Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2012, Article ID 648067, 17 pages
doi:10.1155/2012/648067

Research Article

Blowup Analysis for a Nonlocal Diffusion Equation with Reaction and Absorption

Yulan Wang,[1] Zhaoyin Xiang,[2] and Jinsong Hu[1]

[1] *School of Mathematics and Computer Engineering, Xihua University, Chengdu 610039, China*
[2] *School of Mathematical Sciences, University of Electronic Science and Technology of China, Chengdu 610054, China*

Correspondence should be addressed to Yulan Wang, wangyulan-math@163.com

Received 21 January 2012; Accepted 20 April 2012

Academic Editor: Junjie Wei

We investigate a nonlocal reaction diffusion equation with absorption under Neumann boundary. We obtain optimal conditions on the exponents of the reaction and absorption terms for the existence of solutions blowing up in finite time, or for the global existence and boundedness of all solutions. For the blowup solutions, we also study the blowup rate estimates and the localization of blowup set. Moreover, we show some numerical experiments which illustrate our results.

1. Introduction

In this paper, we devote our attention to the singularity analysis of the following nonlocal diffusion equation:

$$\begin{gathered} u_t(x,t) = \int_\Omega J(x-y)\big(u(y,t) - u(x,t)\big)dy + u^p(x,t) - ku^q(x,t), \quad x \in \Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad x \in \Omega. \end{gathered} \tag{1.1}$$

Here Ω is a bounded connected and smooth domain, which contains the origin, and $J : \mathbb{R}^N \to \mathbb{R}$ is a nonnegative, bounded, symmetric radially and strictly decreasing function with $\int_{\mathbb{R}^N} J(z)dz = 1$, and p, q, k are all positive constants. We take the initial datum, $u_0(x)$, nonnegative and nontrivial.

Equation in (1.1) is called a nonlocal diffusion equation in the sense that the diffusion of the density u at a point x and time t does not only depend on $u(x,t)$, but on all the values of u in a neighborhood of x through the convolution term. Maybe the simplest linear version of nonlocal model (1.1) is

$$u_t(x,t) = \int_{R^N} J(x-y)\left(u(y,t)-u(x,t)\right)dy = J * u(x,t) - u(x,t). \tag{1.2}$$

In recent years, the linear equation (1.2) and its variations have been widely used to model diffusion process, for example, in biology, dislocations dynamics, phase transition model, material science, and network model and so forth (see [1–7] and the references therein). The idea hidden inside those model is simple to understand. As stated in [6], if $u(x,t)$ is thought of as a density at the point x and time t and $J(x-y)$ is thought of as the probability distribution of jumping from location y to location x, then the convolution $(J * u)(x,t) := \int_{\mathbb{R}^N} J(x-y)u(y,t)dy$ is the rate at which individuals are arriving at x from all other places and $-u(x,t) = -\int_{\mathbb{R}^N} J(x-y)u(x,t)dy$ is the rate at which they are leaving location x to travel to all other sites.

In the past decades, some works have shown that (1.2) shares many properties with the classical heat equation

$$u_t - \Delta u = 0 \tag{1.3}$$

such as the existence of constant bounded stationary solutions and the maximum principle, but there is no regularizing effect in general for the former [8]. Therefore, as mentioned in [9–11], it is an interesting topic to compare the properties of solutions to the nonlocal diffusion equation with that of the corresponding local diffusion cases.

To motivate our works, we would like to remark that, in recent years, (1.3) and its variations have been extensively studied. In particular, a considerable effort has been devoted to studying the blowup properties of the following classical diffusion equation with reaction $(\alpha > 0)$ and/or absorption $(k > 0)$ under Dirichlet or Neumann boundary

$$u_t - \Delta u = \alpha u^p - k u^q, \quad x \in \Omega,\ t > 0, \tag{1.4}$$

which provides a simple biological or physical model. For instance, by constructing the self-similar weak subsolutions, Bedjaoui and Souplet [12] obtained the following conclusion for (1.4) with $\alpha = 1$ under Dirichlet or Neumann boundary: if $p < \max\{q,1\}$, then all solutions are global. If $p > \max\{q,1\}$, there exist solutions of (1.4) which blow up in finite time. In the critical case $p = \max\{q,1\}$, the results may depend on the size of the coefficient k. In the Dirichlet boundary case, Xiang et al. [13] also studied the blowup rate estimates and obtained the following results: if $p > \max\{q,1\}$ and the solution $u(x,t)$ of (1.4) blows up at T, then there exists constants $C > c > 0$ such that

$$\max_{\overline{\Omega}\times[0,t]} u(x,\tau) \ge c(T-t)^{1/(p-1)}, \qquad \max_{\overline{\Omega}\times[0,t]} u(x,\tau) \le C(T-t)^{1/(p-1)} \quad \text{if } 1 < p < 1 + \frac{2}{N+1}. \tag{1.5}$$

In this paper, we will deal with the blowup properties of the nonlocal diffusion problem (1.1) and compare them with that of problem (1.4). In our model (1.1), the absorption term $-ku^q(x,t)$ represents a rate of consumption due to an internal reaction, and we are integrating in Ω and thus imposing the condition that the diffusion takes place only in Ω, which means that the individual may enter or leave the domain. This is so called Neumann boundary conditions, see [14, 15]. We remark that García-Melián and Quirós [16] recently proved the existence of a critical exponent of Fujita type for the nonlocal diffusion problem

$$\begin{aligned} u_t(x,t) &= \int_{R^N} J(x-y)(u(y,t)-u(x,t))dy + u^p(x,t) \\ &= J * u(x,t) - u(x,t) + u^p(x,t). \end{aligned} \tag{1.6}$$

As mentioned in [17], nonlocal diffusion systems are more accurate than the classical diffusion systems in modeling the spatial diffusion of the individuals in some biology areas, such as embryological development process. For more study about the nonlocal diffusion operator, we refer to [8, 18–22] and references therein.

It is noteworthy that the method used in [12, 13] for problem (1.4) is invalid in our current setting due to the appearance of the nonlocal diffusion term. For example, instead of constructing self-similar weak subsolutions, we will use technique of integration to prove the finite time blowup. As far as the blowup rate is concerned, the scaling argument in [13] is not applicable.

We now state our main results. Our first result determines the complete classification of the parameters for which the solution blows up in finite time or exists globally.

Theorem 1.1. (i) *If $p \le \max\{q,1\}$, then all solutions of* (1.1) *are global. Moreover, if $p < q$ or $p = q$ and $k \ge 1$, all solutions are uniformly bounded, while if $p = q$ and $k < 1$, there exist unbounded global solutions.*

(ii) *If $p > \max\{q,1\}$, then* (1.1) *with large initial data have solutions blowing up in finite time, while the solutions of* (1.1) *with small initial data exist globally.*

Once we have obtained the values of the parameters for which blowup occurs, the next step is to concern the blowup rate, that is the speed at which solutions are blowing up. Different from the result of problem (1.4), we could have a unified upper and low estimate here.

Theorem 1.2. *Let $p > \max\{q,1\}$ and $u(x,t)$ be a solution of* (1.1) *blowing up at time T. Then*

$$\lim_{t\to T}(T-t)^{1/(p-1)}\max_{x\in\overline{\Omega}} u(x,t) = (p-1)^{-1/(p-1)}. \tag{1.7}$$

Remark 1.3. From this result we find that the nonlocal diffusion term plays no role when determining the blowup rate and the blowup rate is just same as that of the ODE $u_t = u^p$. And this phenomena is the same as that of local diffusion case.

Next we consider the spacial location of the blowup set. As usual, the blowup set of solution $u(x,t)$ is defined as follows:

$$B(u) = \left\{x \in \overline{\Omega};\ \text{there exist}\,(x_n,t_n) \longrightarrow (x,T)\ \text{such that}\ u(x_n,t_n) \longrightarrow \infty\right\}, \tag{1.8}$$

where T is the maximal existence time of u. For a general domain Ω we can localize the blowup set near any pint in Ω just by taking an initial condition being very large near that point and not so large in the rest of the domain. This is the following result.

Theorem 1.4. *Let $p > \max\{q,2\}$. For any $x_0 \in \Omega$ and $\varepsilon > 0$, there exists an initial data u_0 such that the corresponding solution $u(x,t)$ of* (1.1) *blows up at finite time T and its blowup set $B(u)$ is contained in $B_\epsilon(x_0) = \{x \in \overline{\Omega}; \|x - x_0\| < \epsilon\}$.*

Next we consider the radially symmetric case. In this case, single point blowup occurs.

Theorem 1.5. *Let $p > \max\{q,2\}$ and $\Omega = B_R = \{|x| < R\}$. If the initial data $u_0 \in C^1(\overline{B_R})$ is a radial nonnegative function with a unique maximum at the origin, that is, $u_0 = u_0(r) \geq 0$, $u_0'(r) < 0$ for $0 < r \leq R$, $u'(0) = 0$ and $u_0''(0) < 0$, then the blowup set $B(u)$ of the solution u of* (1.1) *consists only of the original point $x = 0$.*

The remainder of this paper is organized as follows. In Section 2, we give the existence and uniqueness of the solutions as well as the comparison principle. In Section 3, we prove the blowup and global existence condition. And then we prove the blowup rate and blowup set results in Sections 4 and 5, respectively. And in the last section we will give some numerical experiments to demonstrate our results.

2. Existence, Uniqueness, and Comparison Principle

We begin our study of problem (1.1) with a result of existence and uniqueness of continuous solutions and comparison principle.

Firstly, existence and uniqueness of solutions is a consequence of Banach's fixed point theorem. We look for $u \in C^1((0,T);C(\overline{\Omega})) \cap C([0,T);C(\overline{\Omega}))$ satisfying (1.1). Fix $t_0 > 0$, and consider the Banach space $X_{t_0} = u \in C^1((0,T);C(\overline{\Omega})) \cap C([0,T);C(\overline{\Omega}))$ with the norm

$$\|\omega\|_{X_{t_0}} = \max_{0\leq t\leq t_0}\|\omega(\cdot,t)\|_{C(\overline{\Omega})} + \max_{0<t\leq t_0}\|\omega_t(\cdot,t)\|_{C(\overline{\Omega})}. \tag{2.1}$$

We define the following operator $T : X_{t_0} \rightarrow X_{t_0}$

$$\begin{aligned} T_{\omega_0}(\omega)(x,t) = \omega_0(x) &+ \int_0^t\int_\Omega J(x-y)(\omega(y,s)-\omega(x,s))\,dy\,ds \\ &+ \int_0^t\left(|\omega|^{p-1}\omega(x,s) - k|\omega|^{q-1}\omega(x,s)\right)ds. \end{aligned} \tag{2.2}$$

Similar to [10, 15] we could prove the solution to (1.1) is a fixed point of operator T in a convenient ball of X_{t_0}. Thus, we could obtain the following result.

Theorem 2.1. *For every $u_0 \in C(\overline{\Omega})$ there exists a unique solution u of* (1.1) *such that $u \in C^1((0,T);C(\overline{\Omega}))\cap C([0,T);C(\overline{\Omega}))$ and T (finite or infinite) is the maximal existence time of solution.*

Next, we will study the comparison principle. As usual we first give the definition of supersolution and subsolution.

Definition 2.2. A function $\overline{u} \in C^1((0,T);C(\overline{\Omega}))\cap C([0,T);C(\overline{\Omega}))$ is a supersolution of (1.1) if it satisfies

$$\begin{gathered} \overline{u}_t(x,t) \geq \int_\Omega J(x-y)(\overline{u}(y,t)-\overline{u}(x,t))dy + \overline{u}^p(x,t) - k\overline{u}^q(x,t), \\ \overline{u}(x,0) \geq u_0(x). \end{gathered} \tag{2.3}$$

Subsolutions are defined similarly by reversing the inequalities.

To obtain the comparison principle for problem (1.1), we first give a maximum principle.

Lemma 2.3. *Suppose that $w(x,t) \in C^1((0,T);C(\overline{\Omega}))\cap C([0,T);C(\overline{\Omega}))$ is nontrivial and satisfies*

$$w_t(x,t) \geq \int_\Omega J(x-y)(w(y,t)-w(x,t))dy + c_1 w(x,t), \quad x \in \Omega,\ t > 0 \tag{2.4}$$

with $w(x,0) \geq 0$ for $x \in \overline{\Omega}$, where c_1 is a bounded function, then $w(x,t) > 0$ for $x \in \Omega$ and $t > 0$.

Proof. We first show $w(x,t) \geq 0$ for $x \in \Omega, t > 0$. Assume that $w(x,t)$ is negative somewhere. Let $\theta(x,t) = e^{-\lambda t}w(x,t)$ $(\lambda > 0,\ \lambda \geq 2\sup|c_1|)$. If we take (x_0,t_0) a point where θ attains its negative minimum, there holds $t_0 > 0$ and

$$\begin{aligned} \theta_t(x_0,t_0) &= -\lambda e^{-\lambda t_0}w(x_0,t_0) + e^{-\lambda t_0}w_t(x_0,t_0) \\ &\geq e^{-\lambda t_0}\int_\Omega J(x_0-y)(w(y,t_0)-w(x_0,t_0))dy + (-\lambda + c_1)w(x_0,t_0) > 0, \end{aligned} \tag{2.5}$$

which is a contradiction. Thus $\theta(x,t) \geq 0$ for $x \in \Omega,\ t > 0$. And so does $w(x,t)$.

Now, we suppose $\theta(x_1,t_1) = 0$ for some (x_1,t_1); that is, θ attains its minimum at (x_1,t_1) from the first step. Notice that the hypotheses on J imply $J(0) > 0$, so that $\theta(x_1,t_1) = 0$ implies that $\theta(x,t_1) = 0$ for x in a neighborhood of x_1. Thus a standard connectedness argument yields $\theta \equiv 0$. This is a contradiction. So we obtain our conclusion. □

Lemma 2.4. *If $p \geq 1$, $q \geq 1$ and $\overline{u}$, $\underline{u}$ are super and subsolutions to* (1.1), *respectively then $\overline{u}(x,t) \geq \underline{u}(x,t)$ for every $(x,t) \in \overline{\Omega}\times[0,T)$.*

Proof. Letting $w(x,t) = \overline{u} - \underline{u}$, it is easy to verify that $w(x,t)$ satisfies (2.4) when $p \geq 1,\ q \geq 1$. We could obtain our conclusion from Lemma 2.3. □

Remark 2.5. When $p < 1$ or $q < 1$, the conclusion is also validity if $\overline{u}$ and $\underline{u}$ are bounded away from 0.

3. Blowup and Global Existence

In this section, we will analyze the blowup condition and give the proof of Theorem 1.1.

Proof of Theorem 1.1. (i) We only need to look for a global supersolution of (1.1). Indeed, it is easy to construct spacial homogeneous global supersolution of (1.1). To see this, we set $\overline{u} = Ce^{\alpha t}$, where C and α are positive constants to be determined.

For any given initial data u_0, we note that $\overline{u}(t_0) \geq \|u_0\|_\infty$ for t_0 sufficiently large and $\overline{u}$ is bounded away from 0. Thus by the comparison principle and Remark 2.5, to make $\overline{u}$ a supersolution of (1.1), we only need to show the existence of C and α satisfying

$$C^p e^{p\alpha t} \leq kC^q e^{q\alpha t} + \alpha C e^{\alpha t}. \tag{3.1}$$

If $p \leq 1 < q$, for any given α, we can take $C = k^{1/(p-q)}$ such that (3.1) holds.

If $q \leq 1$ and thus $p \leq 1$, we can choose C and α satisfying $C^{p-1} = \alpha$, which make (3.1) validity.

Next, we show all global solutions are uniformly bounded when $p < q$ or $p = q$ and $k \geq 1$. In fact, (1.1) has constant supersolution $\overline{u} = A$ whenever $p < q$ or $p = q$ and $k \geq 1$. To see this, we choose A large enough such that

$$kA^q \geq A^p, \quad A \geq \|u_0\|_\infty, \tag{3.2}$$

which imply that $\overline{u}$ is a supersolution of (1.1).

At last we show there exist global unbounded solutions when $p = q$ and $k < 1$. Indeed, let

$$f(t) = \begin{cases} \left((1-k)(1-p)t + f^{1-p}(0)\right)^{1/(1-p)}, & p = q < 1, \\ e^{((1-k)/2)t}, & p = q = 1. \end{cases} \tag{3.3}$$

It is easy to see that if $f(0) \leq \max_\Omega u_0(x)$, $f(t)$ is a subsolution of (1.1). It is obvious that when $p < 1$, $f(t)$ is unbounded.

(ii) We first show that if the initial data $u_0(x)$ is large enough, solutions of (1.1) blow up in finite time.

In the case of $p > q > 1$. Integrating $(1.1)_1$ in Ω and applying Fubini's theorem, we get

$$\frac{d}{dt}\int_\Omega u(x,t)dx = \int_\Omega u^p(x,t)dx - k\int_\Omega u^q(x,t)dx. \tag{3.4}$$

Using Hölder's inequality, we could get

$$\begin{aligned}\frac{d}{dt}\int_\Omega u(x,t)dx &\geq \int_\Omega u^p(x,t)dx - k|\Omega|^{(p-q)/p}\left(\int_\Omega u^p(x,t)dx\right)^{q/p} \\ &= \left(\int_\Omega u^p(x,t)dx\right)^{q/p}\left[\left(\int_\Omega u^p(x,t)dx\right)^{(p-q)/p} - k|\Omega|^{(p-q)/p}\right],\end{aligned} \tag{3.5}$$

where $|\Omega|$ is assumed to be the measure of Ω. Given positive constant $m > k^{1/(p-q)}$ and $u_0 \geq m$, we have by the comparison principle that the solution $u(x,t)$ of problem (1.1) satisfies $u(x,t) \geq m$. Thus

$$\frac{d}{dt}\int_\Omega u(x,t)dx \geq \left(\int_\Omega u^p(x,t)dx\right)^{q/p}\left(m^{p-q}|\Omega|^{(p-q)/p} - k|\Omega|^{(p-q)/p}\right). \tag{3.6}$$

Then we use Jensen's inequality to obtain

$$\frac{d}{dt}\int_\Omega u(x,t)dx > C\left(\int_\Omega u(x,t)dx\right)^q, \tag{3.7}$$

where C is a positive constant independent of the solution u. From this inequality, we could easily obtain that $u(x,t)$ blow up in finite time.

In the case of $p > 1 \geq q$, it follows from $u^q \leq u+1$ and Jensen's inequality that

$$\begin{aligned}\int_\Omega u^p(x,t)dx - k\int_\Omega u^q(x,t)dx &\geq \int_\Omega u^p(x,t)dx - k\int_\Omega u(x,t)dx - k|\Omega| \\ &\geq |\Omega|^{1-p}\left(\int_\Omega u(x,t)dx\right)^p - k\int_\Omega u(x,t)dx - k|\Omega|.\end{aligned} \tag{3.8}$$

Substituting this inequality into (3.4), we obtain

$$\frac{d}{dt}\int_\Omega u(x,t)dx \geq |\Omega|^{1-p}\left(\int_\Omega u(x,t)dx\right)^p - k\int_\Omega u(x,t)dx - k|\Omega|. \tag{3.9}$$

Therefore, if we take the initial data u_0 large enough such that $|\Omega|^{1-p}(\int_\Omega u_0(x)dx)^p - k\int_\Omega u_0(x)dx - k|\Omega| > 0$, then $\int_\Omega u(x,t)dx$ blows up in finite time. So does $u(x,t)$.

Next we show when the initial data $u_0(x)$ is small, solutions of (1.1) exist globally. Consider constant B. Let $0 < B \leq k^{1/(p-q)}$. Then $B_t \geq B^p - kB^q$. Henceforth, if $u_0(x) \leq B$, B is a supersolution of (1.1). From the comparison principle, we know solutions of (1.1) are global in this case. □

4. Blowup Rate Estimate

In this section, we study the blowup rate and prove Theorem 1.2.

Proof of Theorem 1.2. Let $U(t) = u(x(t),t) = \max_{x\in\overline{\Omega}} u(x,t)$. It is easy to see that $U(t)$ is Lipschitz continuous and thus it is differential almost everywhere [23]. From the first equality of (1.1) we have

$$U'(t) \le \int_{\Omega} J(x-y)(u(y,t)-u(x(t),t))dy + u^p(x(t),t) - ku^q(x(t),t) \le u^p(x(t),t) \tag{4.1}$$

at any point of differentiability of $U(t)$. Here we used $\nabla u(x(t),t) = 0$. Noticing that $p > 1$ and integrating (4.1) from t to T, we obtain

$$\max_{x\in\overline{\Omega}} u(x,t) \ge (p-1)^{-1/(p-1)}(T-t)^{-1/(p-1)}. \tag{4.2}$$

Next we will establish the upper estimate. For any $(x,t) \in \overline{\Omega} \times [0,T)$, we have

$$u_t(x,t) \ge -u(x,t) + u^p(x,t) - ku^q(x,t) = u^p(x,t)\left(1 - u^{-(p-1)}(x,t) - ku^{-(p-q)}(x,t)\right). \tag{4.3}$$

In particular,

$$U'(t) \ge U^p(t)\left(1 - U(t)^{-(p-1)} - kU(t)^{-(p-q)}\right). \tag{4.4}$$

From the lower estimate (4.2) we get

$$U'(t) \ge U^p(t)\left(1 - (p-1)(T-t) - k(p-1)^{(p-q)/(p-1)}(T-t)^{(p-q)/(p-1)}\right). \tag{4.5}$$

Integrating in (t,T), we get

$$\max_{x\in\overline{\Omega}} u(x,t) \le \left((p-1)(T-t) - \frac{(p-1)^2}{2}(T-t)^2 - k\frac{(p-1)^{(3p-q-2)/(p-1)}}{2p-q-1}(T-t)^{(2p-q-1)/(p-1)}\right)^{-1/(p-1)}, \tag{4.6}$$

combining with (4.2), the conclusion of Theorem 1.2 is proved if one takes the limit as $t \to T$. □

5. Blowup Set

Next we will concern the blowup set for the solution to problem (1.1). We will first localize the blowup set near any point in $\overline{\Omega}$ just by taking an initial condition being very large near that point and not so large in the rest of the domain.

Proof of Theorem 1.4. Given $x_0 \in \overline{\Omega}$ and $\varepsilon > 0$, we could construct an initial condition u_0 such that

$$B(u) \subset B_\varepsilon(x_0) = \left\{ x \in \overline{\Omega} : \|x - x_0\| < \varepsilon \right\}. \tag{5.1}$$

In fact, we will consider u_0 concentrated near x_0 and small away from x_0.

Let φ be a nonnegative smooth function such that $\operatorname{supp}(\varphi) \subset B_{\varepsilon/2}(x_0)$ and $\varphi(x) > 0$ for $x \in B_{\varepsilon/2}(x_0)$.

Next, let

$$u_0(x) = M\varphi(x) + \delta. \tag{5.2}$$

We want to choose M large and δ small such that (5.1) holds.

First we can assume that T is as small as we need by taking M large enough. Indeed, we have

$$T \le \frac{C(\Omega, p, \varphi)}{M^{q-1}} \quad \text{or} \quad T \le \frac{C(\Omega, p, \varphi)}{M^{p-1}} \tag{5.3}$$

from the proof of Theorem 1.1.

Now, from the proof of blowup rate, we have

$$\begin{aligned}
&\max_{x \in \overline{\Omega}} u(x,t) \\
&\quad \le \left((p-1)(T-t) - \frac{(p-1)^2}{2}(T-t)^2 - k\frac{(p-1)^{(3p-q-2)/(p-1)}}{2p-q-1}(T-t)^{(2p-q-1)/(p-1)} \right)^{-1/(p-1)} \\
&\quad \le C(T-t)^{-1/(p-1)}.
\end{aligned} \tag{5.4}$$

Henceforth, for any $\overline{x} \in \Omega$,

$$\begin{aligned}
u_t(\overline{x}, t) &= \int_\Omega J(x,y)(u(y,t) - u(\overline{x},t))dy + u^p(\overline{x},t) - ku^q(\overline{x},t) \\
&\le \int_\Omega J(\overline{x},y)u(y,t)dy + u^p(\overline{x},t) \le C(T-t)^{-1/(p-1)} + u^p(\overline{x},t).
\end{aligned} \tag{5.5}$$

Therefore, if $\overline{x} \in \overline{\Omega} \setminus B_\varepsilon(x_0)$, then $u(\overline{x}, t)$ is a subsolution to

$$\begin{aligned}
w_t &= C(T-t)^{-1/(p-1)} + w^p(t), \\
w(0) &= \delta,
\end{aligned} \tag{5.6}$$

which shows

$$u(\overline{x}, t) \leq w(t). \tag{5.7}$$

Next, we only need to prove that a solution w to (5.6) remains bounded up to $t = T$, provided that δ and T are small enough.

Let

$$z(s) = (T-t)^{1/(p-1)} w(t), \quad s = -\ln(T-t). \tag{5.8}$$

Then $z(s)$ satisfies

$$z'(s) = Ce^{-s} - \frac{1}{p-1} z(s) + z^p(s), \quad z(-\ln T) = \delta T^{1/(p-1)}, \tag{5.9}$$

which show that for T and δ small (T is small if M is large), we have

$$CT - \frac{1}{p-1}\delta T^{1/(p-1)} + \delta^p T^{p/(p-1)} < 0. \tag{5.10}$$

So $z'(s) < 0$ for all $s > -\ln T$. From this and Lemma 4.2 of [24], we know

$$z(s) \longrightarrow 0, \qquad s \longrightarrow \infty. \tag{5.11}$$

Combining the equation verified by z we obtain that, for given positive constant $\gamma(< 1/p(p-1))$, there exists $s_0 > 0$ such that

$$z'(s) \leq Ce^{-s} - \left(\frac{1}{p-1} - \gamma\right) z(s) \tag{5.12}$$

for $s > s_0$.

Let $v(s)$ be a solution of

$$v'(s) = Ce^{-s} - \left(\frac{1}{p-1} - \gamma\right) v(s) \tag{5.13}$$

with $v(s_0) \geq z(s_0)$. Integrating this equation we get

$$v(s) = C_1 e^{-s} + C_2 e^{-(1/(p-1)-\gamma)s}. \tag{5.14}$$

By a comparison argument we could get that for every $s > s_0$,

$$z(s) \leq v(s) = C_1 e^{-s} + C_2 e^{-(1/(p-1)-\gamma)s}. \tag{5.15}$$

Now we go back to $z'(s) = Ce^{-s} - (1/(p-1))z(s) + z^p(s)$. We have

$$z'(s) + \left(1/\left(p-1\right)\right)z(s) = Ce^{-s} + z^p(s), \tag{5.16}$$

then

$$\left(e^{(1/(p-1))s}z(s)\right)' = e^{(1/(p-1))s}\left(Ce^{-s} + z^p\right). \tag{5.17}$$

Integrating form s_0 to s, one could get

$$\begin{aligned} z(s) &= e^{-(1/(p-1))s}\left(C_1 + \int_{s_0}^{s} e^{(1/(p-1))\sigma}\left(Ce^{-\sigma} + z^p(s)\right)d\sigma\right) \\ &= e^{-(1/(p-1))s}\left(C_1 + \int_{s_0}^{s} e^{-((p-2)/(p-1))\sigma}\left(C + e^{\sigma}z^p(s)\right)d\sigma\right). \end{aligned} \tag{5.18}$$

Using (5.15) and $\gamma < 1/p(p-1)$, we have

$$e^s z^p \le C_1^p e^{-(p-1)s} + C_2^p e^{-(p/(p-1)-p\gamma-1)s} \longrightarrow 0 \tag{5.19}$$

as $s \longrightarrow +\infty$.

And thus from (5.18), we get

$$\begin{aligned} z(s) &\le e^{-(1/(p-1))s}\left(C_1 + C_3\int_{s_0}^{s} e^{-((p-2)/(p-1))\sigma}d\sigma\right) \\ &\le C_1e^{-(1/(p-1))s} + C_4e^{-s}. \end{aligned} \tag{5.20}$$

As $p > 2$, we have

$$z(s) \le Ce^{-(1/(p-1))}s. \tag{5.21}$$

This implies that $w(t) \le C$, for $0 \le t < T$. From the boundedness of w and (5.7) we get $u(\overline{x}, t) \le w(t) \le C$ for every $\overline{x} \in \overline{\Omega} \setminus B_\epsilon(x_0)$, as we wished. □

Next, we will consider the radial symmetric case, that is, the proof of Theorem 1.5. For the convenience of writing, we only deal with the one dimensional case, $\Omega = (-l, l)$. The radial case is analogous; we leave the details to the reader.

Proof of Theorem 1.5. Under the hypothesis on the initial condition imposed in Theorem 1.5 we have that the solution is symmetric and $u_x < 0$ in $(0, l] \times (0, T)$ from the stand parabolic theorem and Lemma 4.1 of [10]. Therefore the solution has a unique maximum at the origin for every $t \in (0, T)$.

Let us perform the following change of variables

$$z(x,s) = (T-t)^{1/(p-1)}u(x,t), \quad s = -\ln(T-t). \tag{5.22}$$

Our remainder proof consist of two steps.

Step 1. We first prove the only blowup point that verifies the blowup estimate (1.7) is $x = 0$. And this shows that for $x \neq 0$, $z(x,s)$ does not converge to $C_p = (p-1)^{-1/(p-1)}$ as $s \to +\infty$.

We conclude by contradiction. Assume that $(T-t)^{1/(p-1)}u(x_0,t) \to C_p$ for a $x_0 > 0$.

Let $v(t) = u(0,t) - u(x_0,t)$. Then

$$\begin{aligned} v'(t) = &\int_{-l}^{l} J(-y)(u(y,t)-u(0,t))dy - \int_{-l}^{l} J(x_0-y)(u(y,t)-u(x_0,t))dy \\ &+ p\xi^{p-1}(t)v(t) - kq\eta^{q-1}(t)v(t), \end{aligned} \tag{5.23}$$

where $\xi(t)$ and $\eta(t)$ are between $u(0,t)$ and $u(x_0,t)$. Hence

$$\begin{aligned} v'(t) \geq &\int_{-l}^{l} (J(-y)-J(x_0-y))u(y,t)dy \\ &+ \int_{-l}^{l} (J(y-x_0)-J(y))u(0,t)dy - v(t) + p\xi^{p-1}(t)v(t) - kq\eta^{q-1}(t)v(t) \\ = &\int_{-l}^{l} (J(y-x_0)-J(y))(u(0,t)-u(y,t))dy - v(t) + p\xi^{p-1}(t)v(t) - kq\eta^{q-1}(t)v(t) \\ \geq &\left(-C_1 + p\xi^{p-1}(t) - kq\eta^{q-1}(t)\right)v(t), \end{aligned} \tag{5.24}$$

for some positive constant.

Integrating the above inequality, we obtain

$$\ln(v)(t) - \ln(v)(t_0) \geq \int_{t_0}^{t} \left(-C_1 + p\xi^{p-1}(s) - kq\eta^{q-1}(s)\right)ds. \tag{5.25}$$

Remember that $(T-t)^{1/(p-1)}u(x_0,t) \to C_p$, $(T-t)^{1/(p-1)}u(0,t) \to C_p$, we have

$$\lim_{t\to T}\xi(t)(T-t)^{1/(p-1)} = \lim_{t\to T}\eta(t)(T-t)^{1/(p-1)} = C_p. \tag{5.26}$$

And this implies that

$$\int_{t_0}^{t}\left(-C_1+p\xi^{p-1}(s)-kq\eta^{q-1}(s)\right)ds \geq p\int_{t_0}^{t}\frac{C_p^{p-1}-\delta_1}{T-s}ds - kq\int_{t_0}^{t}\left(C_p^{q-1}+\delta_2\right)(T-s)^{-(q-1)/(p-1)}ds - C_2. \tag{5.27}$$

$p>q$ implies that $(T-s)^{-(q-1)/(p-1)} \leq \delta_3(T-s)^{-1}$ as $s \to T$ for given $\delta_3>0$. Hence

$$\begin{aligned}\int_{t_0}^{t}\left(-C_1+p\xi^{p-1}(s)-kq\eta^{q-1}(s)\right)ds &\geq p\int_{t_0}^{t}\frac{C_p^{p-1}-\delta}{T-s}ds - C_2\\ &= -p\left(C_p^{p-1}-\delta\right)\ln(T-t)-C_2\end{aligned} \tag{5.28}$$

for some $\delta>0$.

Hence

$$v(t) \geq C(T-t)^{-p(C_p^{p-1}-\delta)} = C(T-t)^{p\delta-p/(p-1)}. \tag{5.29}$$

Using this fact, we have

$$0=\lim_{t\to T}(T-t)^{1/(p-1)}v(t) \geq C\lim_{t\to T}(T-t)^{1/(p-1)-p/(p-1)+p\delta}=+\infty. \tag{5.30}$$

This contradiction proves our claim.

Step 2. We will show the only possible blowup point is $x=0$.

Remembering the transform (5.22), $z(x,s)$ satisfies

$$z_s = e^{-s}\int_{-l}^{l}J(x-y)\left(z(y,s)-z(x,s)\right)dy - \frac{1}{p-1}z + z^p - ke^{((q-p)/(p-1))s}z^q. \tag{5.31}$$

Note that the blowup rate of u implies that $z(x,s) \leq C$ for every $(x,s) \in [-l,l]\times(-\ln T,\infty)$. Therefore,

$$z_s(x,s) \leq Ce^{-s} - \frac{1}{p-1}z(x,s) + z^p(x,s). \tag{5.32}$$

From this we know that if there exists s_0 such that $z^p(x,s_0)-(1/(p-1))z(x,s_0) < -Ce^{-s_0}$, then $z(x,s) \to 0$ as $s \to \infty$ (see Lemma 4.2 in [24]).

Moreover, if there exists s_0 such that $z^p(x,s_0)-(1/(p-1))z(x,s_0) > Ce^{-s_0}$ then $z(x,s)$ blows up in finite time $\overline{s}$. This follows from Lemma 4.3 of [24] using that

$$z_s(x,s) \geq -Ce^{-s} - \frac{1}{p-1}z(x,s) + z^p(x,s). \tag{5.33}$$

Thus if $z(x,s)$ does not converge to zero and does not blow up in finite time, then $z(x,s)$ satisfies

$$Ce^{-s} \geq z^p(x,s) - \frac{1}{p-1} z(x,s) \geq -Ce^{-s}. \tag{5.34}$$

Henceforth,

$$z^p(x,s) - \frac{1}{p-1} z(x,s) \longrightarrow 0 \quad (s \longrightarrow +\infty). \tag{5.35}$$

As $z(x,s)$ is continuous, bounded and does not go to zero, we conclude that $z(x,s) \to C_p$.

Now we could conclude that $z(x,s)$ verifies $z(x,s) \to 0$ $(s \to +\infty)$, or $z(x,s) \to C_p$ $(s \to +\infty)$, or $z(x,s)$ blows up in finite time.

From Step 1 we know for $x \neq 0$, $z(x,s)$ is bounded and does not converge to C_p, so $z(x,s) \to 0$ as $s \to +\infty$. Combined with inequality (5.32), we could get

$$z_s(x,s) \leq Ce^{-s} - \left(\frac{1}{p-1} - \theta\right) z(x,s) \tag{5.36}$$

for any $\theta > 0$.

By a comparison argument as in the proof of Theorem 1.4, it follows that

$$z(x,s) \leq C_1 e^{-s} + C_2 e^{-(1/(p-1)-\theta)s}. \tag{5.37}$$

Going back to the equation verified by $z(x,t)$ we obtain

$$\begin{aligned} \left(e^{(1/(p-1))s} z(x,s)\right)_s = e^{(1/(p-1))s} \Bigg(& e^{-s} \int_{-l}^{l} J(x-y)\big(z(y,s) - z(x,s)\big)dy \\ & + z^p(x,s) - ke^{((q-p)/(p-1))s} z(x,s) \Bigg). \end{aligned} \tag{5.38}$$

Integrating we get

$$\begin{aligned} z(x,s) = e^{-(1/(p-1))s} \Bigg(C_1 + \int_{s_0}^{s} e^{-((p-2)/(p-1))\sigma} \Bigg(& \int_{-l}^{l} J(x-y)\big(z(y,s) - z(x,s)\big)dy \\ & + e^{\sigma} z^p(x,s) - ke^{((q-1)/(p-1))\sigma} z(x,s) d\sigma \Bigg) \Bigg). \end{aligned} \tag{5.39}$$

On the other hand, (5.37) implies that $e^s z^p(x,s) \to 0$ as $s \to \infty$. Henceforth,

$$z(x,s) \le e^{-(1/(p-1))s}\left(C_1 + C_2 \int_{s_0}^{s} e^{-((p-2)/(p-1))\sigma} d\sigma\right). \tag{5.40}$$

Using that $p > 2$, one could have

$$z(x,s) \le C_3 e^{-(1/(p-1))s}. \tag{5.41}$$

Remembering the transform (5.22), we have

$$u(x,t) = e^{(1/(p-1))s} z(x,s) \le c_3. \tag{5.42}$$

And so our proof is complete. □

6. Numerical Experiments

At the end of this paper, we will use several numerical examples to demonstrate our results about the location of blowup points. For this purpose, we discretize the problem in the spacial variable to obtain an ODE system. Taking $\Omega = [-4,4]$ and $-4 = x_{-N} < \cdots < x_N = 4$, $N = 100$, we consider the following system:

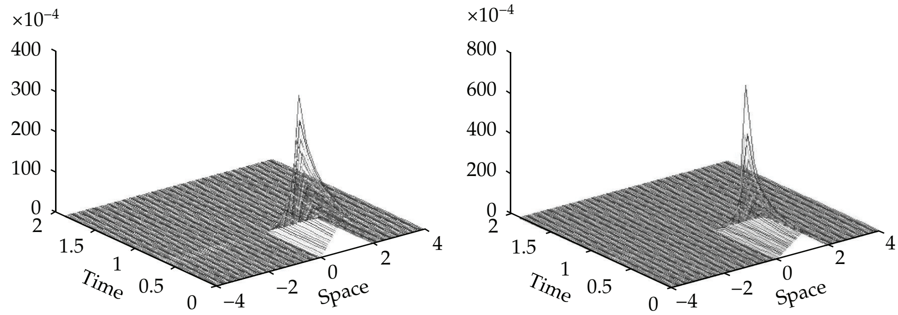
$$\begin{gathered} u_i'(t) = \sum_{j=-N}^{N} J(x_i - x_j)\left(u_j(t) - u_i(t)\right) + (u_i)^p(t) - k(u_i)^q(t), \\ u_i(0) = u_0(x_i). \end{gathered} \tag{6.1}$$

Next we choose $p = 3$, $q = 1$, $k = 1$ and

$$J(z) = \begin{cases} 1, & |z| \le \dfrac{1}{10}, \\ 0, & |z| > \dfrac{1}{10}. \end{cases} \tag{6.2}$$

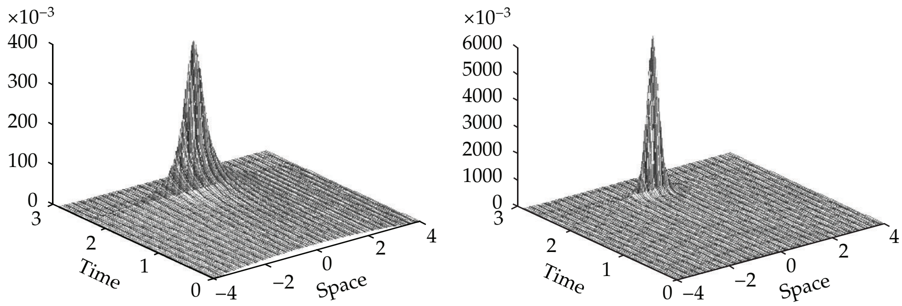
In Figure 1 we choose a nonsymmetric initial condition very large near the point $x_0 = 1$, $u_0(x) = 1/4 + 100(1-|x-1|)_+$. We observe that the blowup set is localized in a neighborhood of $x_0 = 1$.

Next we choose a symmetric initial condition with a unique maximum at the origin, $u_0(x) = 16 - x_0^2$. We observe that the solution blows up only at the origin, Figure 2.



(a) (b)

Figure 1: Evolution in time, nonsymmetric datum.



(a) (b)

Figure 2: Evolution in time, symmetric datum.

Acknowledgments

Y. Wang is supported by the Key Scientific Research Foundation of Xihua University (no. Z0912611), the Scientific Research Found of Sichuan Provincial Education Department (no. 09ZB081) and the Research Fund of Key Disciplinary of Application Mathematics of Xihua University (no. XZD0910-09-1). Z. Xiang is supported by NNSF of China (11101068), the Sichuan Youth Science & Technology Foundation (2011JQ0003), the Fundamental Research Funds for the Central Universities (ZYGX2009X019) and the SRF for ROCS, SEM. J. Hu is supported by the Scientific Research Found of Sichuan Provincial Education Department (no. 11ZB009).

References

[1] P. W. Bates, P. C. Fife, X. Ren, and X. Wang, “Traveling waves in a convolution model for phase transitions,” *Archive for Rational Mechanics and Analysis*, vol. 138, no. 2, pp. 105–136, 1997.

[2] P. W. Bates, “On some nonlocal evolution equations arising in materials science,” in *Nonlinear Dynamics and Evolution Equations*, H. Brunner, X. Zhao, and X. Zou, Eds., vol. 48 of *Fields Institute Communications*, pp. 13–52, American Mathematical Society, Providence, RI, USA, 2006.

[3] P. W. Bates and J. Han, “The Dirichlet boundary problem for a nonlocal Cahn-Hilliard equation,” *Journal of Mathematical Analysis and Applications*, vol. 311, no. 1, pp. 289–312, 2005.

[4] P. W. Bates and J. Han, "The Neumann boundary problem for a nonlocal Cahn-Hilliard equation," *Journal of Differential Equations*, vol. 212, no. 2, pp. 235–277, 2005.

[5] G. B. Ermentrout and J. B. McLeod, "Existence and uniqueness of travelling waves for a neural network," *Proceedings of the Royal Society of Edinburgh A*, vol. 123, no. 3, pp. 461–478, 1993.

[6] P. Fife, "Some nonclassical trends in parabolic and parabolic-like evolutions," in *Trends in Nonlinear Analysis*, pp. 153–191, Springer, Berlin, Germany, 2003.

[7] C. Lederman and N. Wolanski, "Singular perturbation in a nonlocal diffusion problem," *Communications in Partial Differential Equations*, vol. 31, no. 1–3, pp. 195–241, 2006.

[8] E. Chasseigne, M. Chaves, and J. D. Rossi, "Asymptotic behavior for nonlocal diffusion equations," *Journal de Mathématiques Pures et Appliquées. Neuvième Série*, vol. 86, no. 3, pp. 271–291, 2006.

[9] M. Bogoya, R. Ferreira, and J. D. Rossi, "A nonlocal nonlinear diffusion equation with blowing up boundary conditions," *Journal of Mathematical Analysis and Applications*, vol. 337, no. 2, pp. 1284–1294, 2008.

[10] M. Pérez-Llanos and J. D. Rossi, "Blow-up for a non-local diffusion problem with Neumann boundary conditions and a reaction term," *Nonlinear Analysis: Theory, Methods & Applications A*, vol. 70, no. 4, pp. 1629–1640, 2009.

[11] J. Terra and N. Wolanski, "Asymptotic behavior for a nonlocal diffusion equation with absorption and nonintegrable initial data: the supercritical case," *Proceedings of the American Mathematical Society*, vol. 139, no. 4, pp. 1421–1432, 2011.

[12] N. Bedjaoui and P. Souplet, "Critical blowup exponents for a system of reaction-diffusion equations with absorption," *Zeitschrift für Angewandte Mathematik und Physik*, vol. 53, no. 2, pp. 197–210, 2002.

[13] Z. Xiang, Q. Chen, and C. Mu, "Blow-up rate estimates for a system of reaction-diffusion equations with absorption," *Journal of the Korean Mathematical Society*, vol. 44, no. 4, pp. 779–786, 2007.

[14] C. Cortazar, M. Elgueta, J. D. Rossi, and N. Wolanski, "How to approximate the heat equation with Neumann boundary conditions by nonlocal diffusion problems," *Archive for Rational Mechanics and Analysis*, vol. 187, no. 1, pp. 137–156, 2008.

[15] C. Cortazar, M. Elgueta, J. D. Rossi, and N. Wolanski, "Boundary fluxes for nonlocal diffusion," *Journal of Differential Equations*, vol. 234, no. 2, pp. 360–390, 2007.

[16] J. García-Melián and F. Quirós, "Fujita exponents for evolution problems with nonlocal diffusion," *Journal of Evolution Equations*, vol. 10, no. 1, pp. 147–161, 2010.

[17] J. D. Murray, *Mathematical Biology: I. An introduction*, vol. 17 of *Interdisciplinary Applied Mathematics*, Springer, New York, NY, USA, 3rd edition, 2002.

[18] M. Bogoya, "A nonlocal nonlinear diffusion equation in higher space dimensions," *Journal of Mathematical Analysis and Applications*, vol. 344, no. 2, pp. 601–615, 2008.

[19] A. Buades, B. Coll, and J. M. Morel, "Image denoising methods—a new nonlocal principle," *SIAM Review*, vol. 52, no. 1, pp. 113–147, 2010.

[20] C. Cortazar, M. Elgueta, and J. D. Rossi, "Nonlocal diffusion problems that approximate the heat equation with Dirichlet boundary conditions," *Israel Journal of Mathematics*, vol. 170, pp. 53–60, 2009.

[21] L. I. Ignat and J. D. Rossi, "Decay estimates for nonlocal problems via energy methods," *Journal de Mathématiques Pures et Appliquées*, vol. 92, no. 2, pp. 163–187, 2009.

[22] Y. Lou, X. Zhang, S. Osher, and A. Bertozzi, "Image recovery via nonlocal operators," *Journal of Scientific Computing*, vol. 42, no. 2, pp. 185–197, 2010.

[23] A. Friedman and B. McLeod, "Blow-up of positive solutions of semilinear heat equations," *Indiana University Mathematics Journal*, vol. 34, no. 2, pp. 425–447, 1985.

[24] P. Groisman and J. D. Rossi, "Asymptotic behaviour for a numerical approximation of a parabolic problem with blowing up solutions," *Journal of Computational and Applied Mathematics*, vol. 135, no. 1, pp. 135–155, 2001.