Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2012, Article ID 648983, 11 pages
doi:10.1155/2012/648983

Research Article

Generalized Lower and Upper Approximations in Quantales

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Received 8 August 2011; Revised 21 October 2011; Accepted 30 October 2011

Academic Editor: Jin L. Kuang

We introduce the concepts of set-valued homomorphism and strong set-valued homomorphism of a quantale which are the extended notions of congruence and complete congruence, respectively. The properties of generalized lower and upper approximations, constructed by a set-valued mapping, are discussed.

1. Introduction

The concept of Rough set was introduced by Pawlak [1] as a mathematical tool for dealing with vagueness or uncertainty. In Pawlak's rough sets, the equivalence classes are the building blocks for the construction of the lower and upper approximations. It soon invoked a natural question concerning a possible connection between rough sets and algebraic systems. Biswas and Nanda [2] introduced the notion of rough subgroups. Kuroki [3] and Qimei [4] introduced the notions of a rough ideal and a rough prime ideal in a semigroup, respectively. Davvaz in [5] introduced the notion of rough subring with respect to an ideal of a ring. Rough modules have been investigated by Davvaz and Mahdavipour [6]. Rasouli and Davvaz studied the roughness in MV-algebra [7]. In [8–12], the roughness of various hyperstructures are discussed. Further, some authors consider the rough set in a fuzzy algebraic system, see [13–16]. The concept of quantale was introduced by Mulvey [17] in 1986 with the purpose of studying the spectrum of C^*-algebra, as well as constructive foundations for quantum mechanics. There are abundant contents in the structure of quantales, because quantale can be regarded as the generalization of frame. Since quantale theory provides a powerful tool in studying noncommutative structures, it has wide applications, especially in studying noncommutative C^*-algebra theory, the ideal theory of commutative ring, linear logic, and so on.

The quantale theory has aroused great interests of many researchers, and a great deal of new ideas and applications of quantale have been proposed in twenty years [18–24].

The majority of studies on rough sets for algebraic structures such as semigroups, groups, rings, and modules have been concentrated on a congruence relation. An equivalence relation is sometimes difficult to be obtained in real-world problems due to the vagueness and incompleteness of human knowledge. From this point of view, Davvaz [25] introduced the concept of set-valued homomorphism for groups. And then, Yamak et al. [26, 27] introduced the concepts of set-valued homomorphism and strong set-valued homomorphism of a ring and a module. In this paper, the concepts of set-valued homomorphism and strong set-valued homomorphism in quantales are introduced. We discuss the properties of generalized lower and upper approximations in quantales.

2. Preliminaries

In this section, we give some basic notions and results about quantales and rough set theory (see [19, 22, 25, 28]), which will be necessary in the next sections.

Definition 2.1. A *quantale* is a complete lattice Q with an associative binary operation $\circ$ satisfying

$$a \circ \left(\bigvee_{i\in I} b_i\right) = \bigvee_{i\in I}(a \circ b_i), \qquad \left(\bigvee_{i\in I} a_i\right) \circ b = \bigvee_{i\in I}(a_i \circ b), \tag{2.1}$$

for all $a, b, a_i, b_i \in Q (i \in I)$.

An element $e \in Q$ is called a *left (right) unit* if and only if $e \circ a = a$ $(a \circ e = a)$, e is called a *unit* if it is both a right and left unit.

A quantale Q is called a *commutative quantale* if $a \circ b = b \circ a$ for all $a, b \in Q$.

A quantale Q is called an *idempotent quantale* if $a \circ a = a$ for all $a \in Q$.

A subset S of Q is called a *subquantale* of Q if it is closed under $\circ$ and arbitrary sups.

In a quantale Q, we denote the top element of Q by 1 and the bottom by 0. For $A, B \subseteq Q$, we write $A \circ B$ to denote the set $\{a \circ b \mid a \in A, b \in B\}$, $A \vee B$ to denote $\{a \vee b \mid a \in A, b \in B\}$ and $\bigvee_{i\in I} A_i = \{\bigvee_{i\in I} a_i \mid a_i \in A_i\}$.

Definition 2.2. Let Q be a quantale, a subset $\emptyset \neq I \subseteq Q$ is called a *left (right) ideal* of Q if

(1) $a, b \in I$ implies $a \vee b \in I$,

(2) $a \in I, b \in Q$ and $b \leq a$ imply $b \in I$ for all,

(3) $a \in Q$ and $x \in I$ imply $a \circ x \in I$ $(x \circ a \in I)$.

A subset $I \subseteq Q$ is called an *ideal* if it is both a left and a right ideal.

Let X be a subset of Q, we write $\downarrow X = \{y \in Q \mid y \leq x \text{ for some } x \in X\}$, X is a *lower set* if and only if $X = \downarrow X$. It is obvious that an ideal I is a directed lower set. For every (left, right) ideal I of Q, it is easy to see that $0 \in I$.

An ideal of Q is called a *prime ideal* if $a \circ b \in I$ implies $a \in I$ or $b \in I$ for all $a, b \in Q$.

An ideal I of Q is called a *semi-prime ideal* if $a \circ a \in I$ implies $a \in I$ for all $a \in Q$.

An ideal I of Q $(I \neq Q)$ is called a *primary ideal* if for all $a, b \in Q$, $a \circ b \in I$ and $a \notin I$ imply $b^n \in I$ for some $n > 0$. ($b^n = \underbrace{b \circ b \circ \cdots \circ b}_{n}$).

Definition 2.3. A nonempty subset $M \subseteq Q$ is called an *m-system* of Q, if for all $a, b \in M$, $\downarrow (a \circ 1 \circ b) \cap M \neq \emptyset$.

A nonempty subset $S \subseteq Q$ is called a *multiplicative set* of Q, if $a \circ b \in S$ for all $a, b \in S$.

Every ideal of Q is both an m-system and a multiplicative set.

Definition 2.4. Let Q be a quantale, an equivalence relation θ on L is called a *congruence* on Q if for all $a, b, c, d, a_i, b_i \in Q$ $(i \in I)$, we have

(1) $a\theta b, c\theta d \Rightarrow (a \circ c)\theta(b \circ d)$,

(2) $a_i \theta b_i (i \in I) \Rightarrow (\bigvee_{i\in I} a_i)\theta(\bigvee_{i\in I} b_i)$.

It is obvious that $[a]_\theta \circ [b]_\theta \subseteq [a \circ b]_\theta$, $\bigvee_{i\in I}[a_i]_\theta \subseteq [\bigvee_{i\in I} a_i]_\theta$ for all $a, b, a_i \in Q$ $(i \in I)$.

Definition 2.5. Let Q be a quantale, a congruence θ on Q is called a *complete congruence*, if

(1) $[a]_\theta \circ [b]_\theta = [a \circ b]_\theta$ for all $a, b \in Q$,

(2) $\bigvee_{i\in I}[a_i]_\theta = [\bigvee_{i\in I} a_i]_\theta$ for all $a_i \in Q$ $(i \in I)$.

Definition 2.6. Let $(Q_1, \circ_1)$ and $(Q_2, \circ_2)$ be two quantales. A map $f : Q_1 \to Q_2$ is said to be a *homomorphism* if

(1) $f(a \circ_1 b) = f(a) \circ_2 f(b)$ for all $a, b \in Q_1$;

(2) $f(\bigvee_{i\in I} a_i) = \bigvee_{i\in I} f(a_i)$ for all $a_i \in Q_1$ $(i \in I)$.

Definition 2.7. Let U and W be two nonempty universes. Let T be a set-valued mapping given by $T : U \to P(W)$, where $P(W)$ denotes the set of all subsets of W. Then the triple (U, W, T) is referred to as a generalized approximation space. For any set $A \subseteq W$, the generalized lower and upper approximations, $T_-(A)$ and $T^-(A)$, are defined by

$$T_-(A) = \{x \in U \mid T(x) \subseteq A\}, \qquad T^-(A) = \{x \in U \mid T(x) \cap A \neq \emptyset\}. \tag{2.2}$$

The pair $(T_-(A), T^-(A))$ is referred to as a *generalized rough set.*

From the definition, the following theorems can be easily derived.

Theorem 2.8. *Let U, W be nonempty universes and $T : U \to P^*(W)$ be a set-valued mapping, where $P^*(W)$ denotes the set of all nonempty subsets of W. If $A \subseteq W$, then $T_-(A) \subseteq T^-(A)$.*

If $W = U$ and $R_T = \{(x, y) \mid y \in T(x)\}$ is an equivalence relation on U, then the pair (U, R_T) is the Pawlak approximation space. Therefore, a generalized rough set is an extended notion of Pawlak's rough set.

Theorem 2.9. *Let (U, W, T) be a generalized approximation space, its lower and upper approximation operators satisfy the following properties. For all $A, B \in P(W)$,*

(1) $T_-(A) = (T^-(A^c))^c$, $T^-(A) = (T_-(A^c))^c$,

(2) $T_-(W) = U$, $T^-(\emptyset) = \emptyset$,

(3) $T_-(A \cap B) = T_-(A) \cap T_-(B)$, $T^-(A \cup B) = T^-(A) \cup T^-(B)$,

(4) $A \subseteq B \Rightarrow T_-(A) \subseteq T_-(B), T^-(A) \subseteq T^-(B)$,

(5) $T_-(A \cup B) \supseteq T_-(A) \cup T_-(B), T^-(A \cap B) \subseteq T^-(A) \cap T^-(B)$,

where A^c is the complement of the set A.

3. Generalized Rough Subsets in Quantales

In this paper, $(Q_1, \circ_1)$ and $(Q_2, \circ_2)$ are two quantales.

Theorem 3.1. *Let $T : Q_1 \rightarrow P(Q_2)$ be a set-valued mapping and $\emptyset \neq A, B \subseteq Q_2$. Then*

(1) $T^-(A) \cup T^-(B) \subseteq T^-(A \vee B)$, *if* $0 \in A \cap B$,

(2) $T^-(A) \cup T^-(B) \subseteq T^-(A \circ_2 B)$, *if* $e \in A \cap B$,

(3) $T_-(A) \cap T_-(B) \subseteq T_-(A \vee B)$,

(4) $T_-(A) \cap T_-(B) \subseteq T_-(A \vee B)$, *if Q_2 is an idempotent quantale,*

(5) $T_-(A) \cup T_-(B) \subseteq T_-(A \vee B)$, *if* $0 \in A \cap B$,

(6) $T_-(A) \cup T_-(B) \subseteq T_-(A \circ_2 B)$, *if* $e \in A \cap B$.

Proof. (1) Suppose that $a \in A$, we have $a = a \vee 0 \in A \vee B$ for $0 \in B$. So $A \subseteq A \vee B$. Similarly, $B \subseteq A \vee B$. So $A \cup B \subseteq A \vee B$. By Theorem 2.9, we have $T^-(A) \cup T^-(B) = T^-(A \cup B) \subseteq T^-(A \vee B)$.

(2) Suppose that $a \in A$, we have $a = a \circ_2 e \in A \circ_2 B$ for $e \in B$. So $A \subseteq A \circ_2 B$. Similarly, $B \subseteq A \circ_2 B$. So $A \cup B \subseteq A \circ_2 B$. By Theorem 2.9, we have $T^-(A) \cup T^-(B) = T^-(A \cup B) \subseteq T^-(A \circ_2 B)$.

(3) It is obvious that $A \cap B \subseteq A \vee B$. By Theorem 2.9, we have $T_-(A) \cap T_-(B) = T_-(A \cap B) \subseteq T_-(A \vee B)$.

(4) Since Q_2 is an idempotent quantale, we have $A \cap B \subseteq A \circ B$. By Theorem 2.9, we have $T_-(A) \cap T_-(B) = T_-(A \cap B) \subseteq T_-(A \circ_2 B)$.

(5) and (6) The proofs are similar to (1) and (2), respectively. □

Definition 3.2. A set-valued mapping $T : Q_1 \rightarrow P(Q_2)$ is called a *set-valued homomorphism* if

(1) $T(a) \circ_2 T(b) \subseteq T(a \circ_1 b)$ for all $a, b \in Q_1$,

(2) $\bigvee_{i \in I} T(a_i) \subseteq T(\bigvee_{i \in I} a_i)$ for all $a_i \in Q_1 (i \in I)$.

T is called a *strong set-valued homomorphism* if the equalities in (1), (2) hold.

Example 3.3. (1) Let θ be a congruence on Q_2. Then the set-valued mapping $T : Q_1 \rightarrow P(Q_2)$ defined by $T(x) = [x]_\theta$ is a set-valued homomorphism but not necessarily a strong set-valued homomorphism. If θ is complete, then T is a strong set-valued homomorphism.

(2) Let f be a quantale homomorphism from Q_1 to Q_2. Then the set-valued mapping $T : Q_1 \rightarrow P(Q_2)$ defined by $T(a) = \{f(a)\}$ is a strong set-valued homomorphism.

Theorem 3.4. *Let $T : Q_1 \rightarrow P(Q_2)$ be a set-valued homomorphism and $\emptyset \neq A, B \subseteq Q_2$. Then*

(1) $T^-(A) \vee T^-(B) \subseteq T^-(A \vee B)$,

(2) $T^-(A) \circ_1 T^-(B) \subseteq T^-(A \circ_2 B)$,

(3) $T^-(A) \cap T^-(B) \subseteq T^-(A \vee B)$,

(4) $T^-(A) \cap T^-(B) \subseteq T^-(A \circ_2 B)$, *if Q_1 is an idempotent quantale.*

Proof. (1) Suppose that $c \in T^-(A) \vee T^-(B)$, there exist $a \in T^-(A), b \in T^-(B)$ such that $c = a \vee b$. So there exist $x \in A \cap T(a)$ and $y \in B \cap T(b)$. Hence $x \vee y \in A \vee B$ and $x \vee y \in T(a) \vee T(b)$. Since T is a set-valued homomorphism, we have $x \vee y \in T(a \vee b)$. Therefore, $T(a \vee b) \cap (A \vee B) \neq \emptyset$ which implies that $c = a \vee b \in T^-(A \vee B)$.

(2) The proof is similar to (1).

(3) Suppose that $x \in T^-(A) \cap T^-(B)$, there exist $a \in A \cap T(x)$ and $b \in B \cap T(x)$. Since T is a set-valued homomorphism, we have $a \vee b \in T(x) \vee T(x) \subseteq T(x)$. So $a \vee b \in T(x) \cap (A \vee B)$ which implies that $x \in T^-(A \vee B)$.

(4) Suppose that $x \in T^-(A) \cap T^-(B)$, there exist $a \in A \cap T(x)$ and $b \in B \cap T(x)$. Since T is a set-valued homomorphism and Q_1 is idempotent, we have $a \vee b \in T(x) \circ_2 T(x) \subseteq T(x \circ_1 x) = T(x)$. So $a \vee b \in T(x) \cap (A \vee B)$ which implies that $x \in T^-(A \circ_2 B)$. □

Theorem 3.5. *Let $T : Q_1 \to P(Q_2)$ be a strong set-valued homomorphism and $\emptyset \neq A, B \subseteq Q_2$. Then*

(1) $T_-(A) \vee T_-(B) \subseteq T_-(A \vee B)$,

(2) $T_-(A) \circ_1 T_-(B) \subseteq T_-(A \circ_2 B)$.

Proof. (1) Suppose that $c \in T_-(A) \vee T_-(B)$, there exist $a \in T_-(A), b \in T_-(B)$ such that $c = a \vee b$. Hence $T(a) \subseteq A$ and $T(b) \subseteq B$. Since T is a strong set-valued homomorphism, we have $T(a \vee b) = T(a) \vee T(b) \subseteq A \vee B$ which implies that $c = a \vee b \in T_-(A \vee B)$.

(2) The proof is similar to (1). □

4. Generalized Rough Ideals in Quantales

Theorem 4.1. *Let $T : Q_1 \to P(Q_2)$ be a set-valued mapping. If I and J are, respectively, a right and a left ideal of Q_2, then*

(1) $T^-(I \circ_2 J) \subseteq T^-(I) \cap T^-(J)$,

(2) $T_-(I \circ_2 J) \subseteq T_-(I) \cap T_-(J)$,

(3) $T^-(I \wedge J) \subseteq T^-(I) \cap T^-(J)$,

(4) $T_-(I) \wedge T_-(J) = T_-(I \wedge J)$,

(5) $T^-(I) \cup T^-(J) \subseteq T^-(I \vee J)$,

(6) $T_-(I) \cup T_-(J) \subseteq T_-(I \vee J)$.

If Q_1 is an idempotent quantale and T is a set-valued homomorphism, then the equalities in (1)–(3) hold.

Proof. Since I and J are, respectively, a right and a left ideal of Q_2, we have $I \circ_2 J \subseteq I \cap J$, $I \wedge J = I \cap J$ and $o \in I \cap J$. By Theorem 2.9, we get the conclusion (1)–(4). By Theorem 3.1, we get (5) and (6).

If Q_1 is idempotent, we first show that $T^-(I) \cap T^-(J) \subseteq T^-(I \circ_2 J)$. Suppose that $x \in T^-(I) \cap T^-(J)$, there exist $y \in I, z \in J$ such that $y, z \in T(x)$. So, $y \circ_2 z \in I \circ_2 J$ and $y \circ_2 z \in T(x) \circ_2 T(x)$. Since T is a set-valued homomorphism and Q_1 is idempotent, we have $y \circ_2 z \in T(x \circ_1 x) = T(x)$. Therefore, $x \in T^-(I \circ_2 J)$. So the equality in (1) holds. Since Q_1 is idempotent, we have $I \cap J \subseteq I \circ_2 J$. So $I \cap J = I \circ_2 J = I \cap J$. By Theorem 2.9, we get $T_-(I \circ_2 J) = T_-(I \cap J) = T_-(I) \cap T_-(J)$. Since the equality in (1) holds, we have $T^-(I \wedge J) = T^-(I \circ_2 J) = T^-(I) \cap T^-(J)$. □

Lemma 4.2. *Let $T : Q_1 \to P(Q_2)$ be a set-valued homomorphism. If A is a lower set of Q_2, then $T_-(A)$ is a lower set of Q_1.*

Proof. Suppose $x \leq y \in T_-(A)$, then $T(y) \subseteq A$. Let $z \in T(x), a \in T(y)$, we have $z \vee a \in T(x) \vee T(y) \subseteq T(x \vee y) = T(y) \subseteq A$. Since A is a lower set, we have $z \in A$. Therefore, $T(x) \subseteq A$ which implies that $x \in T_-(A)$. □

Lemma 4.3. *Let $T : Q_1 \to P(Q_2)$ be a strong set-valued homomorphism. If A is a lower set of Q_2, then $T^-(A)$ is a lower set of Q_1.*

Proof. Suppose that $x \leq y \in T^-(A)$, there exists $z \in T(y) \cap A$. Since T is a strong set-valued homomorphism, we have $T(x) \vee T(y) = T(x \vee y) = T(y)$. So there exist $a \in T(x), b \in T(y)$ such that $z = a \vee b$. Since A is a lower set, we have $a \in A$. Thus $T(x) \cap A \neq \emptyset$ which follows that $x \in T^-(A)$. □

Lemma 4.4. *Let $T : Q_1 \to P(Q_2)$ be a set-valued homomorphism and A a nonempty subset of Q_2. If A is closed under arbitrary (resp., finite) sups, then $T^-(A)$ is closed under arbitrary (resp., finite) sups.*

Proof. Let $B \subseteq T^-(A)$. For each $b \in B$, we have $b \in T^-(A)$, then there exist $x_b \in T(b) \cap A$. Since T is a set-valued homomorphism, we have $\bigvee_{b\in B} x_b \in \bigvee_{b\in B} T(b) \subseteq T(\bigvee B)$. And we have $\bigvee_{b\in B} x_b \in A$ for A is closed under arbitrary sups. So $T(\bigvee B) \cap A \neq \emptyset$ which implies that $\bigvee B \in T^-(A)$. □

Lemma 4.5. *Let $T : Q_1 \to P(Q_2)$ be a strong set-valued homomorphism and A a nonempty subset of Q_2. If A is closed under arbitrary (resp., finite) sups, then $T_-(A)$ is closed under arbitrary (resp., finite) sups.*

Proof. Let $B \subseteq T_-(A)$. For each $b \in B$, we have $T(b) \subseteq A$. Since T is a strong set-valued homomorphism, we have $T(\bigvee B) = \bigvee T(B)$. Suppose $z \in T(\bigvee B) = \bigvee T(B)$, there exist $x_b \in T(b) \subseteq A$ $(b \in B)$ such that $z = \bigvee_{b\in B} x_b$. Since A is closed under arbitrary sups, we have $z = \bigvee_{b\in B} x_b \in A$. So $T(\bigvee B) \subseteq A$ which implies that $\bigvee B \in T_-(A)$. □

Theorem 4.6. *Let $T : Q_1 \to P(Q_2)$ be a set-valued homomorphism. If A is a subquantale of Q_2, then $T^-(A)$ is a subquantale of Q_1.*

Proof. Since T is a set-valued homomorphism and A is a subquantale, by Theorems 3.4 and 2.9, we have $T^-(A) \circ_1 T^-(A) \subseteq T^-(A \circ_2 A) \subseteq T^-(A)$. So, $T^-(A)$ is closed under $\circ_1$.

Since A is closed under arbitrary sups, by Lemma 4.4, we get $T^-(A)$ is closed under arbitrary sups. □

Theorem 4.7. *Let $T : Q_1 \to P(Q_2)$ be a strong set-valued homomorphism. If A is a subquantale of Q_2, then $T_-(A)$ is a subquantale of Q_1.*

Proof. The proof is similar to Theorem 4.6. □

Theorem 4.8. *Let $T : Q_1 \to P^*(Q_2)$ be a strong set-valued homomorphism and I a right (left) ideal of Q_2. Then $T^-(I)$ is, if it is nonempty, a right (left) ideal of Q_1.*

Proof. Suppose that $a, b \in T^-(I)$. Since I is closed under finite sups, by Lemma 4.4, we have $T^-(I)$ is closed under finite sups. So $a \vee b \in T^-(I)$.

Table 1

$\circ_1$	0	a	1
0	0	a	1
a	0	a	1
1	0	a	1

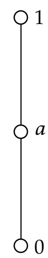

Figure 1

Since I is a lower set, by Lemma 4.3, we have $T^{-}(I)$ is a lower set.

Suppose $a \in Q_1, x \in T^{-}(I)$, there exists $y \in I \cap T(x)$. Since I is a right ideal of Q_2, we have $y \circ_2 b \in I$ for each $b \in T(a) \subseteq Q_2$. So $y \circ_2 b \in T(x) \circ_2 T(a) \subseteq T(x \circ_1 a)$. Therefore, $T(x \circ_1 a) \cap I \neq \emptyset$ which implies that $x \circ_1 a \in T^{-}(I)$. □

Theorem 4.9. *Let $T : Q_1 \to P(Q_2)$ be a strong set-valued homomorphism and I a right (left) ideal of Q_2. Then $T_{-}(I)$ is, if it is nonempty, a right (left) ideal of Q_1.*

Proof. Suppose $a, b \in T_{-}(I)$. Since I is closed under finite sups, by Lemma 4.5, we have $T_{-}(I)$ is closed under finite sups. So, $a \vee b \in T_{-}(I)$.

Since I is a lower set, by Lemma 4.2, we have $T_{-}(I)$ is a lower set.

Suppose $a \in Q_1, x \in T_{-}(I)$, we have $T(x) \subseteq I$. Let $y \in T(x \circ_1 a)$. Since T is a strong set-valued homomorphism, we have $y \in T(x) \circ_2 T(a)$, then there exist $y_1 \in T(x) \subseteq I$, $y_2 \in T(a)$ such that $y = y_1 \circ_2 y_2$. Since I is a right ideal, we have $y = y_1 \circ_2 y_2 \in I$. Therefore, $T(x \circ_1 a) \subseteq I$ which implies that $x \circ_1 a \in T_{-}(I)$. □

Definition 4.10. A subset $A \subseteq Q_2$ is called a *generalized rough ideal* (subquantale) of Q_1 if $T_{-}(A)$ and $T^{-}(A)$ are ideals (subquantales) of Q_1.

The following corollary follows from Theorems 4.6–4.9.

Corollary 4.11. *Let $T : Q_1 \to P(Q_2)$ be a strong set-valued homomorphism and I an ideal (a subquantale) of Q_2. If $T_{-}(I)$ and $T^{-}(I)$ are nonempty, then I is a generalized rough ideal (subquantale) of Q_1.*

From the above, we know that an ideal is a generalized rough ideal with respect to a strong set-valued homomorphism. The following example shows that the converse does not hold in general.

Example 4.12. Let $Q_1 = \{0, a, 1\}$ and $Q_2 = \{0', b, c, 1'\}$ be quantales shown in Figures 1 and 2 and Tables 1 and 2.

Let $T : Q_1 \to P(Q_2)$ be a strong set-valued homomorphism as defined by $T(0) = \{0'\}$, $T(a) = \{b, c\}, T(1) = \{1'\}$. Let $A = \{0', b, c\} \subseteq Q_2$, $B = \{b, c\} \subseteq Q_2$, then $T_{-}(A) = \{0, a\} = T^{-}(A)$

Table 2

$\circ_2$	$0'$	b	c	$1'$
$0'$	$0'$	b	c	$1'$
b	$0'$	b	c	$1'$
c	$0'$	b	c	$1'$
$1'$	$0'$	b	c	$1'$

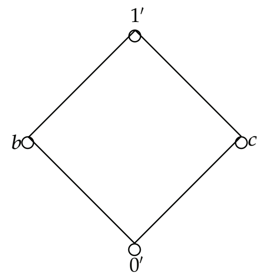

Figure 2

and $T_{-}(B) = \{a\} = T^{-}(B)$. It is obvious that A is a generalized rough ideal of Q_1 but A is not an ideal of Q_2 and B is a generalized rough subquantale of Q_1 but B is not a subquantale of Q_2.

Theorem 4.13. *Let $T : Q_1 \to P(Q_2)$ be a strong set-valued homomorphism and I a prime ideal of Q_2. Then $T^{-}(I)$ is, if it is nonempty, a prime ideal of Q_1.*

Proof. By Theorem 4.8, we get $T^{-}(I)$ is an ideal of Q_1.

Let $a \circ_1 b \in T^{-}(I)$, there exists $x \in I \cap T(a \circ_1 b)$. Since T is a strong set-valued homomorphism, we have $x \in T(a) \circ_2 T(b)$, then there exist $y \in T(a), z \in T(b)$ such that $x = y \circ_2 z \in I$. Since I is a prime ideal of Q_2, we have $y \in I$ or $z \in I$. So $T(a) \cap I \neq \emptyset$ or $T(b) \cap I \neq \emptyset$ which implies that $a \in T^{-}(I)$ or $b \in T^{-}(I)$. □

Theorem 4.14. *Let $T : Q_1 \to P(Q_2)$ be a strong set-valued homomorphism and I a prime ideal of Q_2. Then $T_{-}(I)$ is, if it is nonempty, a prime ideal of Q_1.*

Proof. By Theorem 4.9, we get $T_{-}(I)$ is an ideal of Q_1.

Let $a \circ_1 b \in T_{-}(I)$, we have $T(a \circ_1 b) \subseteq I$. Since T is a strong set-valued homomorphism, we have $T(a) \circ_2 T(b) \subseteq I$. We assume that $a \notin T_{-}(I)$, then $T(a) \nsubseteq I$, there exists $x \in T(a)$ but $x \notin I$. If $y \in T(b)$, then $x \circ_2 y \in T(a) \circ_2 T(b) \subseteq I$. Since I is a prime ideal of Q_2, we have $y \in I$. Therefore, $T(b) \subseteq I$ which implies that $b \in T_{-}(I)$. □

We call $I \subseteq Q_2$ is a generalized rough prime ideal of Q_1 if $T_{-}(I)$ and $T^{-}(I)$ are ideals of Q_1.

Theorem 4.15. *Let $T : Q_1 \to P(Q_2)$ be a strong set-valued homomorphism and I a semiprime ideal of Q_2. Then $T_{-}(I)$ is, if it is nonempty, a semiprime ideal of Q_1.*

Proof. By Theorem 4.9, we get $T_-(I)$ is an ideal of Q_1.

Suppose that $a \circ_1 a \in T_-(I)$, we have $T(a \circ_1 a) \subseteq I$. Let $x \in T(a)$, we have $x \circ_2 x \in T(a) \circ_2 T(a) \subseteq T(a \circ_1 a) \subseteq I$. Since I is a semi-prime ideal, we have $x \in I$. So $T(a) \subseteq I$ which implies that $a \in T_-(I)$. □

Theorem 4.16. *Let $T : Q_1 \to P(Q_2)$ be a strong set-valued homomorphism and I a primary ideal of Q_2. Then $T^-(I)$ is, if $T^-(I) \neq \emptyset$ and $T^-(I) \neq Q_1$, a primary ideal of Q_1.*

Proof. By Theorem 4.8, we get $T^-(I)$ is an ideal of Q_1.

Suppose that $a, b \in Q$, $a \circ_1 b \in T^-(I)$ and $a \notin T^-(I)$, there exists $x \in I \cap T(a \circ_1 b)$. Since T is a strong set-valued homomorphism, we have $x \in T(a) \circ_2 T(b)$, there exist $y \in T(a)$, $z \in T(b)$ such that $x = y \circ_2 z \in I$. Since $a \notin T^-(I)$, we get $y \notin I$. Since I is a primary ideal, we have $z^n \in I$ for some $n > 0$ Since T is a strong set-valued homomorphism, we have $z^n \in T(b^n)$. So $T(b^n) \cap I \neq \emptyset$ which implies that $b^n \in T^-(I)$. □

Theorem 4.17. *Let $T : Q_1 \to P(Q_2)$ be a strong set-valued homomorphism and I a primary ideal of Q_2. Let Q_2 be a commutative quantale and $T(x)$ a finite set for each $x \in Q_1$. Then $T_-(I)$ is, if $T_-(I) \neq \emptyset$ and $T_-(I) \neq Q_1$, a primary ideal of Q_1.*

Proof. By Theorem 4.9, we get $T_-(I)$ is an ideal of Q_1.

Suppose $a, b \in Q$, $a \circ_1 b \in T_-(I)$ and $a \notin T_-(I)$, then $T(a \circ_1 b) \subseteq I$ and $T(a) \nsubseteq I$. So there exists $x \in T(a)$ with $x \notin I$. We assume that $T(b) = \{y_1, y_2, \ldots, y_m\}$, then $x \circ_2 y_i \in T(a) \circ_2 T(b) \subseteq T(a \circ_1 b) \subseteq I (i = 1, 2, \ldots, m)$. Since I is a primary ideal, there exists $y_i^{n_i} \in I$ for some $n_i > 0$ $(i = 1, 2, \ldots, m)$. Let $n = \max\{n_i \mid i = 1, 2, \ldots, m\}$. Since I is an ideal, we have $y_i^l \in I$ $(l \geq n)$. Let $z \in T(b^m b^n) = T(b)^{mn}$. Since Q_2 is commutative, there exists $\{i_1, i_2, \ldots, i_s\} \subseteq \{1, 2, \ldots, m\}$ such that $z = y_{i_1}^{t_1} \circ_2 y_{i_2}^{t_2} \circ_2 \cdots \circ_2 y_{i_s}^{t_s}$ with $t_1 + t_2 + \cdots + t_s = mn$, where $t_j > 0$ $(j = 1, 2, \ldots, s)$. Assume that $t_j < n$, $(j = 1, 2, \ldots, s)$, then $t_1 + t_2 + \cdots + t_s < sn \leq mn$. It contradicts with $t_1 + t_2 + \cdots + t_s = mn$. Therefore, there is $1 \leq k \leq s$ such that $t_k \geq n$, $(j = 1, 2, \ldots, s)$, we have $y_{i_k}^{t_k} \in I$. Since I is an ideal, we get $z \in I$. Hence $T(b^{mn}) \subseteq I$ which implies that $b^{mn} \in T_-(I)$. □

Theorem 4.18. *Let $T : Q_1 \to P(Q_2)$ be a set-valued homomorphism. If S is a multiplicative set of Q_2, then $T^-(S)$ is, if it is nonempty, a multiplicative of Q_1. If T is a strong set-valued homomorphism, then $T_-(S)$ is, if it is nonempty, a multiplicative of Q_1.*

Proof. Suppose that $a, b \in T^-(S)$, there exist $x \in T(a) \cap S$, $y \in T(b) \cap S$. Hence $x \circ y \in T(a) \circ_2 T(b) \subseteq T(a \circ_1 b)$. Since S is a multiplicative set, we have $x \circ_2 y \in S$. Therefore, $x \circ_2 y \in T(a \circ_1 b) \cap S$ which implies that $a \circ_1 b \in T^-(S)$.

Suppose $a, b \in T_-(S)$, then $T(a) \subseteq S$, $T(b) \subseteq S$. Let $x \in T(a \circ_1 b)$. Since T is a strong set-valued homomorphism, we have $x \in T(a) \circ_2 T(b)$. There exist $y \in T(a) \subseteq S$, $z \in T(b) \subseteq S$ such that $x = y \circ_2 z$. Since S is a multiplicative set, we have $x = y \circ_2 z \in S$. So $T(a \circ_1 b) \subseteq S$ which implies that $a \circ_1 b \in T_-(S)$. □

We call $A \subseteq Q_2$ a generalized rough multiplicative set (m-system) of Q_1, if $T_-(A)$, $T^-(A)$ are multiplicative sets (m-systems) of Q_1.

Theorem 4.19. *Let $T : Q_1 \to P^*(Q_2)$ be a set-valued homomorphism. If Q_1 is commutative and $A \subseteq Q_2$ with $1 \notin T^-(A)$, then the following statements are equivalent:*

(1) *A is a generalized rough prime ideal of Q_1,*

(2) *A is a generalized rough ideal and A^c is a generalized rough multiplicative set of Q_1,*

(3) *A is a generalized rough ideal and A^c is a generalized rough m-system of Q_1.*

Proof. (1)⇒(2): Since A is a generalized rough prime ideal of Q_1, we get A is a generalized rough ideal of Q_1 and $T_-(A)$, $T^-(A)$ are prime ideals of Q_1. Now we show that $T^-(A^c)$ is a multiplicative set. Let $a, b \in T^-(A^c)$. By Theorem 2.9(1), we have $a, b \in (T_-(A))^c$. Since $T_-(A)$ is a prime ideal, we have $a \circ_1 b \notin T_-(A)$ which implies that $a \circ_1 b \in (T_-(A))^c = T^-(A^c)$. So $T^-(A^c)$ is a multiplicative set. Similarly, $T_-(A^c)$ is a multiplicative set.

(2)⇒(3): Let $a, b \in T^-(A^c)$. Since $1 \notin T^-(A)$ and $T(1) \neq \emptyset$, we have $1 \in T^-(A^c)$. Since $T^-(A^c)$ is a multiplicative set, we have $a \circ_1 1 \circ_1 b \in T^-(A^c)$. So $\downarrow (a \circ_1 1 \circ_1 b) \cap T^-(A^c) \neq \emptyset$. Therefore, $T^-(A^c)$ is an m-system. Similarly, we have $T_-(A^c)$ is a m-system.

(3)⇒(1): Let $a \circ_1 b \in T^-(A)$. Suppose $a \notin T^-(A)$ and $b \notin T^-(A)$, then $a, b \in (T^-(A))^c = T_-(A^c)$. Since $T_-(A^c)$ is an m-system, we have $\downarrow (a \circ_1 1 \circ_1 b) \cap T_-(A^c) \neq \emptyset$. Thus $\downarrow (a \circ_1 1 \circ_1 b) \cap (T^-(A))^c \neq \emptyset$. Since $T^-(A)$ is an ideal and Q_1 is commutative, we have $a \circ_1 1 \circ_1 b \in T^-(A)$ and $\downarrow (a \circ_1 1 \circ_1 b) \subseteq T^-(A)$. It contradicts with $\downarrow (a \circ_1 1 \circ_1 b) \cap (T^-(A))^c \neq \emptyset$. So, $a \in T^-(A)$ or $b \in T^-(A)$. Therefore, $T^-(A)$ is a prime ideal. Similarly, we have $T_-(A)$ as a prime ideal. □

5. Conclusion

The Pawlak rough sets on the algebraic sets such as semigroups, groups, rings, modules, and lattices were mainly studied by a congruence relation. However, the generalized Pawlak rough set was defined for two universes and proposed on generalized binary relations. Can we extended congruence relations to two universes for algebraic sets? Therefore, Davvaz [25] introduced the concept of set-valued homomorphism for groups which is a generalization of the concept of congruence. In this paper, the concepts of set-valued homomorphism and strong set-valued homomorphism of quantales are introduced. We construct generalized lower and upper approximations by means of a set-valued mapping and discuss the properties of them. We obtain that the concept of generalized rough ideal (subquantale) is the extended notion of ideal (subquantale). Many results in [29] are the special case of the results in this paper. It is an interesting research topic of rough set, we will further study it in the future.

Acknowledgments

The authors are grateful to the reviewers for their valuable suggestions to improve the paper. This work was supported by the National Science Foundation of China (no. 11071061).

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