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Research Article

Solutions for p-Laplacian Dynamic Delay Differential Equations on Time Scales

Hua Su,[1] Lishan Liu,[2] and Xinjun Wang[3]

[1] *School of Mathematics and Quantitative Economics, Shandong University of Finance and Economics, Shandong Jinan, 250014, China*
[2] *School of Mathematical Sciences, Qufu Normal University, Qufu 273165, China*
[3] *School of Economics, Shandong University, Jinan 250014, China*

Correspondence should be addressed to Hua Su, jnsuhua@163.com

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Let $\mathbf{T}$ be a time scale. We study the existence of positive solutions for the nonlinear four-point singular boundary value problem with p-Laplacian dynamic delay differential equations on time scales, subject to some boundary conditions. By using the fixed-point index theory, the existence of positive solution and many positive solutions for nonlinear four-point singular boundary value problem with p-Laplacian operator is obtained.

1. Introduction

The study of dynamic equations on time scales goes back to its founder Hilger [1] and is a new area of still fairly theoretical exploration in mathematics. Boundary value problems for delay differential equations arise in a variety of areas of applied mathematics, physics, and variational problems of control theory (see [2, 3]). In recent years, many authors have begun to pay attention to the study of boundary-value problems or with p-Laplacian equations or with p-Laplacian dynamic equations on time scales (see [4–14] and the references therein).

In [7], Sun and Li considered the existence of positive solution of the following dynamic equations on time scales:

$$u^{\Delta\nabla}(t) + a(t)f(t, u(t)) = 0, \quad t \in (0, T), \tag{1.1}$$

$$\beta u(0) - \gamma u^{\Delta}(0) = 0, \qquad \alpha u(\eta) = u(T), \tag{1.2}$$

where $\beta, \gamma \geq 0$, $\beta + \gamma > 0$, $\eta \in (0, \rho(T))$, $0 < \alpha < T/\eta$. They obtained the existence of single and multiple positive solutions of the problem (1.1) and (1.2) by using fixed-point theorem and Leggett-Williams fixed-point theorem (see [15]), respectively.

In [4], Anderson discussed the following dynamic equation on time scales:

$$u^{\Delta\nabla}(t) + a(t)f(u(t)) = 0, \quad t \in (0, T),$$
$$u(0) = 0, \qquad \alpha u(\eta) = u(T). \tag{1.3}$$

He obtained some results for the existence of one positive solution of the problem (1.3) based on the limits $f_0 = \lim_{u\to 0^+} f(u)/u$ and $f_\infty = \lim_{u\to\infty} f(u)/u$.

In [5], Kaufmann studied the problem (1.3) and obtained the existence results of at least two positives solutions.

In [14], Wang et al. discussed the following dynamic equation on time scales by using Avery-Peterson fixed theorem (see [14]):

$$(\phi_p(u'))' + q(t)f(t, u(t), u(t-1), u'(t)) = 0, \quad t \in (0,1), \tag{1.4}$$

$$u(t) = \xi(t), \quad -1 \le t \le 0, \qquad u(1) = 0, \tag{1.5}$$

$$u(t) = \xi(t), \quad -1 \le t \le 0, \qquad u'(1) = 0. \tag{1.5$'$}$$

They obtained some results for the existence three positive solutions of the problem (1.4), (1.5) and (1.4), and (1.5′), respectively.

In [15], Lee and Sim discussed the following equation:

$$(\phi_p(u'))' + \lambda h(t)f(u(t)) = 0, \quad \text{a. e. } t \in (0,1),$$
$$u(0) = u(1) = 0. \tag{1.6}$$

By applying the global bifurcation theorem and figuring the shape of unbounded subcontinua of solutions, they obtain many different types of global existence results of positive solutions.

However, there are not many concerning the p-Laplacian problems on time scales. Especially, for the singular multipoint boundary value problems for p-Laplacian dynamic delay differential equations on time scales, with the author's acknowledge, no one has studied the existence of positive solutions in this case.

Recently, in [16], we have studied the existence of positive solutions for the following nonlinear two-point singular boundary value problem with p-Laplacian operator:

$$(\phi_p(u'))' + a(t)f(u(t)) = 0, \quad 0 < t < 1,$$
$$\alpha\phi_p(u(0)) - \beta\phi_p(u'(0)) = 0, \qquad \gamma\phi_p(u(1)) + \delta\phi_p(u'(1)) = 0. \tag{1.7}$$

By using the fixed-point theorem of cone expansion and compression of norm type, the existence of positive solution and infinitely many positive solutions for nonlinear singular boundary value problem (1.7) with p-Laplacian operator is obtained.

Now, motivated by the results mentioned above, in this paper, we study the existence of positive solutions for the following nonlinear four-point singular boundary value problem with higher-order p-Laplacian dynamic delay differential equations operator on time scales (SBVP):

$$\left(\phi_p\left(u^{\Delta}(t)\right)\right)^{\nabla} + g(t)f(u(t-\tau), u(t)) = 0, \quad 0 < t < T, \ \tau > 0, \tag{1.8}$$

$$\begin{gathered} u(t) = \zeta(t), \quad -\tau \le t \le 0, \\ \alpha\phi_p(u(0)) - \beta\phi_p\left(u^{\Delta}(\xi)\right) = 0, \\ \gamma\phi_p(u(T)) + \delta\phi_p\left(u^{\Delta}(\eta)\right) = 0, \end{gathered} \tag{1.9}$$

or

$$\begin{gathered} u(t) = \zeta(t), \quad -\tau \le t \le 0, \\ u(0) - B_0\left(u^{\Delta}(\xi)\right) = 0, \\ u(T) + B_1\left(u^{\Delta}(\eta)\ \right) = 0, \end{gathered} \tag{1.10}$$

where $\phi_p(s)$ is p-Laplacian operator, that is, $\phi_p(s) = |s|^{p-2}s,\ p > 1$, $\phi_q = \phi_p^{-1}$, $1/p + 1/q = 1$. ξ, $\eta \in (0,T)$, $\tau \in [0,T]$ is prescribed and $\xi < \eta$, $g : (0,T) \to [0,\infty)$, $\alpha > 0, \beta \ge 0, \gamma > 0, \delta \ge 0$ and $B_0,\ B_1$ are both nondecreasing continuous odd functions defined on $(-\infty, +\infty)$.

In this paper, by constructing one integral equation which is equivalent to the problem (1.8), (1.9) and (1.8), and (1.10), we research the existence of positive solutions for nonlinear singular boundary value problem (1.8), (1.9) and (1.8), and (1.10) when g and f satisfy some suitable conditions.

Our main tool of this paper is the following fixed point index theory.

Theorem 1.1 (see [17, 18]). *Suppose that E is a real Banach space, $K \subset E$ is a cone, let $\Omega_r = \{u \in K : \|u\| \le r\}$. Let operator $T : \Omega_r \to K$ be completely continuous and satisfy $Tx \ne x,\ \forall x \in \partial\Omega_r$. Then*

(i) *if $\|Tx\| \le \|x\|,\ \forall x \in \partial\Omega_r$, then $i(T, \Omega_r, K) = 1$;*

(ii) *if $\|Tx\| \ge \|x\|,\ \forall x \in \partial\Omega_r$, then $i(T, \Omega_r, K) = 0$.*

This paper is organized as follows. In Section 2, we present some preliminaries and lemmas that will be used to prove our main results. In Section 3, we discuss the existence of single solution of the systems (1.8) and (1.9). In Section 4, we study the existence of at least two solutions of the systems (1.8) and (1.9). In Section 5, we discuss the existence of single and many solutions of the systems (1.8) and (1.10). In Section 6, we give two examples as the application.

2. Preliminaries and Lemmas

For convenience, we can found some basic definitions in [1, 19, 20].

In the rest of this paper, $\mathbf{T}$ is closed subset of R with $0 \in \mathbf{T}_k, T \in \mathbf{T}^k$. And let $B = \{u \in C_{ld}[-\tau, T]\}$, then B is a Banach space with the norm $\|u\| = \max_{t\in[-\tau,T]}|u(t)|$. And let

$$K = \{u \in B : u(t) \geq 0,\ u(t) \text{ is concave function}, t \in [0, T]\}. \tag{2.1}$$

Obviously, K is a cone in B. Set $K_r = \{u \in K : \|u\| \leq r\}$.

Definition 2.1. $u(t)$ is called a solution of SBVP (1.8) and (1.9) if it satisfies the following:

(1) $u \in C[-\tau, 0] \cap C_{ld}(0, T)$,

(2) $u(t) > 0$ for all $t \in (0, T)$ and satisfies conditions (1.9),

(3) $(\phi_p(u^{\Delta}(t)))^{\nabla} = -g(t)f(u(t-\tau), u(t))$ holds for $t \in (0, T)$.

In the rest of the paper, we also make the following assumptions:

(H_1) $f \in C_{ld}([0, +\infty)^2, [0, +\infty))$,

(H_2) $g(t) \in C_{ld}((0, T), [0, +\infty))$ and there exists $t_0 \in (0, T)$, such that

$$g(t_0) > 0, \qquad 0 < \int_0^T g(s)\ \nabla s < +\infty, \tag{2.2}$$

(H_3) $\zeta(t) \in C([-\tau, 0], \zeta(t) > 0$ on $[-\tau, 0)$ and $\zeta(0) = 0$,

(H_4) $B_0,\ B_1$ are both increasing, continuous, odd functions defined on $(-\infty, +\infty)$, and at least one of them satisfies the condition that there exists one $b > 0$ such that

$$0 < B_i(v) \leq bv, \qquad \forall v \geq 0,\ i = 0 \text{ or } 1. \tag{2.3}$$

It is easy to check that condition (H_2) implies that

$$0 < \int_0^T \phi_q\left(\int_0^s g(s_1)\nabla s_1\right)\Delta s < +\infty. \tag{2.4}$$

We can easily get the following Lemmas.

Lemma 2.2. *Suppose that condition (H_2) holds. Then there exists a constant $\theta \in (0, 1/2)$ that satisfies*

$$0 < \int_\theta^{T-\theta} g(t)\nabla t < \infty. \tag{2.5}$$

Furthermore, the function

$$A(t) = \int_\theta^t \phi_q\left(\int_s^t g(s_1)\nabla s_1\right)\Delta s + \int_t^{T-\theta} \phi_q\left(\int_t^s g(s_1)\nabla s_1\right)\nabla s, \quad t \in [\theta, T-\theta] \tag{2.6}$$

is positive continuous functions on $[\theta, T-\theta]$; therefore, $A(t)$ has minimum on $[\theta, T-\theta]$. Hence, we suppose that there exists $L > 0$ such that $A(t) \geq L,\ t \in [\theta, T-\theta]$.

Proof. At first, it is easily seen that $A(t)$ is continuous on $[\theta, T-\theta]$. Nest, let

$$A_1(t) = \int_\theta^t \phi_q\left(\int_s^t g(s_1)\nabla s_1\right)\Delta s, \qquad A_2(t) = \int_t^{T-\theta} \phi_q\left(\int_t^s g(s_1)\nabla s_1\right)\Delta s. \tag{2.7}$$

Then, from condition (H_2), we have the function $A_1(t)$ is strictly monotone nondecreasing on $[\theta, T-\theta]$ and $A_1(\theta) = 0$, the function $A_2(t)$ is strictly monotone nonincreasing on $[\theta, T-\theta]$ and $A_2(T-\theta) = 0$, which implies $L = \min_{t\in[\theta,T-\theta]} A(t) > 0$. The proof is complete. □

Lemma 2.3 (see [16]). *Let $u \in K$ and θ of Lemma 2.2, then*

$$u(t) \geq \theta\|u\|, \quad t \in [\theta, T-\theta]. \tag{2.8}$$

Lemma 2.4. *Suppose that conditions $(H_1), (H_2), (H_3)$, and (H_4) hold, $u(t) \in B \cap C_{ld}(0,1)$ is a solution of the following boundary value problems:*

$$\left(\phi_p\left(u^\Delta(t)\right)\right)^\nabla + g(t) f(u(t-\tau) + h(t-\tau), u(t)) = 0, \quad 0 < t < T, \tag{2.9}$$

$$\begin{gathered} u(t) = 0, \quad -\tau \leq t \leq 0, \\ \alpha\phi_p(u(0)) - \beta\phi_p\left(u^\Delta(\xi)\right) = 0, \\ \gamma\phi_p(u(T)) + \delta\phi_p\left(u^\Delta(\eta)\right) = 0, \end{gathered} \tag{2.10}$$

or

$$\begin{gathered} u(t) = 0, \quad -\tau \leq t \leq 0, \\ u(0) - B_0\left(u^\Delta(\xi)\right) = 0, \\ u(T) + B_1\left(u^\Delta(\eta)\right) = 0, \end{gathered} \tag{2.10$'$}$$

where

$$h(t) = \begin{cases} \zeta(t), & -\tau \leq t \leq 0, \\ 0, & 0 \leq t \leq T. \end{cases} \tag{2.11}$$

Then, $\overline{u}(t) = u(t) + h(t)$, $-\tau \leq t \leq T$ is a positive solution to the SBVP (1.8) *and* (1.9) *or* (1.8) *and* (1.10).

Proof. It is easy to check that $\overline{u}(t)$ satisfies (1.8) and (1.9) or (1.8) and (1.10). □

So in the rest section of this paper, we focus on SBVP (2.9), (2.10), and (2.9), (2.10$'$).

Lemma 2.5. *Suppose that conditions $(H_1), (H_2), (H_3)$, or $(H_1), (H_2), (H_3), (H_4)$, hold, $u(t) \in B \cap C_{ld}(0,1)$ is a solution of boundary value problems* (2.9), (2.10) *or* (2.9), (2.10$'$), *respectively, if and* *only if $u(t) \in B$ is a solution of the following integral equation, respectively:*

$$u(t) = \begin{cases} \zeta(t), & -\tau \le t \le 0, \\ \displaystyle\int_0^t w(s)\Delta s, & 0 \le t \le T, \end{cases}$$
$$u(t) = \begin{cases} \zeta(t), & -\tau \le t \le 0, \\ \displaystyle\int_0^t \overline{w}(s)\Delta s, & 0 \le t \le T, \end{cases} \tag{2.12}$$

where

$$w(t) = \begin{cases} \phi_q\left(\dfrac{\beta}{\alpha}\displaystyle\int_\xi^\sigma g(s) f(u(s-\tau)+h(s-\tau), u(s))\nabla s\right) \\ \quad +\displaystyle\int_0^t \phi_q\left(\int_s^\sigma g(r) f(u(r-\tau)+h(r-\tau), u(r))\nabla r\right)\Delta s, & 0 \le t \le \sigma, \\ \phi_q\left(\dfrac{\delta}{\gamma}\displaystyle\int_\sigma^\eta g(s) f(u(s-\tau)+h(s-\tau), u(s))\nabla s\right) \\ \quad +\displaystyle\int_t^T \phi_q\left(\int_\sigma^s g(r) f(u(r-\tau)+h(r-\tau), u(r))\nabla r\right)\Delta s, & \sigma \le t \le T, \end{cases} \tag{2.13}$$

$$\overline{w}(t) = \begin{cases} B_0 \circ \phi_q\left(\displaystyle\int_\xi^\varrho g(s) f(u(s-\tau)+h(s-\tau), u(s))\nabla s\right) \\ \quad +\displaystyle\int_0^t \phi_q\left(\int_s^\varrho g(r) f(u(r-\tau)+h(r-\tau), u(r))\nabla r\right)\Delta s, & 0 \le t \le \varrho, \\ B_1 \circ \phi_q\left(\displaystyle\int_\varrho^\eta g(s) f(u(s-\tau)+h(s-\tau), u(s))\nabla s\right) \\ \quad +\displaystyle\int_t^T \phi_q\left(\int_\varrho^s g(r) f(u(r-\tau)+h(r-\tau), u(r))\nabla r\right)\Delta s, & \varrho \le t \le T. \end{cases} \tag{2.13'}$$

Here σ, ϱ is unique solution of the equation, respectively,

$$g_1(t) = g_2(t), \qquad \overline{g}_1(t) = \overline{g}_2(t), \tag{2.14}$$

where

$$\begin{aligned} g_1(t) &= \phi_q\left(\frac{\beta}{\alpha}\int_\xi^\sigma g(s) f(u(s-\tau)+h(s-\tau), u(s))\nabla s\right) \\ &\quad + \int_0^t \phi_q\left(\int_s^\sigma g(r) f(u(r-\tau)+h(r-\tau), u(r))\nabla r\right)\Delta s, \\ g_2(t) &= \phi_q\left(\frac{\delta}{\gamma}\int_\sigma^\eta g(s) f(u(s-\tau)+h(s-\tau), u(s))\nabla s\right) \\ &\quad + \int_t^T \phi_q\left(\int_\sigma^s g(r) f(u(r-\tau)+h(r-\tau), u(r))\nabla r\right)\Delta s, \end{aligned}$$

$$\begin{aligned}
\overline{g}_1(t) &= B_0 \circ \phi_q\left(\int_{\xi}^{\varrho} g(s) f(u(s-\tau)+h(s-\tau),u(s))\nabla s\right) \\
&\quad + \int_0^t \phi_q\left(\int_s^{\varrho} g(r) f(u(r-\tau)+h(r-\tau),u(r))\nabla r\right)\Delta s, \\
\overline{g}_2(t) &= B_1 \circ \phi_q\left(\int_{\varrho}^{\eta} g(s) f(u(s-\tau)+h(s-\tau),u(s))\nabla s\right) \\
&\quad + \int_t^T \phi_q\left(\int_{\varrho}^{s} g(r) f(u(r-\tau)+h(r-\tau),u(r))\nabla r\right)\Delta s.
\end{aligned} \tag{2.15}$$

Equation $g_1(t) = g_2(t)$, $\overline{g}_1(t) = \overline{g}_2(t)$ *has unique solution in* $(0,T)$. *Because* $g_1(t)$, $\overline{g}_1(t)$ *is strictly monotone increasing on* $[0,T)$, *and* $g_1(0) = 0$, $\overline{g}_1(0) = 0$, $g_2(t)$, $\overline{g}_2(t)$ *is strictly monotone decreasing on* $(0,T]$, *and* $g_2(T) = 0$, $\overline{g}_2(T) = 0$.

Proof. We only proof the first section of the results.

Necessity. Obviously, for $t \in (-\tau, 0)$, we have $u(t) = \zeta(t)$.

If $t \in (0,T)$, by the equation of the boundary condition and we have $u^{\Delta}(\xi) \geq 0$, $u^{\Delta}(\eta) \leq 0$, then there exist is a constant $\sigma \in [\xi,\eta] \subset (0,T)$ such that $u^{\Delta}(\sigma) = 0$.

Firstly, by integrating the equation of the problems (2.9) on (σ, t), we have

$$\phi_p\left(u^{\Delta}(t)\right) = \phi_p\left(u^{\Delta}(\sigma)\right) - \int_{\sigma}^{t} g(s) f(u(s-\tau)+h(s-\tau),u(s))\nabla s, \tag{2.16}$$

then

$$u^{\Delta}(t) = -\phi_q\left(\int_{\sigma}^{t} g(s) f(u(s-\tau)+h(s-\tau),u(s))\nabla s\right), \tag{2.17}$$

thus

$$u(t) = u(\sigma) - \int_{\sigma}^{t} \phi_q\left(\int_{\sigma}^{s} g(r) f(u(r-\tau)+h(r-\tau),u(r))\nabla r\right)\Delta s. \tag{2.18}$$

By $u^{\Delta}(\sigma) = 0$ and condition (2.16), $t = \eta$ on (2.16), we have

$$\phi_p\left(u^{\Delta}(\eta)\right) = -\int_{\sigma}^{\eta} g(s) f(u(s-\tau)+h(s-\tau),u(s))\nabla s. \tag{2.19}$$

By the equation of the boundary condition (2.10), we have

$$\phi_p(u(T)) = -\frac{\delta}{\gamma}\phi_p\left(u^{\Delta}(\eta)\right), \tag{2.20}$$

then

$$u(T) = \phi_q\left(\frac{\delta}{\gamma}\int_\sigma^\eta g(s)f(u(s-\tau)+h(s-\tau),u(s))\nabla s\right). \tag{2.21}$$

Then, by (2.18) and let $t = T$ on (2.18), we have

$$\begin{aligned} u(\sigma) = {}& \phi_q\left(\frac{\delta}{\gamma}\int_\sigma^\eta g(s)f(u(s-\tau)+h(s-\tau),u(s))\nabla s\right) \\ &+ \int_\sigma^T \phi_q\left(\int_\sigma^s g(r)f(u(r-\tau)+h(r-\tau),u(r))\nabla r\right)\Delta s. \end{aligned} \tag{2.22}$$

Then

$$\begin{aligned} u(t) = {}& \phi_q\left(\frac{\delta}{\gamma}\int_\sigma^\eta g(s)f(u(s-\tau)+h(s-\tau),u(s))\nabla s\right) \\ &+ \int_t^T \phi_q\left(\int_\sigma^s g(r)f(u(r-\tau)+h(r-\tau),u(r))\nabla r\right)\Delta s. \end{aligned} \tag{2.23}$$

Similarly, for $t \in (0,\sigma)$, by integrating the equation of problems (2.9) on $(0,\sigma)$, we have

$$\begin{aligned} u(t) = {}& \phi_q\left(\frac{\beta}{\alpha}\int_\xi^\sigma g(s)f(u(s-\tau)+h(s-\tau),u(s))\nabla s\right)\Delta s \\ &+ \int_0^t \phi_q\left(\int_s^\sigma g(r)f(u(r-\tau)+h(r-\tau),u(r))\nabla r\right)\Delta s. \end{aligned} \tag{2.24}$$

Therefore, for any $t \in [0,T]$, $u(t)$ can be expressed as equation

$$u(t) = \begin{cases} \zeta(t), & -\tau \le t \le 0, \\ \displaystyle\int_0^t w(s)\Delta s, & 0 \le t \le T, \end{cases} \tag{2.25}$$

where $w(t)$ is expressed as (2.13).

Sufficiency. Suppose that $u(t) = \int_0^t w(s)\Delta s_{n-2}\Delta s,\ 0 \le t \le T$. Then by (2.13), we have

$$u^\Delta(t) = \begin{cases} \phi_q\left(\displaystyle\int_t^\sigma g(s)f(u(s-\tau)+h(s-\tau),u(s))\nabla s\right) \ge 0, & 0 \le t \le \sigma, \\ -\phi_q\left(\displaystyle\int_\sigma^t g(s)f(u(s-\tau)+h(s-\tau),u(s))\nabla s\right) \le 0, & \sigma \le t \le T, \end{cases} \tag{2.26}$$

So, $(\phi_p(u^\Delta)^\nabla + g(t)f(u(t-\tau)+h(t-\tau),u(t)) = 0,\ 0 < t < T$. These imply that (2.9) holds.

Furthermore, by letting $t = 0$ and $t = T$ on (2.13) and (2.26), we can obtain the boundary value equations of (2.10). The proof is complete. □

Now, we define an operetor equation T given by

$$(Tu)(t) = \begin{cases} \zeta(t), & -\tau \leq t \leq 0, \\ \displaystyle\int_0^t w(s)\Delta s, & 0 \leq t \leq T, \end{cases}$$
$$\left(\overline{T}u\right)(t) = \begin{cases} \zeta(t), & -\tau \leq t \leq 0, \\ \displaystyle\int_0^t \overline{w}(s)\Delta s, & 0 \leq t \leq T, \end{cases} \tag{2.27}$$

where $w(t)$, $\overline{w}(t)$ is given by (2.13) and (2.13′).

From the definition of T, $\overline{T}$ and above discussion, we deduce that for each $u \in K$, Tu, $\overline{T}u \in K$. Moreover, we have the following Lemma.

Lemma 2.6. *T, $\overline{T} : K \to K$ is completely continuous.*

Proof. We only proof the completely continuous of T.

Because

$$(Tu)^{\Delta}(t) = w^{\Delta}(t) = \begin{cases} \phi_q\left(\displaystyle\int_t^{\sigma} g(s) f(u(s-\tau) + h(s-\tau), u(s))\nabla s\right) \geq 0, & 0 \leq t \leq \sigma, \\ -\phi_q\left(\displaystyle\int_{\sigma}^{t} g(s) f(u(s-\tau) + h(s-\tau), u(s))\nabla s\right) \leq 0, & \sigma \leq t \leq T, \end{cases} \tag{2.28}$$

is continuous, decreasing on $[0, T]$, and satisfies that $(Tu)^{\Delta}(\sigma) = 0$, then, $Tu \in K$ for each $u \in K$ and $(Tu)(\sigma) = \max_{t\in[0,T]}(Tu)(t)$. This shows that $TK \subset K$. Furthermore, it is easy to check by Arzela-ascoli Theorem that $T : K \to K$ is completely continuous. □

Lemma 2.7. *Suppose that conditions $(H_1), (H_2)$, and (H_3) hold, the solution $u(t) \in K$ of problem* (2.9) *and* (2.10) *satisfy*

$$\max_{0\leq t\leq T} |u(t-\tau) + h(t-\tau)| \leq \max_{-\tau\leq t\leq 0} |\zeta(t)|. \tag{2.29}$$

Proof. Firstly, we can have

$$\begin{aligned} \max_{0\leq t\leq T} |u(t-\tau) + h(t-\tau)| &\leq \max_{0\leq t\leq T} |u(t-\tau)| + \max_{0\leq t\leq T} |h(t-\tau)| \\ &= \max_{-\tau\leq t\leq T-\tau} |u(t)| + \max_{-\tau\leq t\leq T-\tau} |h(t)| \\ &= \max_{-\tau\leq t\leq 0} |\zeta(t)|. \end{aligned} \tag{2.30}$$

The proof is complete. □

For convenience, we set

$$H = \max_{-\tau \le t \le 0} |\zeta(t)|, \qquad \theta^* = \frac{2}{L},$$
$$\theta_* = \frac{1}{(T + \phi_q(\beta/\alpha))\phi_q\left(\int_0^T g(r)\nabla r\right)}, \qquad \theta_{**} = \frac{1}{(b+1)\phi_q\left(\int_0^T g(r)\nabla r\right)}, \tag{2.31}$$

where L is the constant from Lemma 2.2. By Lemma 2.5, we can also set

$$f^0 = \lim_{u_2 \to 0} \max_{0 \le u_1 \le H} \frac{f(u_1, u_2)}{u_2^{p-1}}, \qquad f^\infty = \lim_{u_2 \to \infty} \max_{0 \le u_1 \le H} \frac{f(u_1, u_2)}{u_2^{p-1}},$$
$$f_0 = \lim_{u_2 \to 0} \min_{0 \le u_1 \le H} \frac{f(u_1, u_2)}{u_2^{p-1}} \qquad f_\infty = \lim_{u_2 \to \infty} \min_{0 \le u_1 \le H} \frac{f(u_1, u_2)}{u_2^{p-1}}. \tag{2.32}$$

3. The Existence of Single Positive Solution to (1.8) and (1.9)

In this section, we present our main results.

Theorem 3.1. *Suppose that condition* (H_1), (H_2), *and* (H_3) *hold. Assume that* f *also satisfies*

(A_1): $f(u_1, u_2) \ge (mr)^{p-1}$, *for* $\theta r \le u_2 \le r$, $0 \le u_1 \le H$,

(A_2): $f(u_1, u_2) \le (MR)^{p-1}$, *for* $0 \le u_2 \le R$, $0 \le u_1 \le H$,

where $m \in (\theta^*, \infty)$, $M \in (0, \theta_*)$.

Then, the SBVP (2.9), (2.10) *has a solution* u *such that* $\|u\|$ *lies between* r *and* R. *Furthermore by Lemma 2.4,* $\overline{u}(t) = u(t) + h(t)$, $-\tau \le t \le T$ *is a positive solution to the SBVP* (1.8) *and* (1.9).

Proof of Theorem 3.1. Without loss of generality, we suppose that $r < R$. For any $u \in K$, by Lemma 2.3, we have

$$u(t) \ge \theta \|u\|, \quad t \in [\theta, T - \theta]. \tag{3.1}$$

We define two open subset Ω_1 and Ω_2 of E:

$$\Omega_1 = \{u \in K : \|u\| < r\}, \qquad \Omega_2 = \{u \in K : \|u\| < R\}. \tag{3.2}$$

For any $u \in \partial\Omega_1$, by (3.1), we have

$$r = \|u\| \ge u(t) \ge \theta\|u\| = \theta r, \quad t \in [\theta, T - \theta]. \tag{3.3}$$

For $t \in [\theta, T - \theta]$ and $u \in \partial\Omega_1$, we shall discuss it from three perspectives.

(i) If $\sigma \in [\theta, T - \theta]$, thus for $u \in \partial\Omega_1$, by (A_1) and Lemma 2.4, we have

$$\begin{aligned} 2\|Tu\| &= 2(Tu)(\sigma) \\ &\ge \int_0^\sigma \phi_q\left(\int_s^\sigma g(r) f(u(r-\tau) + h(r-\tau), u(r))\nabla r\right)\Delta s \end{aligned}$$

$$
\begin{aligned}
&+\int_{\sigma}^{T}\phi_q\left(\int_{\sigma}^{s}g(r)f(u(r-\tau)+h(r-\tau),u(r))\nabla r\right)\Delta s\\
&\geq \int_{\theta}^{\sigma}\phi_q\left(\int_{s}^{\sigma}g(r)f(u(r-\tau)+h(r-\tau),u(r))\nabla r\right)\Delta s\\
&\quad+\int_{\sigma}^{T-\theta}\phi_q\left(\int_{\sigma}^{s}g(r)f(u(r-\tau)+h(r-\tau),u(r))\nabla r\right)\Delta s\\
&\geq mrA(\sigma)\geq mrL\geq 2r=2\|u\|.
\end{aligned}
\tag{3.4}
$$

(ii) If $\sigma\in(T-\theta,T]$, thus for $u\in\partial\Omega_1$, by (A_1) and Lemma 2.4, we have

$$
\begin{aligned}
\|Tu\| &= (Tu)(\sigma)\\
&\geq \phi_q\left(\frac{\beta}{\alpha}\int_{\xi}^{\sigma}g(s)f(u(s-\tau)+h(s-\tau),u(s))\nabla s\right)\\
&\quad+\int_{0}^{\sigma}\phi_q\left(\int_{s}^{\sigma}g(r)f(u(r-\tau)+h(r-\tau),u(r))\nabla r\right)\Delta s\\
&\geq \int_{\theta}^{T-\theta}\phi_q\left(\int_{s}^{T-\theta}g(r)f(u(r-\tau)+h(r-\tau),u(r))\nabla r\right)\Delta s\\
&\geq mrA(T-\theta)\geq mrL\geq 2r>r=\|u\|.
\end{aligned}
\tag{3.5}
$$

(iii) If $\sigma\in(0,\theta)$, thus for $u\in\partial\Omega_1$, by (A_1) and Lemma 2.4, we have

$$
\begin{aligned}
\|Tu\| &= (Tu)(\sigma)\\
&\geq \phi_q\left(\frac{\delta}{\gamma}\int_{\sigma}^{\eta}g(s)f(u(s-\tau)+h(s-\tau),u(s))\nabla s\right)\\
&\quad+\int_{\sigma}^{T}\phi_q\left(\int_{\sigma}^{s}g(r)f(u(r-\tau)+h(r-\tau),u(r))\nabla r\right)\Delta s\\
&\geq \int_{\theta}^{T-\theta}\phi_q\left(\int_{\theta}^{s}g(r)f(u(r-\tau)+h(r-\tau),u(r))\nabla r\right)\Delta s\\
&\geq mrA(\theta)\geq mrL\geq 2r>r=\|u\|.
\end{aligned}
\tag{3.6}
$$

Therefore, no matter under which condition, we all have

$$
\|Tu\|>\|u\|,\quad \forall u\in\partial\Omega_1. \tag{3.7}
$$

Then by Theorem 1.1, we have

$$
i(T,\Omega_1,K)=0. \tag{3.8}
$$

On the other hand, for $u \in \partial\Omega_2$, we have $u(t) \le \|u\| = R$, and by (A_2), we know that

$$\begin{aligned}\|Tu\| &= (Tu)(\sigma)\\ &\le \phi_q\left(\frac{\beta}{\alpha}\int_{\xi}^{\sigma} g(s)f(u(s-\tau)+h(s-\tau),u(s))\nabla s\right)\\ &\quad+\int_0^T \phi_q\left(\int_s^{\sigma} g(r)f(u(r-\tau)+h(r-\tau),u(r))\nabla r\right)\Delta s\\ &\le \left(T+\phi_q\left(\frac{\beta}{\alpha}\right)\right)MR\phi_q\left(\int_0^T g(r)\nabla r\right) \le R = \|u\|,\end{aligned} \tag{3.9}$$

thus

$$\|Tu\| \le \|u\|, \quad \forall u \in \partial\Omega_2. \tag{3.10}$$

Then, by Theorem 1.1, we have

$$i(T,\Omega_2,K) = 1. \tag{3.11}$$

Therefore, by (3.8), and (3.11), $r < R$, we have

$$i\left(T,\Omega_2 \setminus \overline{\Omega}_1,K\right) = 1. \tag{3.12}$$

Then operator T has a fixed point $u \in (\Omega_1 \setminus \overline{\Omega}_2)$, and $r \le \|u\| \le R$. This completes the proof of Theorem 3.1. □

Theorem 3.2. *Suppose that condition* (H_1), (H_2), *and* (H_3) *hold. Assume that* f *also satisfies*

(A_3) : $f^0 = \varphi \in [0,(\theta_*/4)^{p-1})$,

(A_4) : $f_\infty = \lambda \in ((2\theta^*/\theta)^{p-1},\infty)$.

Then, the SBVP (2.9), (2.10) *has a solution* u *which is bounded in* $\|\cdot\|$. *Furthermore, by Lemma 2.4,* $\overline{u}(t) = u(t)+h(t)$, $-\tau \le t \le T$ *is a positive solution to the SBVP* (1.8), (1.9).

Proof of Theorem 3.2. First, by $f^0 = \varphi \in [0,(\theta_*/4)^{p-1})$, for $\epsilon = (\theta_*/4)^{p-1}-\varphi$, there exists an adequately small positive number ρ, as $0 \le u_2 \le \rho$, $u_2 \ne 0$, $u_1 \le H$, we have

$$f(u_1,u_2) \le (\varphi+\epsilon)(u_2)^{p-1} \le \left(\frac{\theta_*}{4}\right)^{p-1}\rho^{p-1} = \left(\frac{\theta_*}{4}\rho\right)^{p-1}. \tag{3.13}$$

Then let $R = \rho$, $M = \theta_*/4 \in (0,\theta_*)$, thus by (3.13),

$$f(u_1,u_2) \le (MR)^{p-1}, \quad 0 \le u_2 \le R. \tag{3.14}$$

So condition (A_2) holds.

Next, by condition (A_4), $f_\infty = \lambda \in ((2\theta^*/\theta)^{p-1}, \infty)$, then for $\epsilon = \lambda - (2\theta^*/\theta)^{p-1}$, there exists an appropriately big positive number $r \neq R$, as $u_2 \geq \theta r$, $u_1 \leq H$, we have

$$f(u_1, u_2) \geq (\lambda - \epsilon)(u_2)^{p-1} \geq \left(\frac{2\theta^*}{\theta}\right)^{p-1} (\theta r)^{p-1} = (2\theta^* r)^{p-1}. \tag{3.15}$$

Let $m = 2\theta^* > \theta^*$, thus by (3.15), condition (A_1) holds. Therefore, by Theorem 3.1, we know that the results of Theorem 3.2 hold. The proof of Theorem 3.2 is complete. □

Theorem 3.3. *Suppose that conditions $(H_1), (H_2), (H_3)$ hold. Assume that f also satisfies*

(A_5): $f^\infty = \lambda \in [0, (\theta_*/4)^{p-1})$,

(A_6): $f_0 = \varphi \in ((2\theta^*/\theta)^{p-1}, \infty)$.

Then, the SBVP (2.9), (2.10) has a solution u which is bounded in $\|\cdot\|$. Furthermore by Lemma 2.4, $\overline{u}(t) = u(t) + h(t)$, $-\tau \leq t \leq T$ is a positive solution to the SBVP (1.8), (1.9).

Proof of Theorem 3.3. First, by condition (A_6), $f_0 = \varphi \in ((2\theta^*/\theta)^{p-1}, \infty)$, then for $\epsilon = \varphi - (2\theta^*/\theta)^{p-1}$, there exists an adequately small positive number r, as $0 \leq u_2 \leq r$, $u_2 \neq 0$, $u_1 \leq H$, we have

$$f(u_1, u_2) \geq (\varphi - \epsilon)(u_2)^{p-1} = \left(\frac{2\theta^*}{\theta}\right)^{p-1} (u_2)^{p-1}, \tag{3.16}$$

thus when $\theta r \leq u_2 \leq r$, $u_1 \leq H$, we have

$$f(u_1, u_2) \geq \left(\frac{2\theta^*}{\theta}\right)^{p-1} (\theta r)^{p-1} = (2\theta^* r)^{p-1}. \tag{3.17}$$

Let $m = 2\theta^* > \theta^*$, so by (3.17), condition (A_1) holds.

Next, by condition (A_5): $f^\infty = \lambda \in [0, (\theta_*/4)^{p-1})$, then for $\epsilon = (\theta_*/4)^{p-1} - \lambda$, there exists an suitably big positive number $\rho \neq r$, as $u_2 \geq \rho$, $u_1 \leq H$, we have

$$f(u_1, u_2) \leq (\lambda + \epsilon)(u_2)^{p-1} \leq \left(\frac{\theta_*}{4}\right)^{p-1} (u_2)^{p-1}. \tag{3.18}$$

If f is unbounded, by the continuity of f on $[0, \infty)^2$, then exists constant $R\ (\neq r) \geq \rho$, and a point $(u_{01}, u_{02}) \in [0, \infty)^2$ such that

$$\begin{gathered} \rho \leq u_{02} \leq R, \\ f(u_1, u_2) \leq f(u_{01}, u_{02}), \quad 0 \leq u_2 \leq R,\ u_1 \leq H. \end{gathered} \tag{3.19}$$

Thus, by $\rho \leq u_{02} \leq R$, $u_1 \leq H$, we know

$$f(u_1, u_2) \leq f(u_{01}, u_{02}) \leq \left(\frac{\theta_*}{4}\right)^{p-1} (u_{02})^{p-1} \leq \left(\frac{\theta_*}{4} R\right)^{p-1}. \tag{3.20}$$

Choose $M = \theta_*/4 \in (0, \theta_*)$. Then, we have

$$f(u_1, u_2) \le (MR)^{p-1}, \quad 0 \le u_2 \le R,\ u_1 \le H. \tag{3.21}$$

If f is bounded, we suppose $f(u_1, u_2) \le \overline{M}^{p-1}$, $u_2 \in [0, \infty)$, $\overline{M} \in R_+$, there exists an appropriately big positive number $R > (4/\theta_*)\overline{M}$, then choose $M = \theta_*/4 \in (0, \theta_*)$, we have

$$f(u_1, u_2) \le \overline{M}^{p-1} \le \left(\frac{\theta_*}{4}R\right)^{p-1} = (MR)^{p-1}, \quad 0 \le u_2 \le R,\ u_1 \le H. \tag{3.22}$$

Therefore, condition (A_2) holds. Therefore, by Theorem 3.1, we know that the results of Theorem 3.3 holds. The proof of Theorem 3.3 is complete. □

4. The Existence of Many Positive Solutions to (1.8) and (1.9)

Next, we will discuss the existence of many positive solutions.

Theorem 4.1. *Suppose that conditions $(H_1), (H_2), (H_3)$, and (A_2) in Theorem 3.1 hold. Assume that f also satisfies*

(A_7): $f_0 = +\infty$,

(A_8): $f_\infty = +\infty$.

Then, the SBVP (2.9), (2.10) has at last two solutions u_1, u_2 such that

$$0 < \|u_1\| < R < \|u_2\|. \tag{4.1}$$

Furthermore, by Lemma 2.4, $\overline{u}_1(t) = u_1(t) + h(t)$, $\overline{u}_2(t) = u_2(t) + h(t)$, $-\tau \le t \le T$ is a positive solution to the SBVP (1.8), (1.9).

Proof of Theorem 4.1. First, by condition (A_7), for any $N > 2/\theta L$, there exists a constant $\rho_* \in (0, R)$ such that

$$f(u_1, u_2) \ge (Nu_2)^{p-1}, \quad 0 < u_2 \le \rho_*,\ u_1 \le H. \tag{4.2}$$

Set $\Omega_{\rho_*} = \{u \in K : \|u\| < \rho_*\}$, for any $u \in \partial\Omega_{\rho_*}$, by (4.2) and Lemma 2.3, similar to the previous proof of Theorem 3.1, we can have from three perspectives

$$\|Tu\| \ge \|u\|, \quad \forall u \in \partial\Omega_{\rho_*}. \tag{4.3}$$

Then by Theorem 1.1, we have

$$i(T, \Omega_{\rho_*}, K) = 0. \tag{4.4}$$

Next, by condition (A_8), for any $\overline{N} > 2/\theta L$, there exists a constant $\rho_0 > 0$ such that

$$f(u_1, u_2) \ge \left(\overline{N}u_2\right)^{p-1}, \quad u_2 > \rho_0,\ u_1 \le H. \tag{4.5}$$

We choose a constant $\rho^* > \max\{R, \rho_0/\theta\}$, obviously $\rho_* < R < \rho^*$. Set $\Omega_{\rho^*} = \{u \in K : \|u\| < \rho^*\}$. For any $u \in \partial\Omega_{\rho^*}$, by Lemma 2.3, we have

$$u(t) \geq \theta\|u\| = \theta\rho^* > \rho_0, \quad t \in [\theta, T-\theta]. \tag{4.6}$$

Then by (4.5) and also similar to the previous proof of Theorem 3.1, we can also have from three perspectives

$$\|Tu\| \geq \|u\|, \quad \forall u \in \partial\Omega_{\rho^*}. \tag{4.7}$$

Then by Theorem 1.1, we have

$$i\left(T, \Omega_{\rho^*}, K\right) = 0. \tag{4.8}$$

Finally, set $\Omega_R = \{u \in K : \|u\| < R\}$, For any $u \in \partial\Omega_R$, by (A_2), Lemma 2.3 and also similar to the latter proof of Theorem 3.1, we can also have

$$\|Tu\| \leq \|u\|, \quad \forall u \in \partial\Omega_R. \tag{4.9}$$

Then by Theorem 1.1, we have

$$i(T, \Omega_R, K) = 1. \tag{4.10}$$

Therefore, by (4.4), (4.8), (4.10), $\rho_* < R < \rho^*$, we have

$$i\left(T, \Omega_R \setminus \overline{\Omega}_{\rho_*}, K\right) = 1, \qquad i\left(T, \Omega_{\rho^*} \setminus \overline{\Omega}_R, K\right) = -1. \tag{4.11}$$

Then T has fixed-point $u_1 \in \Omega_R \setminus \overline{\Omega}_{\rho_*}$, and fixed-point $u_2 \in \Omega_{\rho^*} \setminus \overline{\Omega}_R$. Obviously, $u_1,\ u_2$ are all positive solutions of problem (2.9), (2.10) and $\rho_* < \|u_1\| < R < \|u_2\| < \rho^*$. The proof of Theorem 4.1 is complete. □

Theorem 4.2. *Suppose that conditions $(H_1), (H_2), (H_3)$, and (A_1) in Theorem 3.1 hold. Assume that f also satisfies*

(A_9): $f^0 = 0$,

(A_{10}): $f^\infty = 0$.

Then, the SBVP (2.9), (2.10) has at last two solutions $u_1,\ u_2$ such that $0 < \|u_1\| < r < \|u_2\|$. Furthermore, by Lemma 2.4, $\overline{u}_1(t) = u_1(t) + h(t),\ \overline{u}_2(t) = u_2(t) + h(t), -\tau \leq t \leq T$ is a positive solution to the SBVP (1.8), (1.10).

Proof of Theorem 4.2. First, by $f^0 = 0$, for $\epsilon_1 \in (0, \theta_*)$, there exists a constant $\rho_* \in (0, r)$ such that

$$f(u_1, u_2) \leq (\epsilon_1 u_2)^{p-1}, \quad 0 < u_2 \leq \rho_*,\ u_1 \leq H. \tag{4.12}$$

Set $\Omega_{\rho_*} = \{u \in K : \|u\| < \rho_*\}$, for any $u \in \partial\Omega_{\rho_*}$, by (4.12), we have

$$\begin{aligned}
\|Tu\| &= (Tu)(\sigma) \\
&\leq \phi_q\left(\frac{\beta}{\alpha}\int_\xi^\sigma g(s)f(u(s-\tau)+h(s-\tau),u(s))\nabla s\right) \\
&\quad + \int_0^T \phi_q\left(\int_s^\sigma g(r)f(u(r-\tau)+h(r-\tau),u(r))\nabla r\right)\Delta s \\
&\leq \phi_q\left(\frac{\beta}{\alpha}\int_\xi^\sigma g(s)f(u(s-\tau)+h(s-\tau),u(s))\nabla s\right) \\
&\quad + T\phi_q\left(\int_0^T g(r)f(u(r-\tau)+h(r-\tau),u(r))\nabla r\right) \\
&\leq \left(T+\phi_q\left(\frac{\beta}{\alpha}\right)\right)\epsilon_1\rho_*\phi_q\left(\int_0^T g(r)\nabla r\right) \leq \rho_* = \|u\|,
\end{aligned} \tag{4.13}$$

that is

$$\|Tu\| \leq \|u\|, \quad \forall u \in \partial\Omega_{\rho_*}. \tag{4.14}$$

Then by Theorem 1.1, we have

$$i\left(T,\Omega_{\rho_*},K\right) = 1. \tag{4.15}$$

Next, let $f^*(x) = \max_{0\leq u_2\leq x, u_1\leq H} f(u_1,u_2)$, and note that $f^*(x)$ is monotone increasing with respect to $x \geq 0$. Then from $f^\infty = 0$, it is easy to see that

$$\lim_{x\to\infty}\frac{f^*(x)}{x^{p-1}} = 0. \tag{4.16}$$

Therefore, for any $\epsilon_2 \in (0,\theta_*)$, there exists a constant $\rho^* > r$ such that

$$f^*(x) \leq (\epsilon_2 x)^{p-1}, \quad x \geq \rho^*. \tag{4.17}$$

Set $\Omega_{\rho^*} = \{u \in K : \|u\| < \rho^*\}$, for any $u \in \partial\Omega_{\rho^*}$, by (4.17), we have

$$\begin{aligned}
\|Tu\| &= (Tu)(\sigma) \\
&\leq \phi_q\left(\frac{\beta}{\alpha}\int_\xi^\sigma g(s)f(u(s-\tau)+h(s-\tau),u(s))\nabla s\right) \\
&\quad + \int_0^T \phi_q\left(\int_s^\sigma g(r)f(u(r-\tau)+h(r-\tau),u(r))\nabla r\right)\Delta s
\end{aligned}$$

$$
\begin{aligned}
&\le \phi_q\left(\frac{\beta}{\alpha}\int_{\xi}^{\sigma} g(s)f(u(s-\tau)+h(s-\tau),u(s))\nabla s\right)\\
&\quad + T\phi_q\left(\int_0^T g(r)f(u(r-\tau)+h(r-\tau),u(r))\nabla r\right)\\
&\le \left(T+\phi_q\left(\frac{\beta}{\alpha}\right)\right)\phi_q\left(\int_0^T g(r)f^*(\rho^*)\nabla r\right)\\
&\le \left(T+\phi_q\left(\frac{\beta}{\alpha}\right)\right)\epsilon_2\rho^*\phi_q\left(\int_0^T g(r)\nabla r\right)\le r^*=\|u\|,
\end{aligned}
\tag{4.18}
$$

that is

$$\|Tu\|\le\|u\|,\quad \forall u\in\partial\Omega_{\rho^*}. \tag{4.19}$$

Then by Theorem 1.1, we have

$$i\left(T,\Omega_{\rho^*},K\right)=1. \tag{4.20}$$

Finally, set $\Omega_r=\{u\in K:\|u\|<r\}$. For any $u\in\partial\Omega_r$, by (A_1), Lemma 2.3 and also similar to the previous proof of Theorem 3.1, we can also have

$$\|Tu\|\ge\|u\|,\quad \forall u\in\partial\Omega_r. \tag{4.21}$$

Then by Theorem 1.1, we have

$$i(T,\Omega_r,K)=0. \tag{4.22}$$

Therefore, by (4.15), (4.20), (4.22), $\rho_*<r<\rho^*$, we have

$$i\left(T,\Omega_r\setminus\overline{\Omega}_{\rho_*},K\right)=-1,\qquad i\left(T,\Omega_{\rho^*}\setminus\overline{\Omega}_r,K\right)=1. \tag{4.23}$$

Then T have fixed point $u_1\in\Omega_r\setminus\overline{\Omega}_{\rho_*}$, and fixed point $u_2\in\Omega_{\rho_*}\setminus\overline{\Omega}_r$. Obviously, $u_1,\ u_2$ are all positive solutions of problem (1.8), (1.9) and $\rho_*<\|u_1\|<r<\|u_2\|<\rho^*$. The proof of Theorem 4.2 is complete. □

Similar to Theorem 3.1, we also obtain the following Theorems.

Theorem 4.3. *Suppose that conditions $(H_1),(H_2),(H_3)$ and (A_2) in Theorem 3.1, (A_4) in Theorem 3.2 and (A_6) in Theorem 3.3 hold. Then, the SBVP* (2.9), (2.10) *has at last two solutions $u_1,\ u_2$ such that $0<\|u_1\|<R<\|u_2\|$. Furthermore by Lemma 2.4, $\overline{u}_1(t)=u_1(t)+h(t)$, $\overline{u}_2(t)=u_2(t)+h(t)$, $-\tau\le t\le T$ is a positive solution to the SBVP* (1.8), (1.10).

Theorem 4.4. *Suppose that conditions $(H_1),(H_2),(H_3)$ and (A_1) in Theorem 3.1, (A_3) in Theorem 3.2 and (A_5) in Theorem 3.3 hold. Then, the SBVP* (2.9), (2.10) *have at last two solutions*

u_1, u_2 such that $0 < \|u_1\| < r < \|u_2\|$. *Furthermore by Lemma 2.4,* $\overline{u}_1(t) = u_1(t) + h(t)$, $\overline{u}_2(t) = u_2(t) + h(t)$, $-\tau \le t \le T$ *is a positive solution to the SBVP* (1.8), (1.10).

5. The Existence of Many Positive Solutions to (1.8) and (1.10)

In the following, we will deal with problem (1.8), (1.10), the method is similar to that in Sections 3 and 4, so we omit many proof in this section.

Theorem 5.1. *Suppose that condition* $(H_1), (H_2), (H_3), (H_4)$ *hold. Assume that* f *also satisfies*

(A_1'): $f(u_1, u_2) \ge (mr)^{p-1}$, *for* $\theta r \le u_2 \le r$, $0 \le u_1 \le H$,

(A_2'): $f(u_1, u_2) \le (MR)^{p-1}$, *for* $0 \le u_2 \le R$, $0 \le u_1 \le H$,

where $m \in (\theta^*, \infty)$, $M \in (0, \theta_{**})$. *Then, the SBVP* (2.9), (2.13) *has a solution* u *such that* $\|u\|$ *lies between* r *and* R. *Furthermore by Lemma 2.4,* $\overline{u}(t) = u(t) + h(t)$, $-\tau \le t \le T$ *is a positive solution to the SBVP* (1.8), (1.10).

Theorem 5.2. *Suppose that condition* $(H_1), (H_2), (H_3), (H_4)$ *hold. Assume that* f *also satisfies*

(A_3') : $f^0 = \varphi \in [0, (\theta_{**}/4)^{p-1})$,

(A_4'): $f_\infty = \lambda \in ((2\theta^*/\theta)^{p-1}, \infty)$.

Then, the SBVP (2.9), (2.13) *has a solution* u *which is bounded in* $\|\cdot\|$. *Furthermore by Lemma 2.4,* $\overline{u}(t) = u(t) + h(t)$, $-\tau \le t \le T$ *is a positive solution to the SBVP* (1.8), (1.10).

Theorem 5.3. *Suppose that condition* $(H_1), (H_2), (H_3), (H_4)$ *hold. Assume that* f *also satisfies*

(A_5'): $f^\infty = \lambda \in [0, (\theta_{**}/4)^{p-1})$,

(A_6'): $f_0 = \varphi \in ((2\theta^*/\theta)^{p-1}, \infty)$.

Then, the SBVP (2.9), (2.13) *has a solution* u *which is bounded in* $\|\cdot\|$. *Furthermore by Lemma 2.4,* $\overline{u}(t) = u(t) + h(t)$, $-\tau \le t \le T$ *is a positive solution to the SBVP* (1.8), (1.10).

Theorem 5.4. *Suppose that conditions* $(H_1), (H_2), (H_3), (H_4)$ *and* (A_2') *in Theorem 5.1 hold. Assume that* f *also satisfies*

(A_7'): $f_0 = +\infty$,

(A_8'): $f_\infty = +\infty$.

Then, the SBVP (2.9), (2.13) *has at least two solutions* u_1, u_2 *such that*

$$0 < \|u_1\| < R < \|u_2\|. \tag{5.1}$$

Furthermore by Lemma 2.4, $\overline{u}_1(t) = u_1(t) + h(t)$, $\overline{u}_2(t) = u_2(t) + h(t)$, $-\tau \le t \le T$ *is a positive solution to the SBVP* (1.8), (1.10).

Theorem 5.5. *Suppose that conditions* $(H_1), (H_2), (H_3), (H_4)$ *and* (A_1') *in Theorem 5.1 hold. Assume that* f *also satisfies*

(A_9'): $f^0 = 0$,

(A_{10}'): $f^\infty = 0$.

Then, the SBVP (2.9), (2.13) *has at least two solutions* u_1, u_2 *such that* $0 < \|u_1\| < r < \|u_2\|$. *Furthermore by Lemma 2.4,* $\overline{u}_1(t) = u_1(t) + h(t)$, $\overline{u}_2(t) = u_2(t) + h(t)$, $-\tau \le t \le T$ *is a positive solution to the SBVP* (1.8), (1.10).

Theorem 5.6. *Suppose that conditions* $(H_1), (H_2), (H_3), (H_4)$ *and* (A_2) *in Theorem 5.1,* (A_4) *in Theorem 5.2 and* (A_6') *in Theorem 3.3 hold. Then, the SBVP* (2.9), (2.13) *has at least two solutions* u_1, u_2 *such that* $0 < \|u_1\| < R < \|u_2\|$. *Furthermore by Lemma 2.4,* $\overline{u}_1(t) = u_1(t) + h(t)$, $\overline{u}_2(t) = u_2(t) + h(t)$, $-\tau \le t \le T$ *is a positive solution to the SBVP* (1.8), (1.10).

Theorem 5.7. *Suppose that conditions* $(H_1), (H_2), (H_3), (H_4)$ *and* (A_1') *in Theorem 5.1,* (A_3') *in Theorem 5.2 and* (A_5') *in Theorem 3.3 hold. Then, the SBVP* (2.9), (2.13) *has at least two solutions* u_1, u_2 *such that* $0 < \|u_1\| < r < \|u_2\|$. *Furthermore by Lemma 2.4,* $\overline{u}_1(t) = u_1(t) + h(t)$, $\overline{u}_2(t) = u_2(t)+ h(t)$, $-\tau \le t \le T$ *is a positive solution to the SBVP* (1.8), (1.10).

6. Application

In the section, we present two simple examples to explain our result.

Example 6.1. Let $\mathbf{T} = \{1 - (1/2)^{\mathbf{N}}\} \cup \{1\}$, where $\mathbf{N}$ denotes the set of all nonnegative integers. Consider the following 3-order singular boundary value problem (SBVP) with p-Laplacian

$$\begin{gathered}
\left(\phi_p\left(u^{\Delta}\right)\right)^{\nabla}(t) + \frac{1}{20}t^{-1/2}u^{1/2}(t) \cdot \left[\frac{1}{5} + \frac{(94/5)e^{2u(t)}}{120u(t-1) + 7e^{u(t)} + e^{2u(t)}}\right] = 0, \quad 0 < t < 1, \\
u(t) = t^2 - 1, \quad -1 \le t \le 0, \\
\phi_p(u(0)) - \phi_p\left(u^{\Delta}\left(\frac{1}{4}\right)\right) = 0, \qquad \phi_p(u(1)) + \delta\phi_p\left(u^{\Delta}\left(\frac{1}{2}\right)\right) = 0,
\end{gathered} \tag{6.1}$$

where

$$\begin{gathered}
\alpha = \gamma = 1, \qquad \beta = 1, \qquad p = \frac{3}{2}, \qquad \delta \ge 0, \qquad \xi = \frac{1}{4}, \qquad \eta = \frac{1}{2}, \qquad \theta = \frac{1}{4}, \\
\tau = T = 1.
\end{gathered} \tag{6.2}$$

So, by Lemma 2.4, we discuss the following SBVP:

$$\begin{gathered}
\left(\phi_p\left(u^{\Delta}\right)\right)^{\nabla}(t) + \frac{1}{20}t^{-1/2}u^{1/2}(t) \cdot \left[\frac{1}{5} + \frac{(94/5)e^{2u(t)}}{120[u(t-1) + h(t-1)] + 7e^{u(t)} + e^{2u(t)}}\right] = 0, \quad 0 < t < 1, \\
u(t) = 0, \quad -1 \le t \le 0, \\
\phi_p(u(0)) - \phi_p\left(u^{\Delta}\left(\frac{1}{4}\right)\right) = 0, \qquad \phi_p(u(1)) + \delta\phi_p\left(u^{\Delta}\left(\frac{1}{2}\right)\right) = 0,
\end{gathered} \tag{6.3}$$

where

$$h(t)=\begin{cases} t^2-1, & -1\le t\le 0,\\ 0, & 0\le t\le 1,\end{cases}\qquad g(t)=\frac{1}{20}t^{-1/2},\qquad \zeta(t)=t^2-1,$$
$$f(u_1,u_2)=(u_2)^{1/2}\left[\frac{1}{5}+\frac{(94/5)e^{2u_2}}{120u_1+7e^{u_2}+e^{2u_2}}\right]. \tag{6.4}$$

Then obviously,

$$q=3,\quad H=\max_{-1\le t\le 0}|\zeta(t)|=1,\qquad f^0=\varphi=\lim_{u_2\to 0^+}\max_{0\le u_1\le 1}\frac{f(u_1,u_2)}{u_2^{p-1}}=\frac{51}{20},$$
$$f_\infty=\lambda=\lim_{u_2\to\infty}\min_{0\le u_1\le 1}\frac{f(u_1,u_2)}{u_2^{p-1}}=\frac{95}{5},\qquad \int_0^T g(t)\nabla t=\frac{1}{10}, \tag{6.5}$$

so conditions $(H_1), (H_2), (H_3)$ hold.

Next,

$$\theta_*=\frac{1}{(T+\phi_q(\beta/\alpha))\phi_q\left(\int_0^T g(r)\nabla r\right)}=50, \tag{6.6}$$

then $(\theta_*/4)^{p-1}=5\sqrt{2}/2>51/20$, that is, $\varphi\in[0,(\theta_*/4)^{p-1})$, so condition (A_3) holds.

For $\theta=1/4$, it is easy to see by calculating that

$$L=\min_{t\in[\theta,T-\theta]}A(t)=\frac{1}{16}\left(\frac{7}{36}+\frac{\sqrt{3}}{3}\right). \tag{6.7}$$

Because of

$$\left(\frac{2\theta^*}{\theta}\right)^{p-1}=96\times\left(\frac{1}{7+12\sqrt{3}}\right)^{1/2}<\frac{95}{5}, \tag{6.8}$$

then

$$\lambda\in\left(\left(\frac{2\theta^*}{\theta}\right)^{p-1},\infty\right), \tag{6.9}$$

so condition (A_4) holds. Then by Theorem 3.2, SBVP (6.3) has at least a positive solution $u(t)$. So, $\overline{u}(t)=u(t)+h(t),\ -1<t<1$ is the positive solution of SBVP (6.1).

Example 6.2. Consider the following 3-order singular boundary value problem (SBVP) with p-Laplacian:

$$\left(\phi_p\left(u^{\Delta}\right)\right)^{\nabla}(t)+\frac{1}{64\pi^4}t^{-1/2}(1-t)\left[u(t-1)+u^2(t)+u^4(t)\right]=0,\quad 0<t<1,$$
$$u(t)=-te^t,\quad -1\le t\le 0,\tag{6.10}$$
$$2\phi_p(u(0))-\phi_p\left(u^{\Delta}\left(\frac{1}{4}\right)\right)=0,\qquad \phi_p(u(1))+\delta\phi_p\left(u^{\Delta}\left(\frac{1}{2}\right)\right)=0,$$

where

$$\beta=\gamma=1,\qquad \alpha=2,\qquad p=4,\qquad \delta\ge 0,\qquad p=4,\qquad \xi=\frac{1}{4},\qquad \eta=\frac{1}{3},$$
$$\theta=\frac{1}{4},\qquad \tau=T=1.\tag{6.11}$$

So, by Lemma 2.4, we discuss the following SBVP:

$$(\phi_p(u^{\Delta}))^{\nabla}(t)+\frac{1}{64\pi^4}t^{-1/2}(1-t)\left[[u(t-1)+h(t-1)]+u^2(t)+u^4(t)\right]=0,\quad 0<t<1,$$
$$u(t)=0,\quad -1\le t\le 0,$$
$$2\phi_p(u(0))-\phi_p\left(u^{\Delta}\left(\frac{1}{4}\right)\right)=0,\qquad \phi_p(u(1))+\delta\phi_p\left(u^{\Delta}\left(\frac{1}{2}\right)\right)=0,\tag{6.12}$$

where

$$h(t)=\begin{cases}\zeta(t), & -1\le t\le 0,\\ 0, & 0\le t\le 1,\end{cases}\qquad \zeta(t)=-te^t,$$
$$g(t)=\frac{1}{64\pi^4}t^{-1/2}(1-t),\qquad f(u_1,u_2)=u_1+u_2^2+u_2^4.\tag{6.13}$$

Then obviously,

$$q=\frac{4}{3},\qquad \int_0^1 g(t)\nabla t=\frac{1}{64\pi^3},\qquad H=\max_{-1\le t\le 0}|\zeta(t)|=e,\qquad f_\infty=+\infty,\qquad f_0=+\infty,\tag{6.14}$$

so conditions $(H_1),(H_2),(H_3),(A_7),(A_8)$ hold.

Next,

$$\phi_q\left(\int_0^1 a(t)\nabla t\right)=\frac{1}{4\pi},\qquad \theta_*=\frac{4\pi}{1+\sqrt[3]{4}},\tag{6.15}$$

we choose $R = 3$, $M = 2$, and for $\theta = 1/4$, because of the monotone increasing of $f(u_1, u_2, u_3)$ on $[0, \infty)^3$, then

$$f(u_1, u_2) \leq f(e, 3) = e + 90, \quad 0 \leq u_2 \leq 3, \quad 0 \leq u_1 \leq e. \tag{6.16}$$

Therefore, by

$$M \in (0, \theta_*), \quad (MR)^{p-1} = (6)^3 = 216, \tag{6.17}$$

we know that

$$f(u_1, u_2, u_3) \leq (MR)^{p-1}, \quad 0 \leq u_2 \leq 3, \ 0 \leq u_1 \leq e, \tag{6.18}$$

so condition (A_2) holds. Then by Theorem 4.1, SBVP (6.12) has at least two positive solutions v_1, v_2 and $0 < \|v_1\| < 3 < \|v_2\|$. Then, by Lemma 2.4, $\overline{v}_1(t) = v_1(t) + h(t)$, $\overline{v}_2(t) = v_2(t) + h(t)$, $t \in (-1, 1)$ are the positive solutions of the SBVP (6.10).

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