^\infty\{a_\alpha,2\}(\Omega)))^n$. The spaces considered in this paper are assumed to be real.

3. Mixed Neumann Problem for Infinite Order Parabolic System with Multiple Time Lags

The object of this section is to formulate the following mixed initial boundary value Neumann problem for infinite order parabolic system with multiple time delays which defines the state of the system model [1, 5–11, 18, 24, 26].

$$\frac{\partial y}{\partial t} + \mathcal{A}(t)y(x,t) + \sum_{i=1}^{m} \int_{a_i}^{b_i} b_i(x,t)y(x,t-h_i)dh_i = u,$$

$$\tag{3.1}$$

$$(x,t) \in \Omega \times (0,T), \quad h_i \in (a_i, b_i),$$

$$y(x,t') = \Phi_0(x,t'), \quad (x,t') \in \Omega \times (-\Delta, 0), \tag{3.2}$$

$$y(x,0) = y_0(x), \quad x \in \Omega, \tag{3.3}$$

$$\frac{\partial y}{\partial v_{\mathcal{A}}}(x,t) = \sum_{s=1}^{l} \int_{c_s}^{d_s} c_s(x,t)y(x,t-k_s)dk_s + v, \quad (x,t) \in \Gamma \times (0,T), \ k_s \in (c_s, d_s), \tag{3.4}$$

$$y(x,t') = \Psi_0(x,t'), \quad (x,t') \in \Gamma \times (-\Delta, 0), \tag{3.5}$$

where $\Omega \subset R^n$ has the same properties as in Section 1. We have

$$y \equiv y(x,t;u), \quad y(0) \equiv y(x,0;u), \quad y(T) \equiv y(x,T;u), \quad u \equiv u(x,t), \quad v \equiv v(x,t),$$

$$Q = \Omega \times (0,T), \quad \overline{Q} = \overline{\Omega} \times [0,T], \quad Q_0 = \Omega \times [-\Delta, 0), \quad \Sigma = \Gamma \times (0,T), \quad \Sigma_0 = \Gamma \times [-\Delta, 0),$$

$$\tag{3.6}$$

(i) T is a specified positive number representing a finite time horizon,

(ii) h_i, k_s are time delays, such that $h_i \in (a_i, b_i)$ and $k_s \in (c_s, d_s)$ where $0 < a_1 < a_2 < \cdots < a_m, 0 < b_1 < b_2 < \cdots < b_m$, for $i = 1, 2, \ldots, m$ and $0 < c_1 < c_2 < \cdots < c_l$, $0 < d_1 < d_2 < \cdots < d_l$, for $s = 1, 2, \ldots, l$,

(iii) $b_i(t), i = 1, 2, \ldots, m$ are given real C^∞ functions defined on $\overline{Q}$,

(iv) $c_s(x,t), s = 1, 2, \ldots, l$ are given real C^∞ functions defined on Σ,

(v) $\Delta = \max\{b_m, d_l\}$,

(vi) y is a function defined on Q such that $\Omega \times (0,T) \ni (x,t) \rightarrow y(x,t) \in R$,

(vii) u, v are functions defined on Q and Σ such that $\Omega \times (0,T) \ni (x,t) \rightarrow u(x,t) \in R$ and $\Gamma \times (0,T) \ni (x,t) \rightarrow v(x,t) \in R$,

(viii) Φ_0, Ψ_0 are initial functions defined on Q_0 and Σ_0, respectively, such that $\Omega \times [-\Delta, 0) \ni (x,t') \rightarrow \Phi_0(x,t') \in R$. $\Gamma \times [-\Delta, 0) \ni (x,t') \rightarrow \Psi_0(x,t') \in R$.

The parabolic operator $(\partial/\partial t) + \mathcal{A}(t)$ in the state equation (3.1) is an infinite order parabolic operator and $\mathcal{A}(t)$ [17, 21, 29–31] is given by:

$$\mathcal{A}y = \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} a_\alpha D^{2\alpha} y(x,t),$$

$$\mathcal{A} = \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} a_\alpha D^{2\alpha} \tag{3.7}$$

is an infinite order self-adjoint elliptic partial differential operator maps $W^{\infty}\{a_\alpha, 2\}(\Omega)$ onto $W^{-\infty}\{a_\alpha, 2\}(\Omega)$.

For this operator we define the bilinear form as follows.

Definition 3.1. For each $t \in (0, T)$, we define a family of bilinear forms on $W^{\infty}\{a_\alpha, 2\}(\Omega)$ by:

$$\pi(t; y, \phi) = \left(\mathcal{A}(t)y, \phi\right)_{L^2(\Omega)}, \quad y, \phi \in W^{\infty}\{a_\alpha, 2\}(\Omega), \tag{3.8}$$

where $\mathcal{A}(t)$ maps $W^{\infty}\{a_\alpha, 2\}(\Omega)$ onto $W^{-\infty}\{a_\alpha, 2\}(\Omega)$ and takes the above form. Then

$$\begin{aligned}
\pi(t; y, \phi) &= \left(\mathcal{A}(t)y, \phi\right)_{L^2(\Omega)} \\
&= \left(\sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} a_\alpha D^{2\alpha} y(x,t), \phi(x)\right)_{L^2(\Omega)} \\
&= \int_\Omega \sum_{|\alpha|=0}^{\infty} a_\alpha D^\alpha y(x) D^\alpha \phi(x) \, dx.
\end{aligned} \tag{3.9}$$

Lemma 3.2. *The bilinear form $\pi(t; y, \phi)$ is coercive on $W^{\infty}\{a_\alpha, 2\}(\Omega)$, that is,*

$$\pi(t; y, y) \geq \lambda \, \|y\|^2_{W^{\infty}\{a_\alpha, 2\}(\Omega)}, \quad \lambda > 0. \tag{3.10}$$

Proof. It is well known that the ellipticity of $\mathcal{A}(t)$ is sufficient for the coerciveness of $\pi(t; y, \phi)$ on $W^{\infty}\{a_\alpha, 2\}(\Omega)$:

$$\pi(t; \phi, \psi) = \int_\Omega \sum_{|\alpha|=0}^{\infty} a_\alpha D^\alpha \phi D^\alpha \psi \, dx. \tag{3.11}$$

Then

$$\pi(t; y, y) = \int_\Omega \sum_{|\alpha|=0}^\infty a_\alpha D^\alpha y D^\alpha y \, dx$$

$$\geq \sum_{|\alpha|=0}^\infty a_\alpha \left\| D^{2\alpha} y(x) \right\|_{L^2(\Omega)}^2 \tag{3.12}$$

$$\geq \lambda \|y\|_{W^\infty\{a_\alpha, 2\}(\Omega)}^2, \quad \lambda > 0.$$

Also we have

$$\forall y, \phi \in W^\infty\{a_\alpha, 2\}(\Omega) \text{ the function } t \longrightarrow \pi(t; y, \phi)$$

$$\text{is continuously differentiable in } (0, T) \text{ and } \pi(t; y, \phi) = \pi(t; \phi, y). \tag{3.13}$$

Equations (3.1)–(3.5) constitute a Neumann problem. Then the left-hand side of the boundary condition (3.4) may be written in the following form:

$$\frac{\partial y(x, t)}{\partial v_{\mathcal{A}}} = \sum_{|\omega|=0}^\infty \left(D^\omega y(x, t) \right) \cos(n, x_k) = q(x, t), \quad x \in \Gamma, \, t \in (0, T), \tag{3.14}$$

where $\partial/\partial v_{\mathcal{A}}$ is a normal derivative at Γ, directed towards the exterior of Ω, and $\cos(n, x_k)$ is the kth direction cosine of n, with n being the normal at Γ exterior to Ω.

Then (3.4) can be written as:

$$q(x, t) = \sum_{s=1}^l \int_{c_s}^{d_s} c_s(x, t) y(x, t - k_s) dk_s + v(x, t), \quad x \in \Gamma, t \in (0, T). \tag{3.15}$$

$\square$

Remark 3.3. We will apply the indication $q(x, t)$ appearing in (3.14) to prove the existence of a unique solution for (3.1)–(3.5).

We will formulate sufficient conditions for the existence of a unique solution of the mixed boundary value problem (3.1)–(3.5) for the cases where the boundary control $v \in L^2(\Sigma)$.

For this purpose, we introduce the Sobolev space $W^{\infty,1}(Q)$ [20, Vol. 2, page 6] defined by:

$$W^{\infty,1}(Q) = L^2(0, T; W^\infty\{a_\alpha, 2\}(\Omega)) \cap W^1\left(0, T; L^2(\Omega)\right), \tag{3.16}$$

which is a Hilbert space normed by

$$
\begin{aligned}
\|y\|_{W^{\infty,1}(Q)} &= \left[\int_0^T \int \|y\|_{W^{\infty}\{a_\alpha,2\}(\Omega)}^2 dt + \|y\|_{W^1(0,T;L^2(\Omega))}^2\right]^{1/2} \\
&= \left[\int_Q \left(\sum_{|\alpha|=0}^{\infty} a_\alpha |D^\alpha y|^2 + \left|\frac{\partial y}{\partial t}\right|^2\right) dx\, dt\right]^{1/2} \\
&= \left[\int_Q \left(a_0 |y|^2 + \sum_{|\alpha|=1}^{\infty} a_\alpha |D^\alpha y|^2 + \left|\frac{\partial y}{\partial t}\right|^2\right) dx\, dt\right]^{1/2}, \quad a_0 > 0,
\end{aligned}
\tag{3.17}
$$

where the space $W^1(0,T;L^2(\Omega))$ denotes the Sobolev space of order 1 of functions defined on $(0,T)$ and taking values in $L^2(\Omega)$ [20, Vol. 1].

The existence of a unique solution for the mixed initial-boundary value problem (3.1)–(3.5) on the cylinder Q can be proved using a constructive method, that is, solving at first equations (3.1)–(3.5) on the subcylinder Q_1 and in turn on Q_2 and so forth, until the procedure covers the whole cylinder Q. In this way, the solution in the previous step determines the next one.

For simplicity, we introduce the following notation:

$$
E_j \hat{=} ((j-1)\lambda, j\lambda), \quad Q_j = \Omega \times E_j, \quad \Sigma_j = \Gamma \times E_j \quad \text{for } j = 1,\ldots,K, \ \lambda = \min\{a_1, c_1\}. \tag{3.18}
$$

Making use of the results of [7, 20] we can prove that the following result holds.

Theorem 3.4. *Let y_0, Φ_0, Ψ_0, v and u be given with $y_0 \in W^{\infty}\{a_\alpha,2\}(\Omega)$, $\Phi_0 \in W^{\infty,1}(Q_0)$, $\Psi_0 \in L^2(\Sigma_0)$, $v \in L^2(\Sigma)$ and $u \in W^{-\infty,-1}(Q)$. Then, there exists a unique solution $y \in W^{\infty,1}(Q)$ for the mixed initial-boundary value problem (3.1)–(3.5). Moreover, $y(\cdot, j\lambda) \in W^{\infty}\{a_\alpha, 2\}(\Omega)$ for $j = 1,\ldots,K$.*

4. Problem Formulation-Optimization Theorems

Now, we formulate the optimal control problem for (3.1)–(3.5) in the context of the Theorem 3.4, that is $v \in L^2(\Sigma)$.

Let us denote by $U = L^2(\Sigma)$ the space of controls. The time horizon T is fixed in our problem.

The performance functional is given by

$$
I(v) = \lambda_1 \int_Q [y(x,t;v) - z_d]^2 dx\, dt + \lambda_2 \int_\Sigma (Nv)v\, d\Gamma dt, \tag{4.1}
$$

where $\lambda_i \geq 0$, and $\lambda_1 + \lambda_2 > 0$, z_d is a given element in $L^2(Q)$; N is a positive linear operator on $L^2(\Sigma)$ into $L^2(\Sigma)$.

Control Contraints

We define the set of admissible controls U_{ad} such that

$$U_{\mathrm{ad}} \text{ is closed, convex subset of } U = L^2(\Sigma). \tag{4.2}$$

Let $y(x,t;v)$ denote the solution of the mixed initial-boundary value problem (3.1)–(3.5) at (x,t) corresponding to a given control $v \in U_{\mathrm{ad}}$. We note from Theorem 3.4 that for any $v \in U_{\mathrm{ad}}$ the performance functional (4.1) is well-defined since $(v) \in W^{\infty,1}(Q) \subset L^2(Q)$.

Making use of the Loins's scheme we will derive the necessary and sufficient conditions of optimality for the optimization problem (3.1)–(3.5), (4.1), (4.2). The solving of the formulated optimal control problem is equivalent to seeking a $v^* \in U_{\mathrm{ad}}$ such that

$$I(v^*) \le I(v), \quad \forall v \in U_{\mathrm{ad}}. \tag{4.3}$$

From the Lion's scheme [19, Theorem 1.3, page 10], it follows that for $\lambda_2 > 0$ a unique optimal control v^* exists. Moreover, v^* is characterized by the following condition:

$$I'(v^*)(v - v^*) \ge 0, \quad \forall v \in U_{\mathrm{ad}}. \tag{4.4}$$

For the performance functional of form (4.1) the relation (4.4) can be expressed as

$$\lambda_1 \int_Q \left(y(v^*) - z_d\right)\left[y(v) - y(v^*)\right] dx\, dt + \lambda_2 \int_\Sigma N v^*(v - v^*) d\Gamma\, dt \ge 0, \quad \forall v \in U_{\mathrm{ad}}. \tag{4.5}$$

In order to simplify (4.5), we introduce the adjoint equation, and for every $v \in U_{\mathrm{ad}}$, we define the adjoint variable $p = p(v) \equiv p(x,t;v)$ as the solution of the equations:

$$\frac{-\partial p(v)}{\partial t} + \mathcal{A}^*(t)p(v) + \sum_{i=1}^m \int_{a_i}^{b_i} b_i(x,t+h_i)p(x,t+h_i;v) dh_i = \lambda_1\left(y(v) - z_d\right), \tag{4.6}$$

$$(x,t) \in \Omega \times (0, T - \Delta), \quad h_i \in (a_i, b_i),$$

$$\frac{-\partial p(v)}{\partial t} + \mathcal{A}^*(t)p(v) = \lambda_1\left(y(v) - z_d\right), \quad (x,t) \in \Omega \times (T - \Delta, T), \tag{4.7}$$

$$p(x,T;v) = 0, \quad x \in \Omega, \tag{4.8}$$

$$p(x,t;v) = 0, \quad (x,t) \in \Omega \times [T - \Delta + \lambda, T), \tag{4.9}$$

$$\frac{\partial p(v)}{\partial v_{\mathcal{A}^*}}(x,t) = \sum_{s=1}^l \int_{c_s}^{d_s} c_s(x,t+k_s)p(x,t+k_s;v) dk_s, \quad (x,t) \in \Gamma \times (0, T - \Delta(T)), \ k_s \in (c_s, d_s), \tag{4.10}$$

$$\frac{\partial p(v)}{\partial v_{\mathcal{A}^*}}(x,t) = 0, \quad (x,t) \in \Gamma \times (T - \Delta(T), T), \tag{4.11}$$

where

$$\frac{\partial p(v)}{\partial v_{\mathcal{A}^*}}(x,t) = \sum_{|\omega|=0}^{\infty} (D^{\omega} p(v)) \cos(n, x_{\omega})(x,t),$$

$$\mathcal{A}^*(t)p(v) = \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} a_{\alpha} D^{2\alpha} p(x,t). \tag{4.12}$$

As in the above section with change of variables, that is, with reversed sense of time. that is, $t' = T - t$, for given $z_d \in L^2(Q)$ and any $v \in L^2(\Sigma)$, there exists a unique solution $p(v) \in W^{\infty,1}(Q)$ for problem (4.6)–(4.11).

The existence of a unique solution for the problem (4.6)–(4.11) on the cylinder $\Omega \times (0,T)$ can be proved using a constructive method. It is easy to notice that for given z_d and u, the problem (4.6)–(4.11) can be solved backwards in time starting from $t = T$, that is, first solving (4.6)–(4.11) on the subcylinder Q_K and in turn on Q_{K-1}, and so forth until the procedure covers the whole cylinder $\Omega \times (0,T)$. For this purpose, we may apply Theorem 3.4 (with an obvious change of variables).

Hence, using Theorem 3.4, the following result can be proved.

Lemma 4.1. *Let the hypothesis of Theorem 3.4 be satisfied. Then for given $z_d \in L^2(\Omega, R^{\infty})$ and any $v \in L^2(\Sigma)$, there exists a unique solution $p(v) \in W^{\infty,1}(Q)$ for the adjoint problem (4.6)–(4.11).*

We simplify (4.5) using the adjoint equation (4.6)–(4.11). For this purpose denoting by $p(0) \equiv p(x,0;v)$ and $p(T) \equiv p(x,T;v)$, respectively, setting $v = v^*$ in (4.6)–(4.11), multiplying both sides of (4.6) and (4.7) by $y(v) - y(v^*)$, then integrating over $\Omega \times (0, T - \Delta)$ and $\Omega \times (T - \Delta, T)$, respectively and then adding both sides of (4.6), (4.11), we get

$$\lambda_1 \int_Q (y(v^*) - z_d) [y(v) - y(v^*)] \, dx \, dt$$

$$= \int_Q \left(-\frac{\partial p(v^*)}{\partial t} + \mathcal{A}^*(t)p(v^*) \right) \times [y(v) - y(v^*)] \, dx \, dt$$

$$+ \int_0^{T-\Delta} \int_\Omega \left(\sum_{i=1}^m \int_{a_i}^{b_i} b_i(x, t+h_i)p(x, t+h_i; v^*) dh_i \right) \times [y(x,t;v) - y(x,t;v^*)] \, dx \, dt$$

$$= \int_0^T \int_\Omega p(v^*) \frac{\partial}{\partial t} [y(v) - y(v^*)] \, dx \, dt$$

$$+ \int_0^T \int_\Omega \mathcal{A}^*(t)p(v^*) [y(v) - y(v^*)] \, dx \, dt$$

$$+ \sum_{i=1}^m \int_{a_i}^{b_i} \int_\Omega \int_0^{T-\Delta} (b_i(x, t+h_i)p(x, t+h_i; v^*)) \times [y(x,t;v) - y(x,t;v^*)] \, dx \, dt \, dh_i. \tag{4.13}$$

Using (3.1), the first integral on the right-hand side of (4.13) can be written as:

$$\int_0^T \int_\Omega p(v^*) \frac{\partial}{\partial t} [y(v) - y(v^*)] dx \, dt$$

$$= -\int_Q p(v^*) \mathcal{A}(t)(y(v) - y(v^*)) dx \, dt$$

$$- \sum_{i=1}^m \int_{a_i}^{b_i} \int_\Omega \int_0^T p(x,t;v^*) b_i(x,t) \times [y(x,t-h_i;v) - y(x,t-h_i;v^*)] dt \, dx \, dh_i$$

$$= -\int_Q p(v^*) \mathcal{A}(t)(y(v) - y(v^*)) dx \, dt$$

$$- \sum_{i=1}^m \int_{a_i}^{b_i} \int_\Omega \int_{-h_i}^{T-h_i} p(x,t'+h_i;v^*) b_i(x,t'+h_i) \times [y(x,t';v) - y(x,t';v^*)] dt' \, dx \, dh_i$$

$$= -\int_\Omega p(v^*) \mathcal{A}(t)(y(v) - y(v^*)) dx \, dt$$

$$- \sum_{i=1}^m \int_{a_i}^{b_i} \int_\Omega \int_{-h_i}^0 p(x,t'+h_i;v^*) b_i(x,t'+h_i) \times [y(x,t';v) - y(x,t';v^*)] dt' \, dx \, dh_i$$

$$- \sum_{i=1}^m \int_{a_i}^{b_i} \int_\Omega \int_0^{T-\Delta} p(x,t'+h_i;v^*) b_i(x,t'+h_i) \times [y(x,t';v) - y(x,t';v^*)] dt' \, dx d \, h_i$$

$$- \sum_{i=1}^m \int_{a_i}^{b_i} \int_\Omega \int_{T-\Delta}^{T-h_i} p(x,t'+h_i;v^*) b_i(x,t'+h_i) \times [y(x,t';v) - y(x,t';v^*)] dt' \, dx \, dh_i$$

$$= -\int_Q p(v^*) \mathcal{A}(t)(y(v) - y(v^*)) dx \, dt$$

$$- \sum_{i=1}^m \int_{a_i}^{b_i} \int_\Omega \int_{-h_i}^0 p(x,t'+h_i;v^*) b_i(x,t'+h_i) \times [y(x,t';v) - y(x,t';v^*)] dt' \, dx \, dh_i$$

$$- \sum_{i=1}^m \int_{a_i}^{b_i} \int_\Omega \int_0^{T-\Delta} p(x,t'+h_i;v^*) b_i(x,t'+h_i) \times [y(x,t';v) - y(x,t';v^*)] dt' \, dx \, dh_i$$

$$- \sum_{i=1}^m \int_{a_i}^{b_i} \int_\Omega \int_{T-\Delta+h_i}^T p(x,t;v^*) b_i(x,t) \times [y(x,t-h_i;v) - y(x,t-h_i;v^*)] dt \, dx \, dh_i.$$

$$(4.14)$$

Using Green's formula, the second integral on the right-hand side of (4.13) can be written as:

$$\int_0^T \int_\Omega \mathcal{A}^*(t) p(v^*) [y(v) - y(v^*)] dx \, dt$$

$$= \int_0^T \int_\Omega p(v^*) \mathcal{A}(t) [y(v) - y(v^*)] dx \, dt$$

$$+ \int_0^T \int_\Gamma p(v^*) \left(\frac{\partial y(v)}{\partial v_{\mathcal{A}}} - \frac{\partial y(v^*)}{\partial v_{\mathcal{A}}} \right) d\Gamma \, dt$$

$$- \int_0^T \int_\Gamma \frac{\partial p(v^*)}{\partial v_{\mathcal{A}^*}} [y(v) - y(v^*)] d\Gamma \, dt.$$

(4.15)

Using the boundary condition (3.2), one can transform the second integral on the right-hand side of (4.15) into the form:

$$\int_0^T \int_\Gamma p(v^*) \left(\frac{\partial y(v)}{\partial v_{\mathcal{A}}} - \frac{\partial y(v^*)}{\partial v_{\mathcal{A}}} \right) d\Gamma \, dt$$

$$= \sum_{s=1}^l \int_{c_s}^{d_s} \int_\Gamma \int_0^T p(x,t;v^*) c_s(x,t) \times [y(x,t-k_s;v) - y(x,t-k_s;v^*)] d\Gamma \, dt \, dk_s$$

$$+ \int_0^T \int_\Gamma p(v^*)(v - v^*) d\Gamma \, dt$$

$$= \sum_{s=1}^l \int_{c_s}^{d_s} \int_\Gamma \int_{-k_s}^{T-k_s} p(x,t'+k_s;v^*) c_s(x,t'+k_s) \times [y(x,t';v) - y(x,t';v^*)] dt' \, d\Gamma \, dk_s$$

$$+ \int_0^T \int_\Gamma p(v^*)(v - v^*) d\Gamma \, dt$$

$$= \sum_{s=1}^l \int_{c_s}^{d_s} \int_\Gamma \int_{-k_s}^0 p(x,t'+k_s;v^*) c_s(x,t'+k_s) \times [y(x,t';v) - y(x,t';v^*)] dt' \, d\Gamma \, dk_s$$

$$+ \sum_{s=1}^l \int_{c_s}^{d_s} \int_\Gamma \int_0^{T-\Delta} p(x,t'+k_s;v^*) c_s(x,t'+k_s) \times [y(x,t';v) - y(x,t';v^*)] dt' \, d\Gamma \, dk_s$$

$$+ \sum_{s=1}^l \int_{c_s}^{d_s} \int_\Gamma \int_{T-\Delta}^{T-k_s} p(x,t'+k_s;v^*) c_s(x,t'+k_s) \times [y(x,t';v) - y(x,t';v^*)] dt' \, d\Gamma \, dk_s$$

$$+ \int_0^T \int_\Gamma p(v^*)(v - v^*) d\Gamma \, dt$$

$$= \sum_{s=1}^l \int_{c_s}^{d_s} \int_\Gamma \int_{-k_s}^0 p(x,t'+k_s;v^*) c_s(x,t'+k_s) \times [y(x,t';v) - y(x,t';v^*)] dt' \, d\Gamma \, dk_s$$

$$+ \sum_{s=1}^l \int_{c_s}^{d_s} \int_\Gamma \int_0^{T-\Delta} p(x,t'+k_s;v^*) c_s(x,t'+k_s) \times [y(x,t';v) - y(x,t';v^*)] dt' \, d\Gamma \, dk_s$$

$$+ \sum_{s=1}^{l} \int_{c_s}^{d_s} \int_{\Gamma} \int_{T-\Delta+k_s}^{T} p(x,t;v^*) c_s(x,t) \times \left[y(x,t-k_s;v) - y(x,t-k_s;v^*) \right] dt \, d\Gamma \, dk_s$$

$$+ \int_{0}^{T} \int_{\Gamma} p(v^*)(v-v^*) d\Gamma \, dt.$$

$$(4.16)$$

The last component in (4.15) can be rewritten as

$$\int_{0}^{T} \int_{\Gamma} \frac{\partial p(v^*)}{\partial v_{\mathcal{A}^*}} \left[y(v) - y(v^*) \right] d\Gamma \, dt$$

$$= \int_{0}^{T-\Delta} \int_{\Gamma} \frac{\partial p(v^*)}{\partial v_{\mathcal{A}^*}} \left[y(v) - y(v^*) \right] d\Gamma \, dt + \int_{T-\Delta}^{T} \int_{\Gamma} \frac{\partial p(v^*)}{\partial v_{\mathcal{A}^*}} \left[y(v) - y(v^*) \right] d\Gamma \, dt.$$

$$(4.17)$$

Substituting (4.16) and (4.17) into (4.15) and then the results into (4.13), we obtain

$$\lambda_1 \int_{Q} \left(y(v^*) - z_d \right) \left[y(v) - y(v^*) \right] dx \, dt$$

$$= \int_{0}^{T} \int_{\Gamma} p(v^*)(v-v^*) d\Gamma \, dt - \int_{Q} p(v^*) \mathcal{A}(t) \left[y(v) - y(v^*) \right] dx \, dt$$

$$- \sum_{i=1}^{m} \int_{a_i}^{b_i} \int_{\Omega} \int_{-h_i}^{0} b_i(x,t+h_i) p(x,t+h_i;v^*) \times \left[y(x,t;v) - y(x,t;v^*) \right] dt \, dx \, dh_i$$

$$- \sum_{i=1}^{m} \int_{a_i}^{b_i} \int_{\Omega} \int_{0}^{T-\Delta} b_i(x,t+h_i) p(x,t+h_i;v^*) \times \left[y(x,t;v) - y(x,t;v^*) \right] dt \, dx \, dh_i$$

$$+ \int_{Q} p(v^*) \mathcal{A}(t) \left[y(v) - y(v^*) \right] dx \, dt$$

$$- \sum_{i=1}^{m} \int_{a_i}^{b_i} \int_{\Omega} \int_{T-\Delta+h_i}^{T} p(x,t;v^*) b_i(x,t) \times \left[y(x,t-h_i;v) - y(x,t-h_i;v^*) \right] dt \, dx \, dh_i$$

$$+ \sum_{s=1}^{l} \int_{c_s}^{d_s} \int_{\Gamma} \int_{-k_s}^{0} c_s(x,t+k_s) p(x,t+k_s;v^*) \times \left[y(x,t;v) - y(x,t;v^*) \right] dt \, d\Gamma \, dk_s$$

$$- \sum_{s=1}^{l} \int_{c_s}^{d_s} \int_{\Gamma} \int_{0}^{T-\Delta} c_s(x,t+k_s) p(x,t+k_s;v^*) \times \left[y(x,t;v) - y(x,t;v^*) \right] dt \, d\Gamma \, dk_s$$

$$- \sum_{s=1}^{l} \int_{c_s}^{d_s} \int_{\Gamma} \int_{T-\Delta+k_s}^{T} c_s(x,t) p(x,t;v^*) \times \left[y(x,t-k_s;v) - y(x,t-k_s;v^*) \right] dt \, d\Gamma \, dk_s$$

$$-\int_{\Gamma}\int_0^{T-\Delta}\frac{\partial p(v^*)}{\partial v_{\mathcal{A}^*}}\times[y(x,t;v)-y(x,t;v^*)]\,dt\,d\Gamma$$

$$-\int_{\Gamma}\int_{T-\Delta}^{T}\frac{\partial p(v^*)}{\partial v_{\mathcal{A}^*}}\times[y(x,t;v)-y(x,t;v^*)]\,dt\,d\Gamma$$

$$+\sum_{i=1}^{m}\int_{a_i}^{b_i}\int_{\Omega}\int_0^{T-\Delta}b_i(x,t+h_i)p(x,t+h_i;v^*)\times[y(x,t;v)-y(x,t;v^*)]\,dt\,dx\,dh_i.$$

$$(4.18)$$

Afterwards, using the facts that $y(x,t,v) = y(x,t,v^*) = \Phi_0(x,t)$ for $x \in \Omega$ and $t \in [-\Delta,0)$ and $y(x,t,v) = y(x,t,v^*) = \Psi_0(x,t)$ for $x \in \Gamma$ and $t \in [-\Delta,0)$, $p|_\Omega(x,t;v^*) = 0$ and consequently $p|_\Gamma(x,t;v^*) = 0$ for $t \in [T-\Delta+\lambda,T)$, we obtain

$$\lambda_1\int_Q[y(v^*)-z_d]\times(y(v)-y(v^*))\,dx\,dt = \int_0^T\int_\Gamma p(v^*)(v-v^*)\,d\Gamma\,dt. \qquad (4.19)$$

Substituting (4.19) into (4.5) gives

$$\int_0^T\int_\Gamma(p(v^*)+\lambda_2 N v^*)(v-v^*)\,d\Gamma\,dt \geq 0, \quad \forall v \in U_{\text{ad}}. \qquad (4.20)$$

The foregoing result is now summarized.

Theorem 4.2. *For the problem (3.1)–(3.5), with the performance functional (4.1) with $z_d \in L^2(Q)$ and $\lambda_2 > 0$ and with constraints on controls (4.2), there exists a unique optimal control v^* which satisfies the maximum condition (4.20).*

4.1. Mathematical Examples

Example 4.3. Consider now the particular case where $U_{\text{ad}} = U = L^2(\Sigma)$ (no constraints case). Thus the maximum condition (4.20) is satisfied when

$$v^* = -\lambda_2 N^{-1}p(v^*). \qquad (4.21)$$

If N is the identity operator on $L^2(\Sigma)$, then from Lemma 4.1 it follows that $v^* \in W^{\infty,1}(Q)$.

Example 4.4. We can also consider an analogous optimal control problem where the performance functional is given by:

$$I(v) = \lambda_1\int_\Sigma[y(x,t;v)|_\Sigma - z_d]^2\,d\Gamma\,dt + \lambda_2\int_\Sigma(Nv)v\,d\Gamma\,dt, \qquad (4.22)$$

where $z_d \in L^2(\Sigma)$.

From Theorem 3.4 and the Trace Theorem [20, Vol. 2, page 9], for each $v \in L^2(\Sigma)$, there exists a unique solution $y(v) \in W^{\infty,1}(Q)$ with $y|_\Sigma \in L^2(\Sigma)$. Thus, $I(v)$ is well defined. Then, the optimal control v^* is characterized by:

$$\lambda_1 \int_\Sigma (y(v^*)|_\Sigma - z_d)\left[y(v)|_\Sigma - y(v^*)|_\Sigma\right]d\Gamma\, dt + \lambda_2 \int_\Sigma Nv^*(v - v^*)d\Gamma\, dt \geq 0, \quad \forall v \in U_{\text{ad}}. \tag{4.23}$$

We define the adjoint variable $p = p(v^*) = p(x,t;v^*)$ as the solution of the equations:

$$-\frac{\partial p(v^*)}{\partial t} + \mathcal{A}^*(t)p(v^*) + \sum_{i=1}^{m}\int_{a_i}^{b_i} b_i(x,t+h_i)p(x,t+h_i;v^*)dh_i = 0,$$

$$(x,t) \in \Omega \times (0, T-\Delta), \quad h_i \in (a_i, b_i),$$

$$\frac{-\partial p(v^*)}{\partial t} + \mathcal{A}^*(t)p(v^*) = 0, \quad (x,t) \in \Omega \times (T-\Delta, T),$$

$$p(x,T;v^*) = 0, \quad x \in \Omega,$$

$$p(x,t;v^*) = 0, \quad (x,t) \in \Omega \times [T-\Delta+\lambda, T), \tag{4.24}$$

$$\frac{\partial p(v^*)}{\partial v_{\mathcal{A}^*}}(x,t) = \sum_{s=1}^{l}\int_{c_s}^{d_s} c_s(x,t+k_s)p(x,t+k_s;v^*)dk_s + \lambda_1\left(yv^*|_\Sigma - z_{\Sigma d}\right),$$

$$(x,t) \in \Gamma \times (0, T-\Delta(T)), \quad k_s \in (c_s, d_s),$$

$$\frac{\partial p(v^*)}{\partial v_{\mathcal{A}^*}}(x,t) = \lambda_1\left(y(v^*)|_\Sigma - z_{\Sigma d}\right), \quad (x,t) \in \Gamma \times (T-\Delta(T), T).$$

As in the above section, we have the following result.

Lemma 4.5. *Let the hypothesis of Theorem 3.4 be satisfied. Then, for given $z_{\Sigma d} \in L^2(\Sigma)$ and any $v \in L^2(\Sigma)$, there exists a unique solution $p(v^*) \in W^{\infty,1}(Q)$ to the adjoint problem (4.24).*

Using the adjoint equations (4.24) in this case, the condition (4.23) can also be written in the following form:

$$\int_0^T \int_\Gamma (p(v^*) + \lambda_2 Nv^*)(v - v^*)d\Gamma\, dt \geq 0, \quad \forall v \in U_{\text{ad}}. \tag{4.25}$$

The following result is now summarized.

Theorem 4.6. *For the problem (3.1)–(3.5) with the performance function (4.22) with $z_{\Sigma d} \in L^2(\Sigma)$ and $\lambda_2 > 0$, and with constraint (4.2), and with adjoint equations (4.24), there exists a unique optimal control v^* which satisfies the maximum condition (4.25).*

Example 4.7 ($u \in L^2(Q)$). We can also consider an analogous optimal control problem where the performance functional is given by:

$$I(u) = \lambda_1 \int_Q \left[y(x,t;u) - z_d \right]^2 dx\,dt + \lambda_2 \int_Q (Nu)u\,dx\,dt, \tag{4.26}$$

where $z_d \in L^2(Q)$.

From **Theorem 3.4** and the Trace Theorem [20, Vol. 2, page 9], for each $u \in L^2(Q)$, there exists a unique solution $y(u) \in W^{\infty,1}(Q)$. Thus, I is well defined. Then, the optimal control u^* is characterized by:

$$\lambda_1 \int_Q (y(u^*) - z_d)\left[y(u) - y(u^*)\right]dx\,dt + \lambda_2 \int_Q Nu^*(u - u^*)dx\,dt \geq 0, \quad \forall u \in U_{\text{ad}}. \tag{4.27}$$

We define the adjoint variable $p = p(u^*) = p(x,t;u^*)$ as the solution of the equations:

$$-\frac{\partial p(u^*)}{\partial t} + \mathcal{A}^*(t)p(u^*) + \sum_{i=1}^m \int_{a_i}^{b_i} b_i(x,t+h_i)p(x,t+h_i;u^*)dh_i = \lambda_1\big(y(u^*) - z_d\big),$$

$$(x,t) \in \Omega \times (0, T - \Delta), \quad h_i \in (a_i, b_i),$$

$$-\frac{\partial p(u^*)}{\partial t} + \mathcal{A}^*(t)p(u^*) = \lambda_1\big(y(u^*) - z_d\big), \quad (x,t) \in \Omega \times (T - \Delta, T),$$

$$p(x,T;u^*) = 0, \quad x \in \Omega,$$

$$p(x,t;u^*) = 0, \quad (x,t) \in \Omega \times [T - \Delta + \lambda, T), \tag{4.28}$$

$$\frac{\partial p(u^*)}{\partial v_{\mathcal{A}^*}}(x,t) = \sum_{s=1}^l \int_{c_s}^{d_s} c_s(x,t+k_s)p(x,t+k_s;u^*)dk_s,$$

$$(x,t) \in \Gamma \times (0, T - \Delta(T)), \quad k_s \in (c_s, d_s),$$

$$\frac{\partial p(u^*)}{\partial v_{\mathcal{A}^*}}(x,t) = 0, \quad (x,t) \in \Gamma \times (T - \Delta(T), T).$$

As in the above section, we have the following result.

Lemma 4.8. *Let the hypothesis of Theorem 3.4 be satisfied. Then, for given $z_d \in L^2(Q)$ and any $u \in L^2(Q)$, there exists a unique solution $p(u^*) \in W^{\infty,1}(Q)$ to the adjoint problem (4.28).*
     Using the adjoint equations (4.28) in this case, the condition (4.27) can also be written in the following form:

$$\int_Q (p(u^*) + \lambda_2 Nu^*)(u - u^*)dx\,dt \geq 0, \quad \forall u \in U_{ad}. \tag{4.29}$$

The following result is now summarized.

Theorem 4.9. *For the problem* (3.1)–(3.5) *with the performance function* (4.26) *with* $z_d \in L^2(Q)$ *and* $\lambda_2 > 0$, *and with constraint* (4.2), *and with adjoint equations* (4.28), *there exists a unique optimal control* u^* *which satisfies the maximum condition* (4.29).

Example 4.10. We can also consider an analogous optimal control problem where the performance functional is given by:

$$I(u) = \lambda_1 \int_\Sigma \left[y|_\Sigma(x,t;u) - z_{\Sigma d}\right]^2 d\Gamma\, dt + \lambda_2 \int_Q (Nu)u\, dx\, dt, \tag{4.30}$$

where $z_{\Sigma d} \in L^2(\Sigma)$.

From **Theorem 3.4** and the Trace Theorem [20, Vol. 2, page 9], for each $u \in L^2(Q)$, there exists a unique solution $y(u) \in W^{\infty,1}(Q)$ with $y|_\Sigma \in L^2(\Sigma)$. Thus, I is well defined. Then, the optimal control u^* is characterized by:

$$\lambda_1 \int_\Sigma \left(y(u^*) - z_{\Sigma d}\right)\left[y(u) - y(u^*)\right] d\Gamma\, dt + \lambda_2 \int_Q Nu^*(u - u^*) dx\, dt \geq 0, \quad \forall u \in U_{\mathrm{ad}}. \tag{4.31}$$

The above inequality can be simplified by introducing an adjoint equation, the form of which is identical to (4.24). Then using **Theorem 3.4** we can establish the existence of a unique solution $p = p(u^*) = p(x,t;u^*) \in W^{\infty,1}(Q)$ for (4.24).

As in the above section, we have the following result.

Lemma 4.11. *Let the hypothesis of Theorem 3.4 be satisfied. Then, for given* $z_{\Sigma d} \in L^2(\Sigma)$ *and any* $u \in L^2(Q)$, *there exists a unique solution* $p(u^*) \in W^{\infty,1}(Q)$ *to the adjoint problem* (4.24)–(37).

Using the adjoint equations (4.24)–(37) *in this case, the condition* (4.31) *can also be written in the following form:*

$$\int_Q \left(p(u^*) + \lambda_2 Nu^*\right)(u - u^*) dx\, dt \geq 0, \quad \forall u \in U_{ad}. \tag{4.32}$$

The following result is now summarized.

Theorem 4.12. *For the problem* (3.1)–(3.5) *with the performance function* (4.30) *with* $z_{\Sigma d} \in L^2(\Sigma)$ *and* $\lambda_2 > 0$, *and with constraint* (4.2), *and with adjoint equations* (4.24), *there exists a unique optimal control* u^* *which satisfies the maximum condition* (4.32).

5. Generalization

The optimal control problems presented here can be extended to certain different two cases. Case 1: optimal control for 2×2 coupled infinite order parabolic systems with multiple time delays. Case 2: optimal control for $n \times n$ coupled infinite order parabolic systems with multiple time delays. Such extension can be applied to solving many control problems in mechanical engineering.

Case 1 (optimal control for 2×2 coupled infinite order parabolic systems with multiple time delays). We can extend the discussions to study the optimal control for 2×2 coupled infinite order parabolic systems with multiple time delays. We consider the case where $v = (v_1, v_2) \in L^2(\Sigma) \times L^2(\Sigma)$, the performance functional is given by [15, 16]:

$$I(v) = I_1(v) + I_2(v) = \sum_{i=1}^{2} \left(\lambda_1 \int_Q \left[y_i(x, t; v) - z_{id} \right]^2 dx\, dt + \lambda_2 \int_\Sigma (N_i v_i) v_i dx\, dt \right), \qquad (5.1)$$

where $z_d = (z_{1d}, z_{2d}) \in (L^2(Q))^2$.

The following results can now be proved.

Theorem 5.1. *Let y_0, Φ_0, Ψ_0, v, and u be given with $y_0 = (y_{0,1}, y_{0,2}) \in (W^\infty\{\alpha_\alpha, 2\}(\Omega))^2$, $\Psi_0 = (\Psi_{0,1}, \Psi_{0,2}) \in (L^2(\Sigma_0))^2$, $\Phi_0 = (\Phi_{0,1}, \Phi_{0,2}) \in (W^{\infty,1}(Q_0))^2$, $v = (v_1, v_2) \in (L^2(\Sigma))^2$, and $u = (u_1, u_2) \in (W^{-\infty,-1}(Q))^2$. Then, there exists a unique solution $y = (y_1, y_2) \in (W^{-\infty,1}(Q))^2$ for the following mixed initial-boundary value problem:*

$$\frac{\partial y_1}{\partial t} + \left(\sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} a_\alpha D^{2\alpha} + 1 \right) y_1 + \sum_{i=1}^{m} \int_{a_i}^{b_i} b_i(x,t) y_1(x, t - h_i) dh_i - y_2 = u_1,$$

$$in\ Q, h_i \in (a_i, b_i),$$

$$\frac{\partial y_2}{\partial t} + \left(\sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} a_\alpha D^{2\alpha} + 1 \right) y_2 + \sum_{i=1}^{m} \int_{a_i}^{b_i} b_i(x,t) y_2(x, t - h_i) dh_i + y_1 = u_2,$$

$$in\ Q, h_i \in (a_i, b_i),$$

$$y_1(x, t'; u) = \Phi_{0,1}(x, t'), \quad (x, t') \in \Omega \times [-\Delta, 0),$$

$$y_2(x, t'; u) = \Phi_{0,2}(x, t'), \quad (x, t') \in \Omega \times [-\Delta, 0),$$

$$y_1(x, 0; u) = y_{0,1}, \quad x \in \Omega,$$

$$y_2(x, 0; u) = y_{0,2}, \quad x \in \Omega,$$

$$\frac{\partial y_1}{\partial v_{\mathcal{A}}}(x, t) = \sum_{s=1}^{l} \int_{c_s}^{d_s} c_{s1}(x,t) y_1(x, t - k_s) dk_s + v_1, \quad on\ \Sigma, k_s \in (c_s, d_s),$$

$$\frac{\partial y_2}{\partial v_{\mathcal{A}}} = \sum_{s=1}^{l} \int_{c_s}^{d_s} c_{s2}(x,t) y_2(x, t - k_s) dk_s + v_2, \quad on\ \Sigma, k_s \in (c_s, d_s),$$

$$y_1(x, t'; u) = \Psi_{0,1}(x, t'), \quad (x, t') \in \Gamma \times [-\Delta, 0),$$

$$y_2(x, t'; u) = \Psi_{0,2}(x, t'), \quad (x, t') \in \Gamma \times [-\Delta, 0),$$

$$(5.2)$$

where

$$y \equiv y(x, t; u) = (y_1(x, t; u), y_2(x, t; u)) \in \left(W^{\infty, 1}(Q) \right)^2,$$

$$u \equiv u(x, t) = (u_1(x, t), u_2(x, t)) \in \left(\left(W^{\infty, 1}(Q) \right)' \right)^2, \tag{5.3}$$

$$v \equiv v(x, t) = (v_1(x, t), v_2(x, t)) \in \left(L^2(\Sigma) \right)^2.$$

Lemma 5.2. *Let the hypothesis of Theorem 5.1 be satisfied. Then for given $z_d = (z_{1d}, z_{2d}) \in (L^2(Q))^2$ and any $v = (v_1, v_2) \in (L^2(\Sigma))^2$, there exists a unique solution $p(v) = (p_1(v), p_2(v)) \in (W^{\infty, 1}(Q))^2$ for the adjoint problem:*

$$-\frac{\partial p_1(v)}{\partial t} + \left(\sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} a_\alpha D^{2\alpha} + 1 \right) p_1(v) + \sum_{i=1}^{m} \int_{a_i}^{b_i} b_i(x, t + h_i) p_1(x, t + h_i; v) dh_i + p_2(v)$$

$$= \lambda_1 (y_1(v) - z_{1d}), \quad (x, t) \in \Omega \times (0, T - \Delta), \ h_i \in (a_i, b_i),$$

$$-\frac{\partial p_2(v)}{\partial t} + \left(\sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} a_\alpha D^{2\alpha} + 1 \right) p_2(v) + \sum_{i=1}^{m} \int_{a_i}^{b_i} b_i(x, t + h_i) p_2(x, t + h_i; v) dh_i - p_1(v)$$

$$= \lambda_1 (y_2(v) - z_{2d}), \quad (x, t) \in \Omega \times (0, T - \Delta), \ h_i \in (a_i, b_i),$$

$$\frac{\partial p_1(v)}{\partial t} + \left(\sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} a_\alpha D^{2\alpha} + 1 \right) p_1(v) = \lambda_1 (y_1(v) - z_{1d}), \quad (x, t) \in \Omega \times (T - \Delta, T),$$

$$\frac{\partial p_2(v)}{\partial t} + \left(\sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} a_\alpha D^{2\alpha} + 1 \right) p_2(v) = \lambda_1 (y_2(v) - z_{2d}), \quad (x, t) \in \Omega \times (T - \Delta, T),$$

$$p_1(x, T; v) = 0, \quad p_2(x, T; v) = 0, \quad x \in \Omega,$$

$$p_1(x, t; v) = 0, \quad p_2(x, t; v) = 0, \quad (x, t) \in \Omega \times [T - \Delta + \lambda, T),$$

$$\frac{\partial p_1(x, t; v)}{\partial v_{\mathscr{A}^*}} = \sum_{s=1}^{l} \int_{c_s}^{d_s} c_{s1}(x, t + k_s) p_1(x, t + k_s; v) dk_s, \quad (x, t) \in \Gamma \times (0, T - \Delta), \ k_s \in (c_s, d_s),$$

$$\frac{\partial p_2(x, t; v)}{\partial v_{\mathscr{A}^*}} = \sum_{s=1}^{l} \int_{c_s}^{d_s} c_{s2}(x, t + k_s) p_2(x, t + k_s; v) dk_s, \quad (x, t) \in \Gamma \times (0, T - \Delta), \ k_s \in (c_s, d_s),$$

$$\frac{\partial p_1(x, t)}{\partial v_{\mathscr{A}^*}} = 0, \quad \frac{\partial p_2(x, t)}{\partial v_{\mathscr{A}^*}} = 0, \quad (x, t) \in \Gamma \times (T - \Delta(T), T).$$

$$\tag{5.4}$$

Theorem 5.3. *The optimal control* $v^* \equiv v^*(x,t) = (v_1^*(x,t), v_2^*(x,t)) \in (L^2(\Sigma))^2$ *is characterized by the following maximum condition:*

$$\int_0^T \int_\Gamma \left(\left[p_1(v^*) + \lambda_2 N_1 v_1^* \right] (v_1 - v_1^*) + \left[p_2(v^*) + \lambda_2 N_2 v_2^* \right] (v_2 - v_2^*) \right) d\Gamma \, dt \geq 0,$$

$$\forall v = (v_1, v_2) \in \left(L^2(\Sigma) \right)^2,$$

(5.5)

where $p \equiv p(x,t;v) = (p_1(x,t;v), p_2(x,t;v)) \in (W^{\infty,1}(Q))^2$ *is the adjoint state.*

The foregoing result is now summarized.

Theorem 5.4. *For the problem* (5.2) *with the performance function* (5.1) *with* $z_d = (z_{1d}, z_{2d}) \in (L^2(Q))^2$ *and* $\lambda_2 > 0$, *and with constraint:* U_{ad} *is closed, convex subset of* $(L^2(\Sigma))^2$, *and with adjoint problem* (5.4), *then there exists a unique optimal control* $v^* \equiv v^*(x,t) = (v_1^*(x,t), v_2^*(x,t)) \in (L^2(\Sigma))^2$ *which satisfies the maximum condition* (5.5).

Case 2 (optimal control for $(n \times n)$ coupled infinite order parabolic systems with multiple time delays). We will extend the discussion to $(n \times n)$ coupled infinite order parabolic systems. We consider the case where $v = (v_1, v_2, \ldots, v_n) \in ((L^2(\Sigma))^n$, the performance functional is given by [15, 16]:

$$I(v) = \sum_{j=1}^n \left(\lambda_1 \int_Q \left[y_j(x,t;v) - z_{jd} \right]^2 dx \, dt + \lambda_2 \int_\Sigma (N_j v_j) v_j dx \, dt \right),$$

(5.6)

where $z_d = (z_{1d}, z_{2d}, \ldots, z_{nd}) \in (L^2(Q))^n$.

The following results can now be proved.

Theorem 5.5. *Let* $y_0, \Phi_0, \Psi_0, v,$ *and* u *be given with* $y_p = (y_{p,1}, y_{p,2}, \ldots, y_{p,n}) \in (W^\infty\{a_\alpha, 2\}(\Omega))^n,$ $\Phi_0 = (\Phi_{0,1}, \Phi_{0,2}, \ldots, \Phi_{0,n}) \in (W^{\infty,1}(Q_0))^n,$ $\Psi_0 = (\Psi_{0,1}, \Psi_{0,2}, \ldots, \Psi_{0,n}) \in (L^2(\Sigma_0))^n,$ $v = (v_1, v_2, \ldots, v_n) \in (L^2(\Sigma_0))^n,$ *and* $u = (u_1, u_2, \ldots, u_n) \in (W^{-\infty,-1}(Q))^n.$ *Then, there exists a unique solution* $y = (y_1, y_2, \ldots, y_n) \in (W^{\infty,1}(Q_0))^n$ *for the following mixed initial-boundary value problem: for all* $j, j = 1, 2, \ldots, n$ *one has*

$$\frac{\partial y_j}{\partial t} + S(t) y_j(x,t) + \sum_{i=1}^m \int_{a_i}^{b_i} b_i(x,t) y_j(x, t - h_i) dh_i = u_j,$$

$$(x,t) \in \Omega \times (0,T), \quad h_i \in (a_i, b_i),$$

$$y_j(x, t') = \Phi_{0,j}(x, t') \quad (x, t') \in \Omega \times [-\Delta, 0),$$

$$y_j(x, 0) = y_{0,j}(x), \quad x \in \Omega,$$

$$\frac{\partial y_j}{\partial v_S}(x,t) = \sum_{S=1}^{l} \int_{c_s}^{d_s} c_s(x,t) y_j(x,t-k_s) dk_s + v_j,$$

$$(x,t) \in \Gamma \times (0,T), \quad k_s \in (c_s, d_s),$$

$$y_j(x,t') = \Psi_{0,j}(x,t'), \quad (x,t') \in \Gamma \times [-\Delta, 0),$$

$$(5.7)$$

where

$$y \equiv y(x,t;u) = (y_1(x,t;u), y_2(x,t;u), \ldots, y_n(x,t;u)) \in \left(W^{\infty,1}(Q)\right)^n,$$

$$u \equiv u(x,t) = (u_1(x,t), u_2(x,t), \ldots, u_n(x,t)) \in \left(W^{-\infty,-1}(Q)\right)^n, \tag{5.8}$$

$$v \equiv v(x,t) = (v_1(x,t), v_2(x,t), \ldots, v_n(x,t)) \in \left(L^2(\Sigma)\right)^n.$$

The operator $\mathcal{S}(t)$ is an $n \times n$ matrix takes the form [15, 16, 18, 22]:

$$\mathcal{S}(t)$$

$$= \begin{pmatrix} \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} a_\alpha D^{2\alpha} + 1 & -1 & \cdot \ \cdot & -1 \\ 1 & \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} a_\alpha D^{2\alpha} + 1 & \cdot \ \cdot & -1 \\ \cdot & \cdot & \cdot \ \cdot & \cdot \\ \cdot & \cdot & \cdot \ \cdot & \cdot \\ 1 & 1 & \cdot \ \cdot & \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} a_\alpha D^{2\alpha} + 1 \end{pmatrix}_{n \times n}, \tag{5.9}$$

that is

$$\mathcal{S}(t) y_j(x) = \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} a_\alpha D^{2\alpha} y_j(x) + \sum_{r=1}^{n} B_{jr} y_j(x), \quad \forall j, j = 1, 2, \ldots, n, \tag{5.10}$$

where

$$B_{jr} = \begin{cases} 1, & \text{if } j \geq r, \\ -1, & \text{if } j < r. \end{cases} \tag{5.11}$$

Lemma 5.6. *Let the hypothesis of Theorem 5.5 be satisfied. Then for given $z_d = (z_{1d}, z_{2d}, \ldots, z_{nd}) \in (L^2(Q))^n$ and any $v(x,t) = (v_1(x,t), v_2(x,t), \ldots, v_n(x,t)) \in (L^2(\Sigma))^n$, there exists a unique solution*

$$p(v) \equiv p(x,t;v) = (p_1(x,t;v), p_1(x,t;v), \ldots, p_n(x,t;v)) \in \left(W^{\infty,1}(Q)\right)^n, \tag{5.12}$$

for the adjoint problem: for all $j, j = 1, 2, \ldots, n$, one has

$$\frac{-\partial p_j(v)}{\partial t} + \mathcal{S}^*(t)p_j(v) + \sum_{i=1}^{m} \int_{a_i}^{b_i} b_i(x, t + h_i)p_j(x, t + h_i; v)dh_i = \lambda_1(y_j(v) - z_{jd}),$$

$$(x, t) \in \Omega \times (0, T - \Delta), \quad h_i \in (a_i, b_i),$$

$$\frac{\partial p_j(v)}{\partial t} + \mathcal{S}^*(t)p_j(v) = \lambda_j(y_j(v) - z_{jd}), \quad (x, t) \in \Omega \times (T - \Delta, T),$$

$$p_j(x, T, v) = 0, \quad x \in \Omega,$$

$$p(x, t; v) = 0, \quad (x, t) \in \Omega \times [T - \Delta + \lambda, T),$$

$$\frac{\partial p_j(v)}{\partial v_{\mathcal{S}^*}}(x, t) = \sum_{s=1}^{l} \int_{c_s}^{d_s} c_s(x, t + k_s)p_j(x, t + k_s; v)dk_s, \quad (x, t) \in \Gamma \times (0, T - \Delta), \; k_s \in (c_s, d_s),$$

$$\frac{\partial p_j(v)}{\partial v_{\mathcal{S}^*}}(x, t) = 0, \quad (x, t) \in \Gamma \times (T - \Delta, T),$$

$$(5.13)$$

where

$$\mathcal{S}^*(t)p_j(x) = \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} a_\alpha D^{2\alpha} p_j(x) + \sum_{r=1}^{n} B_{rj} p_j(x), \quad \forall j, j = 1, 2, \ldots, n, \qquad (5.14)$$

B_{rj} *are the transpose of* B_{jr}.

Theorem 5.7. *The optimal control* $v^* \equiv v^*(x, t) = (v_1^*(x, t), v_2^*(x, t), \ldots, v_n^*(x, t)) \in (L^2(\Sigma))^n$ *is characterized by the following maximum condition:*

$$\sum_{j=1}^{n} \int_{\Sigma} \left[p_j(v^*) + \lambda_2 N_j v_j^* \right] \left(v_j - v_j^* \right) d\Gamma \, dt \geq 0, \quad \forall v = (v_1, v_2, \ldots, v_n) \in (U_{ad})^n, \qquad (5.15)$$

where

$$p(v^*) \equiv p(x, t; v^*) = (p_1(x, t; v^*), p_1(x, t; v^*), \ldots, p_n(x, t; v^*)) \in \left(W^{\infty, 1}(Q) \right)^n \qquad (5.16)$$

is the adjoint state.

The foregoing result is now summarized.

Theorem 5.8. *For the problem (5.7) with the performance function (5.6) with* $z_d = (z_{1d}, z_{2d}, \ldots, z_{nd}) \in (L^2(Q))^n$ *and* $\lambda_2 > 0$, *and with constraint:* U_{ad} *is closed, convex subset of* $(L^2(\Sigma))^n$, *and with adjoint equations (5.13), then there exists a unique optimal control* $v^* \equiv v^*(x, t) = (v_1^*(x, t), v_2^*(x, t), \ldots, v_n^*(x, t)) \in (L^2(\Sigma))^n$ *which satisfies the maximum condition (5.15).*

In the case of performance functionals (4.1), (4.22), (4.26), (4.30), (5.1), and (5.6) with $\lambda_1 > 0$ and $\lambda_2 = 0$, the optimal control problem reduces to minimization of the functional on a closed and convex subset in a Hilbert space. Then, the optimization problem is equivalent to a quadratic programming one, which can be solved by the use of the well-known Gilbert algorithm.

6. Conclusions

The optimization problem presented in the paper constitutes a generalization of the optimal boundary control problem for second-order parabolic systems with Neumann boundary condition involving constant time lag appearing in the state and in the boundary conditions considered in [1, 5–9, 14–16, 18, 21, 32].

Moreover, the results obtained in this paper (Theorems 4.2, 4.6, 5.4, and 5.8) can be treated as a generalization of the optimization theorems proved by [8–10]. Also the main result of the paper contains necessary and sufficient conditions of optimality for $(n \times n)$ infinite order parabolic systems with multiple time delays given in integral form both in the state equation and in the Neumann boundary condition that give characterization of optimal control (Theorem 5.8). But it is easily seen that obtaining analytical formulas for optimal control are very difficult. This results from the fact that state equations (5.7), adjoint equations (5.13), and maximum condition (5.15) are mutually connected that cause that the usage of derived conditions is difficult. Therefore we must resign from the exact determination of the optimal control and therefore we are forced to use approximation methods.

Also it is evident that by modifying:

  (i) the boundary conditions, (Dirichlet, Neumann, mixed, etc.),

  (ii) the nature of the control (distributed, boundary, etc.),

  (iii) the nature of the observation (distributed, boundary, etc.),

  (iv) the initial differential system,

  (v) the time delays (constant time delays, time-varying delays, multiple time-varying delays, time delays given in the integral form, etc.),

  (vi) the number of variables (finite number of variables, infinite number of variables systems, etc.),

  (vii) the type of equation (elliptic, parabolic, hyperbolic, etc.),

  (viii) the order of equation (second order, Schrödinger, infinite order, etc.),

  (ix) the type of control (optimal control problem, time-optimal control problem, etc.),

an infinity of variations on the above problems are possible to study with the help of [19] and Dubovitskii-Milyutin formalisms [22–27]. Those problems need further investigations and form tasks for future research. These ideas mentioned above will be developed in forthcoming papers.

Acknowledgments

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