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Research Article

Classical Lie Point Symmetry Analysis of a Steady Nonlinear One-Dimensional Fin Problem

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We consider the one-dimensional steady fin problem with the Dirichlet boundary condition at one end and the Neumann boundary condition at the other. Both the thermal conductivity and the heat transfer coefficient are given as arbitrary functions of temperature. We perform preliminary group classification to determine forms of the arbitrary functions appearing in the considered equation for which the principal Lie algebra is extended. Some invariant solutions are constructed. The effects of thermogeometric fin parameter and the exponent on temperature are studied. Also, the fin efficiency is analyzed.

1. Introduction

A search for exact and numerical solutions for models arising in heat flow through extended surfaces continues to be of scientific interest. The literature in this area is sizeable (see, e.g., [1] and references cited therein). Perhaps such interest has been instilled by frequent encounters of fin problems in many engineering applications to enhance heat transfer. Fins play an important role in enhancing heat dissipation from a hot surface. They are used in air conditioning, air-cooled craft engines, refrigeration, cooling of computer processors, cooling of oil carrying pipe line, and so on.

In recent years, many authors have been interested in the steady-state problems [2–5] describing heat flow in one-dimensional longitudinal rectangular fins. The symmetry analysis, in particular, group classification of the unsteady fin problem has attracted some interest (see, e.g., [6–10]).

Few exact solutions exist for one-dimensional problems. Perhaps this is due to highly nonlinearity of the fin models. In fact, existing solutions are constructed only when both thermal conductivity and heat transfer coefficient are given as constant [2]. Recently, in [5], exact solutions of the one-dimensional fin problem given nonlinear thermal conductivity and heat transfer coefficient have been constructed.

In this paper, we determine the cases of thermal conductivity and heat transfer coefficient terms for which extra symmetries are admitted. We then select the realistic cases and analyze the problem. In Section 2, we provide the mathematical formulation of the problem. Symmetry analysis is performed in Section 3. We determine the principal Lie algebra, equivalence transformations, and list the cases for which the principal Lie algebra is extended. In Section 4, we employ symmetry techniques to determine wherever possible, the invariant solutions.

2. Mathematical Models

Consider longitudinal rectangular one-dimensional fin with a cross-sectional area $A_c = \delta \times W$ as shown in Figure 1. The perimeter and length are given by P and L, respectively. The fin is attached to a fixed base surface of temperature T_b and extends into a fluid of temperature T_a. The fin is insulated at the tip. The steady energy balance equation is given by [4]

$$A_c \frac{d}{dX}\left(K(T)\frac{dT}{dX}\right) = PH(T)(T - T_a), \quad 0 \leq X \leq L, \tag{2.1}$$

where K and H are temperature-dependent thermal conductivity and heat transfer coefficient, respectively (see, e.g., [2, 3]). The spatial variable is X.

The relevant boundary conditions are given by

$$T(L) = T_b, \qquad \left.\frac{dT}{dX}\right|_{X=0} = 0. \tag{2.2}$$

Introducing the dimensionless variables

$$x = \frac{X}{L}, \quad \theta = \frac{T - T_a}{T_b - T_a}, \quad h(\theta) = \frac{H(T)}{h_b}, \quad k(\theta) = \frac{K(T)}{k_a}, \quad M^2 = \frac{Ph_bL^2}{k_aA_c} \tag{2.3}$$

reduces (2.1) to the relevant dimensionless energy equation

$$\frac{d}{dx}\left[k(\theta)\frac{d\theta}{dx}\right] - M^2 h(\theta)\theta = 0, \quad 0 \leq x \leq 1, \tag{2.4}$$

and the boundary conditions become

$$\theta(1) = 1, \qquad \theta'(0) = 0. \tag{2.5}$$

Setting $P/A_c = 1/\delta$ leads to the equivalent definition of thermogeometric fin parameter $M = (Bi)^{1/2}E$, where $Bi = \delta h_b/k_a$ is the Biot number, and $E = L/\delta$ is the extension factor

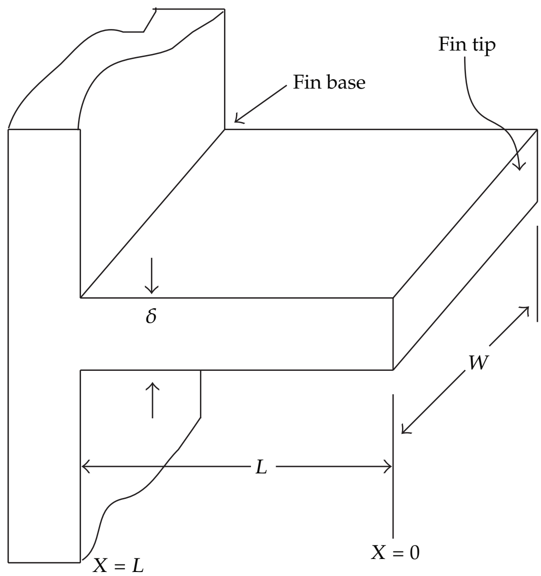

Figure 1: Schematic representation of a one-dimensional fin.

with δ being the fin thickness. Since $h(\theta)$ is an arbitrary function of temperature, we equate the product $h(\theta)\theta$ to $G(\theta)$. Note that the thermogeometric fin parameter M is specified. The parameters h_b and k_a are the heat transfer coefficient at the fin base and the fluid thermal conductivity. The analysis of (2.4) was conducted in [5], wherein the heat transfer coefficient was assumed to be given by the power law function of temperature. In this paper, we allow both the heat transfer coefficient and thermal conductivity to be arbitrary functions of temperature and employ preliminary group classification techniques to determine the forms which lead to exact solutions. We consider the governing equation

$$\frac{d}{dx}\left[k(\theta)\frac{d\theta}{dx}\right] - M^2 G(\theta) = 0, \quad 0 \leq x \leq 1. \tag{2.6}$$

We note that (2.6) is linearizable provided that G is a differential consequence of k. The proof of this statement follows from chain rule [11]. This implies that (2.6) may be linearizable for any k such that its derivative is G. Also, the linearization of (2.6) was performed in [12] wherein approximate techniques were employed to solve the problem. In this paper, we apply Lie point symmetry techniques to analyze the problem.

3. Symmetry Analysis

The theory and applications of symmetry analysis may be found in excellent text such as those of [13–18]. In the next subsections, we construct the equivalence algebra and hence equivalence group of transformations admitted by (2.6). Furthermore we determine the Lie point symmetries admitted by (2.6) with arbitrary functions k and G; that is, we seek the principal Lie algebra. Symmetry technique are algorithmic and tedious. Here we utilize the interactive computer software algebra REDUCE [19] to facilitate the calculations.

3.1. Equivalence Transformations

In brief, an equivalence transformation of a differential equation is an invertible transformation of dependent and independent variables which leave the form of the equation in question unchanged [20]. However the form of the arbitrary functions appearing in the transformed equation may be distinct from those of the original equation. To determine the equivalence transformation, one may seek the equivalence algebra generated by the vector field

$$\widetilde{X} = \xi(x,\theta)\partial_x + \eta(x,\theta)\partial_\theta + \mu^1(x,\theta,k,G)\partial_k + \mu^2(x,\theta,k,G)\partial_G. \tag{3.1}$$

The second prolongation is given by

$$\widetilde{X}^{[2]} = \widetilde{X} + \zeta_x\partial_{\theta'} + \zeta_{xx}\partial_{\theta''} + \omega_x^1\partial_{k_x} + \omega_\theta^1\partial_{k'} + \omega_x^2\partial_{G_x}, \tag{3.2}$$

where

$$\begin{aligned}
\zeta_x &= D_x(\eta) - \theta' D_x(\xi),\\
\zeta_{xx} &= D_x(\zeta_x) - \theta'' D_x(\xi),\\
\omega_x^1 &= \widetilde{D}_x\left(\mu^1\right) - k_x\widetilde{D}_x(\xi) - k'\widetilde{D}_x(\eta),\\
\omega_x^2 &= \widetilde{D}_x\left(\mu^2\right) - G_x\widetilde{D}_x(\xi) - G'\widetilde{D}_x(\eta),\\
\omega_\theta^1 &= \widetilde{D}_\theta\left(\mu^1\right) - k_x\widetilde{D}_\theta(\xi) - k'\widetilde{D}_\theta(\eta),
\end{aligned} \tag{3.3}$$

with D_x and $\widetilde{D}_x$ being the total derivative operator defined by

$$\begin{aligned}
D_x &= \partial_x + \theta'\partial_\theta + \theta''\partial_{\theta'} + \cdots,\\
\widetilde{D}_x &= \partial_x + k_x\partial_k + G_x\partial_G + k_{xx}\partial_{k_x} + \cdots = \partial_x,\\
\widetilde{D}_\theta &= \partial_\theta + k'\partial_k + \cdots,
\end{aligned} \tag{3.4}$$

respectively. The prime implies differentiation with respect to θ. The invariance surface condition is given by

$$\widetilde{X}^{[2]}(2.6)|_{(2.6)} = 0, \quad \widetilde{X}^{[2]}(k_x = 0)|_{k_x=0} = 0, \quad \widetilde{X}^{[2]}(G_x = 0)|_{G_x=0} = 0. \tag{3.5}$$

This system of equations yields the infinite dimensional equivalence algebra spanned by the base vectors

$$\widetilde{X}_1 = \partial_x, \quad \widetilde{X}_2 = x\partial_x - 2G\partial_G, \quad \widetilde{X}_3 = u(\theta)\partial_\theta - u'(\theta)k\partial_k, \quad \widetilde{X}_4 = v(G)(k\partial_k + G\partial_G), \tag{3.6}$$

admitted by (2.6). Here u and v are arbitrary functions of θ and G, respectively.

3.2. Principal Lie Algebra

In this subsection, we seek classical Lie point symmetries generated by the vector field

$$X = \xi(x,\theta)\frac{\partial}{\partial x} + \eta(x,\theta)\frac{\partial}{\partial \theta} \tag{3.7}$$

admitted by the governing equation for any arbitrary functions k and G. We seek invariance in the form

$$X^{[2]}(2.6)|_{(2.6)} = 0. \tag{3.8}$$

Here $X^{[2]}$ is the second prolongation defined by

$$X^{[2]} = X + \zeta_x \frac{\partial}{\partial \theta'} + \zeta_{xx}\frac{\partial}{\partial \theta''}, \tag{3.9}$$

where the prolongation formulae are given above. The principal Lie algebra is one dimensional and spanned by space translation. For nontrivial function k and G, we obtain the determining equations

(1) $k'\xi_\theta - k\xi_{\theta\theta} = 0$,

(2) $k^2\eta_{\theta\theta} + kk'\eta_\theta + kk''\eta - (k')^2\eta - 2k^2\xi_{x\theta} = 0$,

(3) $2k\eta_{x\theta} + 2k'\eta_x - k\xi_{xx} - 3M^2G\xi_\theta = 0$,

(4) $k^2\eta_{xx} + M^2kG\eta_\theta - M^2kG'\eta + M^2Gk'\eta - 2M^2kG\xi_x = 0$.

The determining equation (2.1) implies that $\xi = \phi(\theta) + \psi(x)$ and $k = \phi'(\theta)$, where ϕ and ψ are arbitrary functions of θ and x, respectively. The determining equations (2.2), (2.3), and (2.4) become

(2^*) $\eta_{\theta\theta}\phi'^2 + \phi''\phi'\eta_\theta + (\phi'''\phi' - \phi''^2)\eta = 0$,

(3^*) $2\eta_{x\theta}\phi' + 2\eta_x\phi'' + \phi'\psi'' - 3M^2\phi'G = 0$,

(4^*) $\eta_{xx}\phi'^2 + M^2\eta_\theta\phi'G - M^2G'\phi'\eta + M^2\phi''\eta G - 2M^2\phi'\psi'G = 0$.

It appears that full group classification of (2.6) may be difficult to achieve. Hence, we resort to the preliminary group classification techniques.

3.3. Preliminary Group Classification

We follow the sketch of the preliminary group classification technique as outlined in [20]. We note that the (2.6) admits an infinite equivalence algebra as given in Section 3.1. So we are free to take any finite dimensional subalgebra as large as we desire and use it for preliminary group classification. We choose a five-dimensional equivalence algebra spanned by the vectors

$$\widetilde{X}_1 = \partial_x, \quad \widetilde{X}_2 = x\partial_x - 2G\partial_G, \quad \widetilde{X}_3 = \partial_\theta, \quad \widetilde{X}_4 = \theta\partial_\theta - k\partial_k, \quad \widetilde{X}_5 = k\partial_k + G\partial_G. \tag{3.10}$$

Recall that k and G are θ dependent. Thus, we consider the projections of (3.10) on the space of (θ, k, G). The nonzero projections of operators (3.10) are

$$\mathbf{v_1} = \mathrm{pr}\left(\widetilde{X}_2\right) = -2G\partial_G, \quad \mathbf{v_2} = \mathrm{pr}\left(\widetilde{X}_3\right) = \partial_\theta, \quad \mathbf{v_3} = \mathrm{pr}\left(\widetilde{X}_4\right) = \theta\partial_\theta - k\partial_k,$$
$$\mathbf{v_4} = \mathrm{pr}\left(\widetilde{X}_5\right) = k\partial_k + G\partial_G. \tag{3.11}$$

Proposition 3.1 (see, e.g., [20]). *Let $\mathcal{L}_r$ be an r-dimensional subalgebra of the algebra $\mathcal{L}_4$. Denote by Z_i, $i = 1, \ldots, r$ a basis of $\mathcal{L}_r$ and by W_i the elements of the algebra $\mathcal{L}_5$ such that Z_i is the projections of W_i on* (θ, k, G). *If equations*

$$k = \omega(\theta), \quad G = \varphi(\theta) \tag{3.12}$$

are invariant with respect to the algebra $\mathcal{L}_r$ then the equation

$$\frac{d}{dx}\left(\omega(\theta)\frac{d\theta}{dx}\right) - M^2\varphi(\theta) = 0 \tag{3.13}$$

admits the operator

$$Z_i = \text{ projection of } W_i \quad \text{ on } (x, \theta). \tag{3.14}$$

Proposition 3.2 (see, e.g., [20]). *Let* (3.13) *and equation*

$$\frac{d}{dx}\left(\overline{\omega(\theta)}\frac{d\theta}{dx}\right) - M^2\overline{\varphi(\theta)} = 0 \tag{3.15}$$

be constructed according to Proposition 3.1 *via subalgebras $\mathcal{L}_r$ and $\overline{\mathcal{L}_r}$, respectively. If $\mathcal{L}_r$ and $\overline{\mathcal{L}_r}$, are similar subalgebras in $\mathcal{L}_5$ then* (3.13) *and* (3.15) *are equivalent with respect to the equivalence group G_5 generated by $\mathcal{L}_r$. These propositions imply that the problem of preliminary group classification of* (2.6) *is reduced to the algebraic problem of constructing nonsimilar subalgebras of $\mathcal{L}_4$ or optimal system of subalgebras [20]. We explore methods in [13] to construct the one-dimensional optimal systems. The set of nonsimilar one-dimensional subalgebras is*

$$\left\{\mathbf{v_1} + \alpha\mathbf{v_3} + \beta\mathbf{v_4}, \mathbf{v_3} \pm \mathbf{v_2} + \alpha\mathbf{v_4}, \mathbf{v_3} + \alpha\mathbf{v_4}, \mathbf{v_4} + \alpha\mathbf{v_2}, \mathbf{v_2}\right\}. \tag{3.16}$$

Here α and β are arbitrary constants.

As an example, we apply 1 to one of the element of the optimal system. Since this involves routine calculations of invariants, we list the rest of cases in Table 1, wherein λ, p, and q are arbitrary constants. Note that the power law k was obtained in [5], therefore we omit this case in this manuscript.

Consider the subalgebra

$$\mathbf{v_2} + \mathbf{v_4} = k\partial_k + G\partial_G + \partial_\theta, \tag{3.17}$$

Table 1: Extensions of the principal Lie algebra.

Forms		Symmetries
k	G	$X_1 = \partial_x$.
$e^{p\theta}$	$e^{q\theta}$	$X_2 = x\partial_x + \dfrac{2}{p-q}\partial_\theta, \quad p \neq q$.
p	$e^{q\theta}$	$X_2 = x\partial_x - \dfrac{2}{q}\theta\partial_\theta$.
$(1+\lambda\theta)$	$(1+\lambda\theta)^p$	$X_2 = x\partial_x + \dfrac{2(1+\lambda\theta)}{\lambda(p-2)}\partial_\theta, \quad p \neq -6$ $X_2 = 2\lambda x^2\partial_x + x(1+\lambda\theta)\partial_\theta,$ $X_3 = 2\lambda x\partial_x + (1+\lambda\theta)\partial_\theta, \quad p = -6$

where, without loss of generality, we have assumed α to be unity. A basis of invariants is obtained from the equation

$$\frac{dk}{k} = \frac{dG}{G} = \frac{d\theta}{1}, \tag{3.18}$$

and the forms of k and G are

$$k = e^{\theta}, \quad G = e^{\theta}. \tag{3.19}$$

For simplicity, we have allowed both integration constants to vanish. Further cases are listed in Table 1. By applying Proposition 3.1, we obtain the symmetry generator $X_2 = \partial_\theta$. We shall show in Section 4.2 that, for these forms of k and G, one may obtain seven more Lie point symmetry generators.

4. Symmetry Reductions and Invariant Solutions

The main use of symmetries is to reduce the number of independent variables of the given equation by one. If a partial differential equation (PDE) is reduced to an ordinary differential equation (ODE), one may or may not solve the resulting ODE exactly. If a second-order ODE admits a two-dimensional Lie algebra, then one can use Lie's method of *canonical coordinates* to completely integrate the equation (see, e.g., [21]).

4.1. Example 1

As an illustrative example, we consider the case $k = e^{p\theta}$ and $h = \theta^{-1}e^{q\theta}$, where $p \neq q$. In this case (2.4) admits a non-Abelian two-dimensional Lie algebra spanned by the base vectors listed in Table 1. This noncommuting pair of symmetries leads to the canonical variables

$$t = e^{((p-q)/2)\theta}, \quad u = c_1 e^{((p-q)/2)\theta} + x, \tag{4.1}$$

where c_1 is an arbitrary constant. We have two cases, the "particular" canonical variables when $c_1 = 0$ and the "general" canonical variables given a nonzero c_1, say $c_1 = 1$.

4.1.1. Particular Canonical Form

The corresponding canonical forms of X_1 and X_2 are

$$\Gamma_1 = \partial_u, \qquad \Gamma_2 = t\partial_t + u\partial_u. \tag{4.2}$$

Writing $u = u(t)$ transforms (2.6) to

$$u'' = \frac{u'}{t}\left[\left(\frac{2p}{p-q} - 1\right) - \left(\frac{p-q}{2}\right)M^2 u'^2\right], \quad p \neq q. \tag{4.3}$$

Here prime is the total derivative with respect to t. Three cases arise.

Case 1. For $u' = 0$, we obtain the constant solution which is not related to the original problem. Thus, we ignore it.

Case 2. If the term in the square bracket vanishes, then we obtain in terms of original variables the exact "particular" solution

$$\theta = \left(\frac{2}{p-q}\right)\ln\left[\frac{(p-q)M}{\pm\sqrt{2(p+q)}}\left(x - 1 \pm \frac{\sqrt{2(p+q)}}{(p-q)M}\mathrm{e}^{(p-q)/2}\right)\right]. \tag{4.4}$$

Note that this exact solution satisfies the boundary only at one end. The Neumann's boundary condition leads to a contradiction since the thermogeometric fin parameter is a nonzero constant.

Case 3. Solving the entire equation (4.3) we, obtain the solution in complicated quadratures, and therefore we omit it.

4.1.2. General Canonical Form

In this case, the transformed equations are given by

$$u'' = \frac{(u'-1)}{t}\left[\left(\frac{2p}{p-q} - 1\right) - \left(\frac{p-q}{2}\right)M^2 (u'-1)^2\right], \quad p \neq q. \tag{4.5}$$

Clearly $u' - 1 \to y'$ reduces (4.5) to (4.3). We herein omit further analysis.

4.2. Example 2

We consider as an example (2.6) with thermal conductivity given as exponential function of temperature; that is, $k = \mathrm{e}^{p\theta}$ and heat transfer coefficient is given as the quotient $\theta^{-1}\mathrm{e}^{q\theta}$. Given

$p = q$, then (2.6) admits a maximal eight-dimensional symmetry algebra spanned by the base vectors

$$X_1 = e^{\sqrt{p}Mx+n\theta}\left\{\partial_x + \frac{M}{\sqrt{p}}\partial_\theta\right\},$$
$$X_2 = e^{-\sqrt{p}Mx+p\theta}\left\{\partial_x - \frac{M}{\sqrt{p}}\partial_\theta\right\},$$
$$X_3 = e^{\sqrt{p}Mx-p\theta}\partial_\theta, \quad X_4 = \frac{\sqrt{p}e^{2\sqrt{p}Mx}}{M}\partial_\theta, \tag{4.6}$$
$$X_5 = \partial_x, \quad X_6 = e^{-\sqrt{p}Mx-p\theta}\{\partial_x + \partial_\theta\},$$
$$X_7 = e^{-2\sqrt{p}Mx}\left\{-\frac{\sqrt{p}}{M}\partial_x + \partial_\theta\right\}, \qquad X_8 = \partial_\theta.$$

Equation (2.6) is linearizable or equivalent to $y'' = 0$ (see, e.g., [21]). In fact, we note that the point transformation $\omega = e^{p\theta}$, $p \in \mathbb{R}$ linearizes (2.6) given $p = q$. Following a simple manipulation, we obtain the invariant solutions satisfying the prescribed boundary conditions, namely,

$$\theta = \ln\left[\frac{e^p \cosh(M\sqrt{p}x)}{\cosh(M\sqrt{p})}\right]^{1/p}, \quad p > 0. \tag{4.7}$$

Solution (4.7) is depicted in Figures 2 and 3. Note that, for $p = 0$ and $p < 0$, we obtain solutions which have no physical significance for heat transfer in fins. Therefore, we herein omit such solutions.

The fin efficiency is defined as the ratio of actual heat transfer from the fin surface to the surrounding fluid while the whole fin surface is kept at the same temperature (see, e.g., [1]). Given (4.7) fin efficiency (η) is given by

$$\eta = \int_0^1 \ln\left[\frac{e^p \cosh(M\sqrt{p}x)}{\cosh(M\sqrt{p})}\right]^{1/p} dx. \tag{4.8}$$

We use MAPLE package to evaluate this integral. The plot is depicted in Figure 4.

5. Some Discussions and Concluding Remarks

We considered a one-dimensional fin model describing steady-state heat transfer in longitudinal rectangular fins. Here, the thermal conductivity and heat transfer coefficient are temperature dependent. As such the considered problem is highly nonlinear. This is a significant improvement to the results presented in the literature (see, e.g., [2, 3]). Preliminary group classification led to a number of cases of thermal conductivity and heat transfer coefficient for which extra symmetries are obtained. Exact solutions are constructed when thermal conductivity and heat transfer coefficient increase exponential with temperature. We observed, in Figure 2, that temperature inversely proportional to the values of the

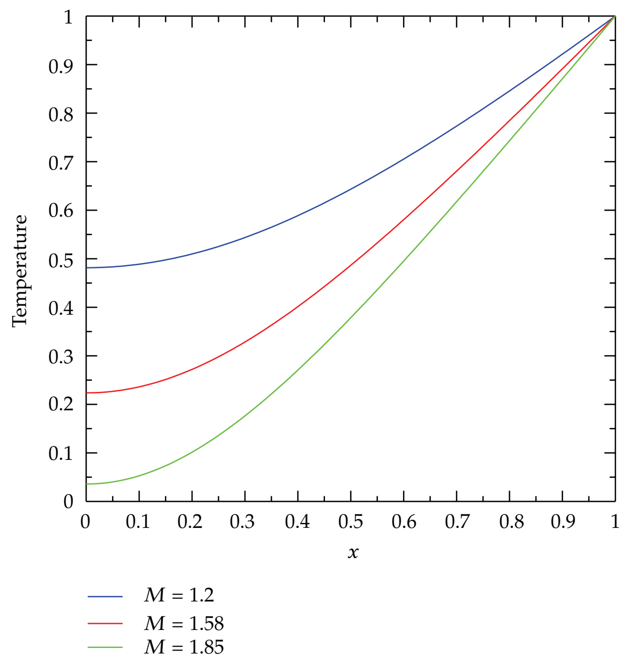

Figure 2: Temperature profile in a fin with varying values of the thermogeometric fin parameter. Here, p is fixed at unity.

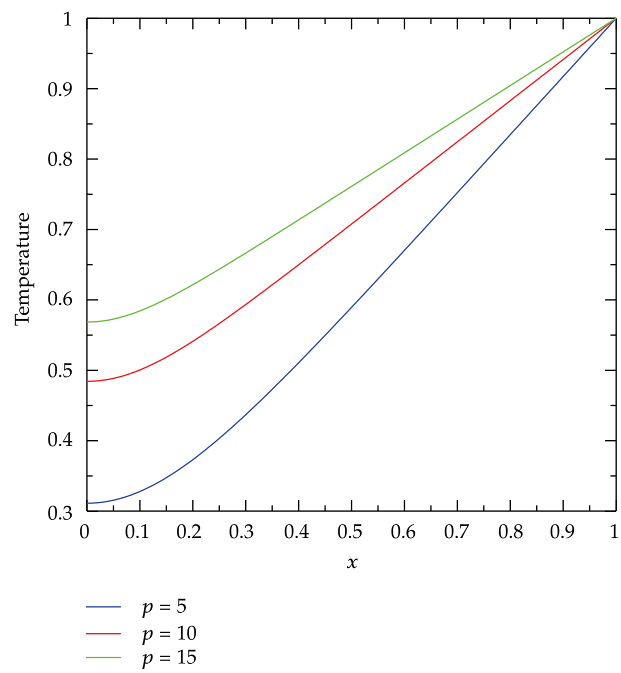

Figure 3: Temperature profile in a fin with varying values of the p. Here the thermogeometric fin parameter is fixed at 1.85.

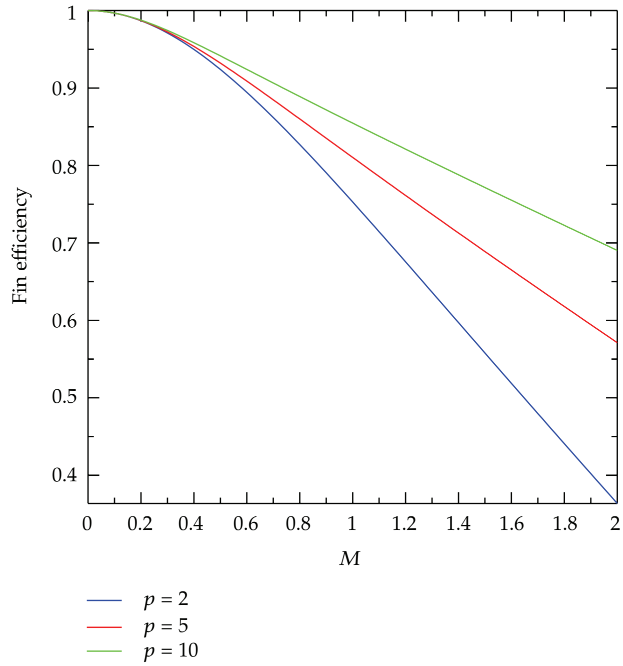

Figure 4: Fin efficiency.

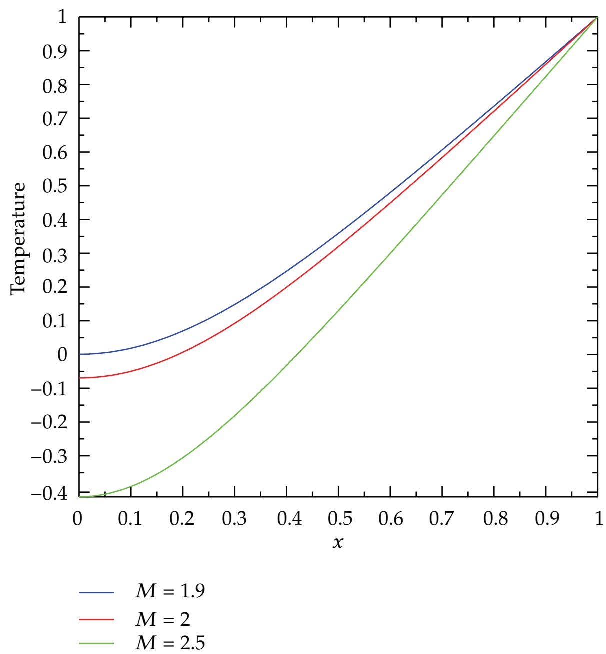

Figure 5: Temperature profile in a fin of varying values of the thermogeometric fin parameter. Here p is fixed at unity.

thermogeometric fin parameter. Furthermore, we observe that for certain values of M, the solution is not physically sound (see also, [22]). One may recall that the thermogeometric fin parameter depends also on heat transfer coefficient at the base of the fin. We notice that the exponential temperature-dependent heat transfer coefficient in this paper leads to lower values of M for which the solutions are realistic. That is, the maximum values of M, say $M_{\max}$ for which the solutions are physically sound, is around 2. We observe, in Figure 5, that as values of M increase beyond 2, the temperature profile becomes negative. This contradicts the rescaling of temperature (the dimensionless temperature). Unlike [5, 23] whereby heat transfer is given by a power law, this value is much higher. The reasons behind this observation is studied elsewhere. In Figure 3, temperature increases with increased values of the exponent p. Furthermore, fin efficiency decreases with increased values of the thermogeometric fin parameter. We observed, in Figure 4, that the maximum value of the thermogeometric fin parameter for which the fin efficiency is realistic is again around 2.

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