Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 658971, 7 pages
doi:10.1155/2012/658971

Research Article

On T-Stability of the Picard Iteration for Generalized φ-Contraction Mappings

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Received 20 May 2012; Accepted 6 June 2012

Academic Editor: Yonghong Yao

We introduce some results on T-stability of the Picard iteration for φ-contraction and generalized φ-contraction mappings on metric spaces.

1. Introduction

It is known that iteration methods are numerical procedures which compute a sequence of gradually accurate iterates to approximate the solution of a class of problems. Such methods are useful tools of applied mathematics for solving real life problems ranging from economics and finance or biology to transportation, network analysis, or optimization. An iteration method is considered to be sound if possesses some qualitative properties such as convergence and stability. That is why several scientists paid and still pay attention to the qualitative study of iteration methods; please, see [1–7].

There are some papers about the stability or different iteration methods. In [3], Harder and Hicks studied the stability of Picard iteration for several contractivity conditions [7], while in [6] Rhoades introduced a contractivity condition independent of that in [7] to obtain stability results for Mann, Kirk, or Massa iteration processes. Meantime, Bosede and Rhoades [2] introduced stability results of Picard and Mann iteration for a general class of functions; also, see [4], while Rezapour et al. [5] studied the almost stability of Mann iteration for φ-contraction mappings and the stability of Picard iteration for mappings satisfying a contractive condition of integral type. In the present paper, we introduce our new results on stability of Picard iteration for φ-contraction and generalized φ-contraction mappings on metric spaces.

2. Previous Notation and Definitions

Let (X, d) be a complete metric space, $T : X \to X$ a map and $x_{n+1} = f(T, x_n)$ an iteration procedure. Suppose that T has at least one fixed point and that sequence $\{x_n\}$ converges to a fixed point $x^* \in X$. We denote the set of fixed points of mapping T by F_T. Let $\{y_n\}$ be an arbitrary sequence in X and $\epsilon_n = d(y_{n+1}, f(T, y_n))$.

If $\lim_{n\to\infty}\epsilon_n = 0$ implies that $\lim_{n\to\infty} y_n = x^*$, then the iteration procedure $x_{n+1} = f(T, x_n)$ is said to be T-stable (e.g., [1, 6]).

If $\{y_n\}$ is a bounded sequence and $\lim_{n\to\infty}\epsilon_n = 0$ implies that $\lim_{n\to\infty} y_n = x^*$, then the iteration procedure $x_{n+1} = f(T, x_n)$ is said to be boundedly T-stable.

In most papers on T-stability, some authors consider the notion of boundedly T-stability instead of T-stability. Here, we mention the Picard iteration methods. Let $x_0 \in X$. The Picard iteration is given by $x_{n+1} = Tx_n$.

The following example illustrates that the notion of T-Stability is different from the notion of boundedly T-stability.

Example 2.1. Consider mapping $T : [0, \infty) \to [0, \infty)$ given by $Tx = (1/2)(x + 1)$ whenever $x \in [0, 1]$ and $Tx = x + 1$ whenever $x > 1$. Put $y_n = n + (1/n)$ for all $n \geq 1$. Note that $\{y_n\}$ is unbounded, while $\lim_{n\to\infty}|y_{n+1} - Ty_n| = 0$.

3. Main Results

Now, we are ready to state and prove our main results.

Definition 3.1 (see [1]). A function $\varphi : [0, \infty) \to [0, \infty)$ is said to be *comparison* if φ is increasing and $\varphi^n(t)$ converges to 0 for all $t \geq 0$.

Note that if φ is comparison, then $\varphi(t) < t$ for all $t > 0$ and $\varphi(0) = 0$.

Definition 3.2 (see [1]). Let (X, d) be a metric space, and let $\varphi : [0, \infty) \to [0, \infty)$ be a comparison function. A mapping $T : X \to X$ is called φ-contraction whenever

$$d(Tx, Ty) \leq \varphi(d(x, y)), \tag{3.1}$$

for all $x, y \in X$.

We say that $\varphi : [0, \infty) \to [0, \infty)$ is a subadditive comparison function whenever φ is comparison and $\varphi(t + s) \leq \varphi(t) + \varphi(s)$ for all $t, s \in [0, \infty)$.

There are many subadditive comparison mappings.

For example, if we consider $\lambda < 1$ and $g : [0, \infty) \to [0, \lambda)$ is a decreasing function, then $\varphi(t) = \int_0^t g(x)dx$ is a comparison function. In fact, φ is increasing because $g > 0$. Also, $\varphi(t) < \min\{t, \lambda\}$. Hence, $\varphi^n(t)$ converges to 0 for all $t \geq 0$. Since g is decreasing, we have

$$\begin{aligned}\varphi(u + v) &= \int_0^{u+v} g(x)dx = \int_0^u g(x)dx + \int_u^{u+v} g(x)dx \\ &\leq \int_0^u g(x)dx + \int_0^v g(x)dx = \varphi(u) + \varphi(v),\end{aligned} \tag{3.2}$$

for all $u, v \geq 0$.

In particular, if we consider $g(t) = \lambda e^{-t}$, it follows that $\varphi(t) = \int_0^t g(x)dx$ is a subadditive comparison function.

Theorem 3.3. *Let (X, d) be a complete metric space, and, $\varphi : [0,\infty) \to [0,\infty)$ a subadditive comparison function. If $T : X \to X$ is a φ-contraction, then the Picard iteration is T-stable.*

Proof. By using Theorem 2.7 in [1], we conclude that T has a unique fixed point q.

Let $\{y_n\}$ be a sequence in X with $\lim_{n\to\infty} d(y_{n+1}, Ty_n) = 0$.

First, we show that $\{y_n\}$ is bounded. If $\{y_n\}$ is not bounded, then there exist subsequence $\{z_n\}$ of $\{y_n\}$ for which $d(z_n, q) \geq n$. Since $\lim_{n\to\infty} d(y_{n+1}, Ty_n) = 0$, we can take a subsequence $\{x_n\}$ of $\{z_n\}$ such that $d(x_{n+1}, Tx_n) \leq 1/n^2$. Now, we have

$$\begin{aligned} d(Tx_n, q) &\leq \varphi(d(x_n, q)) \leq \varphi(d(x_n, Tx_{n-1})) + \varphi^2(d(x_{n-1}, q)) \\ &\leq 1 + \frac{1}{2^2} + \cdots + \frac{1}{n^2} + \varphi^n d(x_1, q) \leq \sum_{i=1}^{\infty} \frac{1}{i^2} + d(x_1, q). \end{aligned} \tag{3.3}$$

Thus, $\{Tx_n\}$ is bounded and so is $\{x_n\}$. This is a contradiction. Therefore $\{y_n\}$ is bounded.

Now, choose $M > 0$ such that $d(y_n, q) < M$ for all $n \geq 1$. For each $\varepsilon > 0$ there exist natural numbers p_0 and N such that

$$\varphi^{p_0}(M) < \varepsilon, \qquad d(y_{n+1}, Ty_n) < \varepsilon, \tag{3.4}$$

for all $n \geq N$. But we have

$$\begin{aligned} d(y_{n+2}, q) &\leq d(y_{n+2}, Ty_{n+1}) + d(Ty_{n+1}, q) \leq d(y_{n+2}, Ty_{n+1}) + \varphi(d(y_{n+1}, q)), \\ d(y_{n+3}, q) &\leq d(y_{n+3}, Ty_{n+2}) + d(Ty_{n+2}, q) \leq d(y_{n+3}, Ty_{n+2}) + \varphi(d(y_{n+2}, q)) \\ &\leq d(y_{n+3}, Ty_{n+2}) + \varphi(d(y_{n+2}, Ty_{n+1})) + \varphi^2(d(y_{n+1}, q)). \end{aligned} \tag{3.5}$$

By continuing this process, we obtain

$$\begin{aligned} d(y_{n+p_0+1}, q) &\leq d(y_{n+p_0+1}, Ty_{n+p_0}) + \varphi(d(y_{n+p_0}, Ty_{n+p_0-1})) \\ &\quad + \cdots + \varphi^{p_0-1}(d(y_{n+1}, Ty_n)) + \varphi^{p_0}(d(y_n, q)) \\ &< d(y_{n+p_0+1}, Ty_{n+p_0}) + \varphi(d(y_{n+p_0}, Ty_{n+p_0-1})) \\ &\quad + \cdots + \varphi^{p_0-1}(d(y_{n+1}, Ty_n)) + \varepsilon. \end{aligned} \tag{3.6}$$

Hence, $\lim_{n\to\infty} \sup d(y_n, q) \leq \varepsilon$. Since $\varepsilon > 0$ was arbitrary, $\lim_{n\to\infty} d(y_n, q) = 0$. □

Definition 3.4 (see [1]). A function $\varphi : \mathbb{R}_+^5 \to \mathbb{R}_+$ is called (5-dimensional) *comparison function* whenever $\varphi(u) \leq \varphi(v)$, for each $u, v \in \mathbb{R}_+^5$ with $u \leq v$, and the function

$$\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+, \qquad \psi(t) = \varphi(t, t, t, t, t) \tag{3.7}$$

satisfies $\lim_{n\to\infty} \psi^n(t) = 0$, for all $t > 0$.

Note that ψ is a comparison function, while the following are 5-dimensional comparison functions:

(i) $\varphi(t) = a\max\{t_1, t_2, t_3, t_4, t_5\}$ for each $t = (t_1, t_2, t_3, t_4, t_5)$, where $a \in [0,1)$,

(ii) $\varphi(t) = a\max\{t_1, t_2, t_3, t_4, (1/2)(t_4 + t_5)\}$, $a \in [0,1)$,

(iii) $\varphi(t) = at_1 + b(t_2 + t_3)$, $a, b \in \mathbb{R}_+$ with $a + 2b < 1$,

(iv) $\varphi(t) = a\max\{t_2, t_3\}$, $a \in (0,1)$.

In the previous four examples, function ψ given by (3.7) is a subadditive comparison function.

Definition 3.5. Let (X, d) be a metric space, and, $\varphi : \mathbb{R}_+^5 \to \mathbb{R}_+$, a 5-dimensional comparison function. A mapping $T : X \to X$ is called *generalized* φ-contraction whenever

$$d(Tx, Ty) \le \varphi(d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)), \tag{3.8}$$

for all x, y in X.

In the sequel, we will use functions φ such that ψ is subadditive.

Lemma 3.6. *Let (X, d) be a metric space, and let $T : X \to X$ be a generalized φ-contraction map. Suppose $\{y_n\}$ is a bounded sequence in X such that $\lim_{n\to\infty} d(y_{n+1}, Ty_n) = 0$. Let p_n be the diameter of the set $A_n = \{y_i\}_{i\ge n} \bigcup \{Ty_i\}_{i\ge n}$. Then, $\lim_{n\to\infty} p_n = 0$. In particular, $\lim_{n\to\infty} d(y_n, Ty_n) = 0$.*

Proof. By using definition of T, for each n and $i, j \ge n$ we have

$$d(Ty_i, Ty_j) \le \varphi(d(y_i, y_j), d(y_i, Ty_i), d(y_j, Ty_j), d(y_i, Ty_j), d(y_j, Ty_i)) \le \psi(p_n). \tag{3.9}$$

Let $\epsilon_i = d(y_{i+1}, Ty_i)$. Then

$$\begin{aligned} d(y_i, y_j) &\le d(y_i, Ty_{i-1}) + d(Ty_{i-1}, Ty_{j-1}) + d(Ty_{j-1}, y_j) \\ &\le \epsilon_{i-1} + \psi(p_{n-1}) + \epsilon_{j-1}, \\ d(y_i, Ty_j) &\le d(y_i, Ty_{i-1}) + d(Ty_{i-1}, Ty_j) \le \epsilon_{i-1} + \psi(p_{n-1}). \end{aligned} \tag{3.10}$$

Let $a_n = \sup_{i\ge n} 2\epsilon_i$. It is easy to see that $\lim_{n\to\infty} a_n = 0$, and we have

$$p_n \le a_n + \psi(p_{n-1}). \tag{3.11}$$

By using (3.11), we observe that

$$\psi(p_n) \le \psi(a_n) + \psi^2(p_{n-1}). \tag{3.12}$$

Since $\{y_n\}$ is bounded, $\{Ty_n\}$ so is. Choose $M > 0$ such that $p_n \leq M$ for all $n \geq 1$. Since ψ is comparison, for each $\varepsilon > 0$ there exists a natural number k_0 such that $\psi^{k_0}(M) < \varepsilon/2$. But, for each $n \geq 1$ we obtain

$$\psi(p_{n+1}) \leq \psi(a_{n+1}) + \psi^2(p_n) \leq \psi(a_{n+1}) + \psi^2(a_n) + \psi^3(p_{n-1}). \tag{3.13}$$

Hence,

$$\psi(p_{n+2}) \leq \psi(a_{n+2}) + \psi^2(a_{n+1}) + \psi^3(a_n) + \psi^4(p_{n-1}). \tag{3.14}$$

Since $\psi(t) < t$, for all $t > 0$, and ψ is increasing, then $\sum_{i=1}^{k+1} \psi^i(a_{n-i+3}) \to 0$, for all natural numbers k. Thus by continuing these relations, for each $k \geq k_0$ we have

$$\psi(p_{n+k}) \leq \sum_{i=1}^{k+1} \psi^i(a_{n-i+3}) + \psi^{k+2}(p_{n-1}) \leq \sum_{i=1}^{k+1} \psi^i(a_{n-i+3}) + \varepsilon. \tag{3.15}$$

It implies that $\lim_{n\to\infty} \sup \psi(p_n) \leq \varepsilon$. Since $\varepsilon > 0$ was arbitrary, $\lim_{n\to\infty} \sup \psi(p_n) = 0$. Therefore by using (3.11), $\lim_{n\to\infty} p_n = 0$. □

Theorem 3.7. *Let (X, d) be a metric space, let $T : X \to X$ be a generalized φ-contraction map, and let $F_T = \{q\}$. Then the Picard iteration is boundedly T-stable.*

Proof. Let $\{y_n\}$ be a bounded sequence in X such that $\lim_{n\to\infty} d(y_{n+1}, Ty_n) = 0$. Choose $M > 0$ such that $d(y_n, q) < M$ for all $n \geq 1$. Observe that

$$\begin{aligned} d(Ty_n, q) &\leq \varphi(d(y_n, q), d(y_n, Ty_n), 0, d(y_n, q), d(q, Ty_n)) \\ &\leq \psi(\max\{d(y_n, q), d(y_n, Ty_n), d(q, Ty_n)\}). \end{aligned} \tag{3.16}$$

If $\max\{d(y_n, q), d(y_n, Ty_n), d(q, Ty_n)\} = d(Ty_n, q)$, then $d(Ty_n, q) = 0$.

Without loss of generality, suppose that the last equality does not hold. Therefore, we get

$$\begin{aligned} d(Ty_n, q) &\leq \psi(\max\{d(y_n, q), d(y_n, Ty_n)\}) \\ &\leq \psi(d(y_n, q) + d(y_n, Ty_n)) \leq \psi(d(y_n, q)) + \psi(d(y_n, Ty_n)). \end{aligned} \tag{3.17}$$

For any given $\varepsilon > 0$, choose $p_0 \in \mathbb{N}$ such that

$$\psi^{p_0}(M) < \varepsilon. \tag{3.18}$$

Now, for each $n \geq 1$ we have

$$\begin{aligned} d(Ty_{n+1}, q) &\leq \psi(d(y_{n+1}, Ty_{n+1})) + \psi(d(y_{n+1}, q)) \\ &\leq \psi(d(y_{n+1}, Ty_{n+1})) + \psi(d(y_{n+1}, Ty_n)) + d(Ty_n, q) \\ &\leq \psi(d(y_{n+1}, Ty_{n+1})) + \psi(d(y_{n+1}, Ty_n)) + \psi^2(d(y_n, Ty_n)) + \psi^2(d(y_n, q)). \end{aligned} \tag{3.19}$$

Similarly

$$\begin{aligned} d(Ty_{n+2}, q) &\leq \psi(d(y_{n+2}, Ty_{n+2})) + \psi(d(y_{n+2}, q)) \\ &\leq \psi(d(y_{n+2}, Ty_{n+2})) + \psi(d(y_{n+2}, Ty_{n+1})) + d(Ty_{n+1}, q) \\ &\leq \psi(d(y_{n+2}, Ty_{n+2})) + \psi(d(y_{n+2}, Ty_{n+1})) + \psi^2(d(y_{n+1}, Ty_{n+1})) \\ &\quad + \psi^2(d(y_{n+1}, Ty_n)) + \psi^3(d(y_n, Ty_n)) + \psi^3(d(y_n, q)). \end{aligned} \tag{3.20}$$

Now for each $p \geq p_0$ we obtain

$$\begin{aligned} d(Ty_{n+p}, q) &\leq \sum_{i=1}^{p} \psi^{p-i+1}(d(y_{n+i}, Ty_{n+i})) + \sum_{i=1}^{p} \psi^{p-i+1}(d(y_{n+i}, Ty_{n+i-1})) \\ &\quad + \psi^{p+1}(d(y_n, q)) + \psi^{p+1}(d(y_n, Ty_n)) \\ &\leq \sum_{i=1}^{p} \psi^{p-i+1}(d(y_{n+i}, Ty_{n+i})) + \sum_{i=1}^{p} \psi^{p-i+1}(d(y_{n+i}, Ty_{n+i-1})) + \psi^{p+1}(d(y_n, Ty_n)) + \varepsilon. \end{aligned} \tag{3.21}$$

If $n \to \infty$, then by a similar method in Lemma 3.6, $\lim_{n\to\infty} \sup d(Ty_n, q) \leq \varepsilon$. Since $\varepsilon > 0$ is arbitrary, $\lim_{n\to\infty} d(Ty_n, q) = 0$.

Finally note that the inequality

$$d(y_n, q) \leq d(y_n, Ty_n) + d(Ty_n, q) \tag{3.22}$$

implies that $\lim_{n\to\infty} d(y_n, q) = 0$.

The proof is complete. □

Remark 3.8. Let (X, d) be a complete metric space, and let $T : X \to X$ be a mapping for which there exists $h \in [0, 1)$ satisfying

$$d(Tx, Ty) \leq h \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}, \tag{3.23}$$

for all x, y in X. If we define $\varphi(t) = h \max\{t_1, t_2, t_3, t_4, t_5\}$, then by using Theorem 3.7, the Picard iteration is boundedly T-stable. Consider that some contractive conditions are special cases of (3.8), and, for each of those, the Picard iteration is boundedly T-stable. For example, Theorem 1 in [6] and Theorems 1 and 2 in [3] are special cases of Theorem 3.7.

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