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Research Article

New Fixed Point Results of Single-Valued Mapping for *c*-Distance in Cone Metric Spaces

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A new concept of the *c*-distance in cone metric space has been introduced recently in 2011. The aim of this paper is to extend and generalize some fixed point results in literature for *c*-distance in cone metric spaces by replacing the constants in contractive conditions with functions. Some supporting examples are given.

1. Introduction

The concept of cone metric spaces is a generalization of metric spaces, where each pair of points is assigned to a member of a real Banach space with a cone, for new results on cone metric spaces see [1–8]. This cone naturally induces a partial order in the Banach spaces. The concept of cone metric space was introduced in the work of Huang and Zhang [9] where they also established the Banach contraction mapping principle in this space. Then, several authors have studied fixed point problems in cone metric spaces. Some of these works are noted in [10–15].

In [16], Cho et al. introduced a new concept of the *c*-distance in cone metric spaces (also see [17]) and proved some fixed point theorems in ordered cone metric spaces. This is more general than the classical Banach contraction mapping principle. Then, Sintunavarat et al. [18] extended and developed the Banach contraction theorem on *c*-distance of Cho et al. [16]. They gave some illustrative examples of main results. Their results improve, generalize, and unify the results of Cho et al. [16] and some results of the fundamental metrical fixed point theorems in the literature, for some new results for *c*-distance see [19–24].

In [21], Fadail et al. proved the following theorems for *c*-distance in cone metric spaces.

Theorem 1.1. *Let (X, d) be a complete cone metric space and q is a c-distance on X. Suppose the mapping $f : X \to X$ satisfies the following contractive condition:*

$$q(fx, fy) \preceq kq(x, y), \tag{1.1}$$

for all $x, y \in X$, where $k \in [0, 1)$ is a constant. Then f has a fixed point $x^* \in X$ and for any $x \in X$, iterative sequence $\{f^n x\}$ converges to the fixed point. If $v = fv$, then $q(v, v) = \theta$. The fixed point is unique.

Theorem 1.2. *Let (X, d) be a complete cone metric space and q is a c-distance on X. Suppose the mapping $f : X \to X$ is continuous and satisfies the following contractive condition:*

$$q(fx, fy) \preceq kq(x, y) + lq(x, fx) + rq(y, fy), \tag{1.2}$$

for all $x, y \in X$, where k, l, r are none negative real numbers such that $k+l+r < 1$. Then f has a fixed point $x^* \in X$ and for any $x \in X$, iterative sequence $\{f^n x\}$ converges to the fixed point. If $v = fv$, then $q(v, v) = \theta$. The fixed point is unique.

Theorem 1.3. *Let (X, d) be a complete cone metric space and q is a c-distance on X. Suppose the mapping $f : X \to X$ satisfies the folowing contractive condition:*

$$(1 - r)q(fx, fy) \preceq kq(x, fy) + lq(x, fx), \tag{1.3}$$

for all $x, y \in X$, where k, l, r are none negative real numbers such that $2k + l + r < 1$. Then f has a fixed point $x^* \in X$ and for any $x \in X$, iterative sequence $\{f^n x\}$ converges to the fixed point. If $v = fv$, then $q(v, v) = \theta$. The fixed point is unique.

The aim of this paper is to continue the study of common coupled fixed points of mappings but now for c-distance in cone metric space. Our results extend and develop some theorems on c-distance of Fadail et al. [21]. In this paper, we do not impose the normality condition for the cones; the only assumption is that the cone P is solid, that is, $\operatorname{int} P \neq \emptyset$.

2. Preliminaries

Let E be a real Banach space and θ denote to the zero element in E. A cone P is a subset of E such that

(1) P is nonempty set closed and $P \neq \{\theta\}$,

(2) if a, b are nonnegative real numbers and $x, y \in P$, then $ax + by \in P$,

(3) $x \in P$ and $-x \in P$ implies $x = \theta$.

For any cone $P \subset E$, the partial ordering $\preceq$ with respect to P is defined by $x \preceq y$ if and only if $y - x \in P$. The notation of $\prec$ stand for $x \preceq y$ but $x \neq y$. Also, we used $x \ll y$ to indicate that $y - x \in \operatorname{int} P$, where $\operatorname{int} P$ denotes the interior of P. A cone P is called normal if there exists a number K such that

$$\theta \preceq x \preceq y \Longrightarrow \|x\| \leq K\|y\|, \tag{2.1}$$

for all $x, y \in E$. The least positive number K satisfying the above condition is called the normal constant of P. It is clear that $K \geq 1$.

Definition 2.1 (see [9]). Let X be a nonempty set and E a real Banach space equipped with the partial ordering $\preceq$ with respect to the cone P. Suppose that the mapping $d : X \times X \to E$ satisfies the following conditions:

(1) $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$,

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$,

(3) $d(x, y) \preceq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

Definition 2.2 (see [9]). Let (X, d) be a cone metric space, $\{x_n\}$ a sequence in X, and $x \in X$.

(1) For all $c \in E$ with $\theta \ll c$, if there exists a positive integer N such that $d(x_n, x) \ll c$ for all $n > N$, then x_n is said to be convergent and x is the limit of $\{x_n\}$. We denote this by $x_n \to x$.

(2) For all $c \in E$ with $\theta \ll c$, if there exists a positive integer N such that $d(x_n, x_m) \ll c$ for all $n, m > N$, then $\{x_n\}$ is called a Cauchy sequence in X.

(3) A cone metric space (X, d) is called complete if every Cauchy sequence in X is convergent.

Lemma 2.3 (see [25]). *(1) If E be a real Banach space with a cone P and $a \preceq \lambda a$ where $a \in P$ and $0 \leq \lambda < 1$, then $a = \theta$.*

(2) If $c \in \operatorname{int} P$, $\theta \preceq a_n$ and $a_n \to \theta$, then there exists a positive integer N such that $a_n \ll c$ for all $n \geq N$.

Next we give the notation of c-distance on a cone metric space which is a generalization of ω-distance of Kada et al. [26] with some properties.

Definition 2.4 (see [16]). Let (X, d) be a cone metric space. A function $q : X \times X \to E$ is called a c-distance on X if the following conditions hold:

(q1) $\theta \preceq q(x, y)$ for all $x, y \in X$,

(q2) $q(x, y) \preceq q(x, y) + q(y, z)$ for all $x, y, z \in X$,

(q3) for each $x \in X$ and $n \geq 1$, if $q(x, y_n) \preceq u$ for some $u = u_x \in P$, then $q(x, y) \preceq u$ whenever $\{y_n\}$ is a sequence in X converging to a point $y \in X$,

(q4) for all $c \in E$ with $\theta \ll c$, there exists $e \in E$ with $\theta \ll e$ such that $q(z, x) \ll e$ and $q(z, y) \ll e$ imply $d(x, y) \ll c$.

Example 2.5 (see [16]). Let $E = \mathbb{R}$ and $P = \{x \in E : x \geq 0\}$. Let $X = [0, \infty)$ and define a mapping $d : X \times X \to E$ by $d(x, y) = |x - y|$ for all $x, y \in X$. Then (X, d) is a cone metric space. Define a mapping $q : X \times X \to E$ by $q(x, y) = y$ for all $x, y \in X$. Then q is a c-distance on X.

Lemma 2.6 (see [16]). *Let (X, d) be a cone metric space and q is a c-distance on X. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X and $x, y, z \in X$. Suppose that u_n is a sequences in P converging to θ. Then the following hold.*

(1) *If $q(x_n, y) \preceq u_n$ and $q(x_n, z) \preceq u_n$, then $y = z$.*

(2) *If $q(x_n, y_n) \preceq u_n$ and $q(x_n, z) \preceq u_n$, then $\{y_n\}$ converges to z.*

(3) *If $q(x_n, x_m) \preceq u_n$ for $m > n$, then $\{x_n\}$is a Cauchy sequence in X.*

(4) *If $q(y, x_n) \preceq u_n$, then $\{x_n\}$ is a Cauchy sequence in X.*

Remark 2.7 (see [16]). (1) $q(x, y) = q(y, x)$ does not necessarily for all $x, y \in X$.
(2) $q(x, y) = \theta$ is not necessarily equivalent to $x = y$ for all $x, y \in X$.

3. Main Results

In this section, we generalize some fixed point results from [21] by replacing the constants in contractive conditions with functions.

Theorem 3.1. *Let (X, d) be a complete cone metric space and q is a c-distance on X. Let $f : X \to X$ be a mapping and suppose that there exists mapping $k : X \to [0, 1)$ such that the following hold:*

(a) *$k(fx) \leq k(x)$ for all $x \in X$,*

(b) *$q(fx, fy) \preceq k(x)q(x, y)$ for all $x, y \in X$.*

Then f has a fixed point $x^* \in X$ and for any $x \in X$, iterative sequence $\{f^n x\}$ converges to the fixed point. If $v = fv$, then $q(v, v) = \theta$. The fixed point is unique.

Proof. Choose $x_0 \in X$. Set $x_1 = fx_0, x_2 = fx_1 = f^2 x_0, \ldots, x_{n+1} = fx_n = f^{n+1} x_0$. Then we have

$$\begin{aligned} q(x_n, x_{n+1}) &= q(fx_{n-1}, fx_n) \\ &\preceq k(x_{n-1}) q(x_{n-1}, x_n) \\ &= k(fx_{n-2}) q(x_{n-1}, x_n) \\ &\preceq k(x_{n-2}) q(x_{n-1}, x_n) \\ &\quad \cdots \\ &\preceq k(x_0) q(x_{n-1}, x_n) \\ &\preceq k^2(x_0) q(x_{n-2}, x_{n-1}) \\ &\preceq \cdots \\ &\preceq (k(x_0))^n q(x_0, x_1). \end{aligned} \tag{3.1}$$

Let $m > n \geq 1$. Then it follows that

$$\begin{aligned} q(x_n, x_m) &\preceq q(x_n, x_{n+1}) + q(x_{n+1}, x_{n+2}) + \cdots + q(x_{m-1}, x_m) \\ &\preceq \left((k(x_0))^n + (k(x_0))^{n+1} + \cdots + (k(x_0))^{m-1}\right) q(x_0, x_1) \\ &\preceq \frac{(k(x_0))^n}{1 - k(x_0)} q(x_0, x_1). \end{aligned} \tag{3.2}$$

Thus, Lemma 2.6(3) shows that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. By q3, we have

$$q(x_n, x^*) \preceq \frac{(k(x_0))^n}{1 - k(x_0)} q(x_0, x_1). \tag{3.3}$$

On the other hand, we have

$$\begin{aligned} q(x_n, fx^*) &= q(fx_{n-1}, fx^*) \\ &\preceq k(x_{n-1}) q(x_{n-1}, x^*) \\ &= k(fx_{n-2}) q(x_{n-1}, x^*) \\ &\preceq k(x_{n-2}) q(x_{n-1}, x^*) \\ &\quad \cdots \\ &\preceq k(x_0) q(x_{n-1}, x^*) \\ &\preceq k(x_0) \frac{(k(x_0))^{n-1}}{1 - k(x_0)} q(x_0, x_1) \\ &= \frac{(k(x_0))^n}{1 - k(x_0)} q(x_0, x_1). \end{aligned} \tag{3.4}$$

By Lemma 2.6(3), (3.3), and (3.4), we have $x^* = fx^*$. Thus, x^* is a fixed point of f.

Suppose that $v = fv$, then we have $q(v, v) = q(fv, fv) \preceq k(v) q(v, v)$. Since $k(v) < 1$, Lemma 2.3(1) shows that $q(v, v) = \theta$.

Finally, suppose there is another fixed point y^* of f, then we have: $q(x^*, y^*) = q(fx^*, fy^*) \preceq k(x^*) q(x^*, y^*)$. Since $k(x^*) < 1$, Lemma 2.3(1) shows that $q(x^*, y^*) = \theta$, and also we have $q(x^*, x^*) = \theta$, hence by Lemma 2.6(1), $x^* = y^*$. Therefore, the fixed point is unique. □

In the above theorem, if $k(x)$ is constant, then we have the following corollary.

Corollary 3.2 ([21, theorem 3.1]). *Let (X, d) be a complete cone metric space and q is a c-distance on X. Suppose the mapping $f : X \to X$ satisfies the following contractive condition:*

$$q(fx, fy) \preceq k q(x, y), \tag{3.5}$$

for all $x, y \in X$, where $k \in [0,1)$ is a constant. Then f has a fixed point $x^* \in X$ and for any $x \in X$, iterative sequence $\{f^n x\}$ converges to the fixed point. If $v = fv$, then $q(v,v) = \theta$. The fixed point is unique.

Theorem 3.3. *Let (X, d) be a complete cone metric space and q is a c-distance on X. Let $f : X \to X$ be a continuous mapping and suppose that there exists mapping $k, l, r : X \to [0,1)$ such that the following hold:*

(a) $k(fx) \le k(x)$, $l(fx) \le l(x)$, $r(fx) \le r(x)$ *for all* $x \in X$,

(b) $(k + l + r)(x) < 1$ *for all* $x \in X$,

(c) $q(fx, fy) \preceq k(x)q(x,y) + l(x)q(x,fx) + r(x)q(y,fy)$ *for all* $x, y \in X$.

Then f has a fixed point $x^* \in X$ and for any $x \in X$, iterative sequence $\{f^n x\}$ converges to the fixed point. If $v = fv$, then $q(v,v) = \theta$. The fixed point is unique.

Proof. Choose $x_0 \in X$. Set $x_1 = fx_0, x_2 = fx_1 = f^2x_0, \ldots, x_{n+1} = fx_n = f^{n+1}x_0$. Then we have

$$\begin{aligned} q(x_n, x_{n+1}) &= q(fx_{n-1}, fx_n) \\ &\preceq k(x_{n-1})q(x_{n-1}, x_n) + l(x_{n-1})q(x_{n-1}, fx_{n-1}) + r(x_{n-1})q(x_n, fx_n) \\ &= k(fx_{n-2})q(x_{n-1}, x_n) + l(fx_{n-2})q(x_{n-1}, x_n) + r(fx_{n-2})q(x_n, x_{n+1}) \\ &\preceq k(x_{n-2})q(x_{n-1}, x_n) + l(x_{n-2})q(x_{n-1}, x_n) + r(x_{n-2})q(x_n, x_{n+1}) \\ &\quad \cdots \\ &\preceq k(x_0)q(x_{n-1}, x_n) + l(x_0)q(x_{n-1}, x_n) + r(x_0)q(x_n, x_{n+1}). \end{aligned} \tag{3.6}$$

So

$$\begin{aligned} q(x_n, x_{n+1}) &\preceq \frac{k(x_0) + l(x_0)}{1 - r(x_0)} q(x_{n-1}, x_n) \\ &= hq(x_{n-1}, x_n) \\ &\preceq h^2 q(x_{n-2}, x_{n-1}) \\ &\preceq \cdots \\ &\preceq h^n q(x_0, x_1), \end{aligned} \tag{3.7}$$

where $h = (k(x_0) + l(x_0))/(1 - r(x_0)) < 1$.

Let $m > n \ge 1$. Then it follows that

$$\begin{aligned} q(x_n, x_m) &\preceq q(x_n, x_{n+1}) + q(x_{n+1}, x_{n+2}) + \cdots + q(x_{m-1}, x_m) \\ &\preceq \left(h^n(x_0) + h^{n+1}(x_0) + \cdots + h^{m-1}(x_0)\right) q(x_0, x_1) \\ &\preceq \frac{h^n(x_0)}{1 - h(x_0)} q(x_0, x_1). \end{aligned} \tag{3.8}$$

Thus, Lemma 2.6(3) shows that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. Since f is continuous, then $x^* = \lim x_{n+1} = \lim f(x_n) = f(\lim x_n) = f(x^*)$. Therefore x^* is a fixed point of f.

Suppose that $v = fv$, then $q(v,v) = q(fv,fv) \preceq k(x_0)q(v,v) + l(x_0)q(v,fv) + r(x_0)q(v,fv) = (k+l+r)(x_0)q(v,v)$. Since $(k+l+r)(x_0) < 1$, Lemma 2.3(1) shows that $q(v,v) = \theta$.

Finally, suppose there is another fixed point y^* of f, then we have

$$\begin{aligned} q(x^*,y^*) &= q(fx^*,fy^*) \\ &\preceq k(x^*)q(x^*,y^*) + l(x^*)q(x^*,fx^*) + r(x^*)q(y^*,fy^*) \\ &= k(x^*)q(x^*,y^*) + l(x^*)q(x^*,x^*) + r(x^*)q(y^*,y^*) \\ &= k(x^*)q(x^*,y^*) \\ &\preceq k(x^*)q(x^*,y^*) + l(x^*)q(x^*,y^*) + r(x^*)q(x^*,y^*) \\ &= (k+l+r)(x^*)q(x^*,y^*). \end{aligned} \tag{3.9}$$

Since $(k+l+r)(x_0) < 1$, Lemma 2.3(1) shows that $q(x^*,y^*) = \theta$, and also we have $q(x^*,x^*) = \theta$, hence by Lemma 2.6(1), $x^* = y^*$. Therefore, the fixed point is unique. □

In Theorem 3.3, if $k(x), l(x)$, and $r(x)$ are constant, then we have the following corollary.

Corollary 3.4 ([21, theorem 3.3]). *Let (X,d) be a complete cone metric space and q is a c-distance on X. Suppose the mapping $f : X \to X$ is continuous and satisfies the following contractive condition:*

$$q(fx,fy) \preceq kq(x,y) + lq(x,fx) + rq(y,fy), \tag{3.10}$$

for all $x,y \in X$, where k, l, r are none negative real numbers such that $k+l+r < 1$. Then f has a fixed point $x^* \in X$ and for any $x \in X$, iterative sequence $\{f^n x\}$ converges to the fixed point. If $v = fv$, then $q(v,v) = \theta$. The fixed point is unique.

Theorem 3.5. *Let (X,d) be a complete cone metric space and q is a c-distance on X. Let $f : X \to X$ be a mapping and suppose that there exists mapping $k,l,r : X \to [0,1)$ such that the following hold:*

(a) $k(fx) \le k(x)$, $l(fx) \le l(x)$, $r(fx) \le r(x)$ *for all* $x \in X$,

(b) $(2k+l+r)(x) < 1$ *for all* $x \in X$,

(c) $(1-r(x))q(fx,fy) \preceq k(x)q(x,fy) + l(x)q(x,fx)$ *for all* $x,y \in X$.

Then f has a fixed point $x^* \in X$ and for any $x \in X$, iterative sequence $\{f^n x\}$ converges to the fixed point. If $v = fv$, then $q(v,v) = \theta$. The fixed point is unique.

Proof. Choose $x_0 \in X$. Set $x_1 = fx_0, x_2 = fx_1 = f^2x_0, \ldots, x_{n+1} = fx_n = f^{n+1}x_0$. Observe that

$$(1-r(x))q(fx,fy) \preceq k(x)q(x,fy) + l(x)q(x,fx), \tag{3.11}$$

equivalently

$$q(fx, fy) \preceq k(x)q(x, fy) + l(x)q(x, fx) + r(x)q(fx, fy). \tag{3.12}$$

Then we have

$$\begin{aligned} q(x_n, x_{n+1}) &= q(fx_{n-1}, fx_n) \\ &\preceq k(x_{n-1})q(x_{n-1}, fx_n) + l(x_{n-1})q(x_{n-1}, fx_{n-1}) + r(x_{n-1})q(fx_{n-1}, fx_n) \\ &= k(fx_{n-2})q(x_{n-1}, x_{n+1}) + l(fx_{n-2})q(x_{n-1}, x_n) + r(fx_{n-2})q(x_n, x_{n+1}) \\ &\preceq k(x_{n-2})q(x_{n-1}, x_{n+1}) + l(x_{n-2})q(x_{n-1}, x_n) + r(x_{n-2})q(x_n, x_{n+1}) \\ &\quad \cdots \\ &\preceq k(x_0)q(x_{n-1}, x_{n+1}) + l(x_0)q(x_{n-1}, x_n) + r(x_0)q(x_n, x_{n+1}) \\ &\preceq k(x_0)q(x_{n-1}, x_n) + k(x_0)q(x_n, x_{n+1}) + l(x_0)q(x_{n-1}, x_n) + r(x_0)q(x_n, x_{n+1}). \end{aligned} \tag{3.13}$$

So

$$\begin{aligned} q(x_n, x_{n+1}) &\preceq \frac{k(x_0) + l(x_0)}{1 - k(x_0) - r(x_0)} q(x_{n-1}, x_n) \\ &= hq(x_{n-1}, x_n) \\ &\preceq h^2 q(x_{n-2}, x_{n-1}) \\ &\preceq \cdots \\ &\preceq h^n q(x_0, x_1), \end{aligned} \tag{3.14}$$

where $h = (k(x_0) + l(x_0))/(1 - k(x_0) - r(x_0)) < 1$.

Let $m > n \geq 1$. Then it follows that

$$\begin{aligned} q(x_n, x_m) &\preceq q(x_n, x_{n+1}) + q(x_{n+1}, x_{n+2}) + \cdots + q(x_{m-1}, x_m) \\ &\preceq \left(h^n + h^{n+1} + \cdots + h^{m-1}\right) q(x_0, x_1) \\ &\preceq \frac{h^n}{1-h} q(x_0, x_1). \end{aligned} \tag{3.15}$$

Thus, Lemma 2.6(3) shows that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$.

By q3, we have

$$q(x_n, x^*) \preceq \frac{h^n}{1-h} q(x_0, x_1). \tag{3.16}$$

On the other hand, we have

$$
\begin{aligned}
q(x_n, fx^*) &= q(fx_{n-1}, fx^*) \\
&\preceq k(x_{n-1})q(x_{n-1}, fx^*) + l(x_{n-1})q(x_{n-1}, fx_{n-1}) + r(x_{n-1})q(fx_{n-1}, fx^*) \\
&= k(fx_{n-2})q(x_{n-1}, fx^*) + l(fx_{n-2})q(x_{n-1}, x_n) + r(fx_{n-2})q(x_n, x_{n+1}) \\
&\preceq k(x_{n-2})q(x_{n-1}, fx^*) + l(x_{n-2})q(x_{n-1}, x_n) + r(x_{n-2})q(x_n, x_{n+1}) \\
&\quad \cdots \\
&\preceq k(x_0)q(x_{n-1}, fx^*) + l(x_0)q(x_{n-1}, x_n) + r(x_0)q(x_n, x_{n+1}) \\
&\preceq k(x_0)q(x_{n-1}, x_n) + k(x_0)q(x_n, fx^*) + l(x_0)q(x_{n-1}, x_n) + r(x_0)q(x_n, fx^*).
\end{aligned}
\tag{3.17}
$$

So

$$
\begin{aligned}
q(x_n, fx^*) &\preceq \frac{k(x_0) + l(x_0)}{1 - k(x_0) - r(x_0)} q(x_{n-1}, x_n) \\
&\preceq \frac{k(x_0) + l(x_0)}{1 - k(x_0) - r(x_0)} h^{n-1} q(x_0, x_1) \\
&= hh^{n-1} q(x_0, x_1) \\
&= h^n q(x_0, x_1) \\
&\preceq \frac{h^n}{1 - h} q(x_0, x_1).
\end{aligned}
\tag{3.18}
$$

By Lemma 2.6(1), (3.16), and (3.18), we have $x^* = fx^*$. Thus, x^* is a fixed point of f.

Suppose that $v = fv$, then we have

$$
\begin{aligned}
q(v, v) &= q(fv, fv) \\
&\preceq k(v)q(v, fv) + l(v)q(v, fv) + r(v)q(v, fv) \\
&= k(v)q(v, v) + l(v)q(v, v) + r(v)q(v, v) \\
&\preceq k(v)q(v, v) + k(v)q(v, v) + l(v)q(v, v) + r(v)q(v, v) \\
&= (2k + l + r)(v)q(v, v).
\end{aligned}
\tag{3.19}
$$

Since $(2k + l + r)(v) < 1$, Lemma 2.3 (1) shows that $q(v, v) = \theta$.

Finally, suppose there is another fixed point y^* of f, then we have

$$\begin{aligned}
q(x^*, y^*) &= q(fx^*, fy^*) \\
&\preceq k(x^*)q(x^*, fy^*) + l(x^*)q(x^*, fx^*) + r(x^*)q(fx^*, fy^*) \\
&\preceq k(x^*)q(x^*, fy^*) + k(x^*)q(x^*, fy^*) + l(x^*)q(x^*, fx^*) + r(x^*)q(fx^*, fy^*) \\
&= k(x^*)q(x^*, y^*) + k(x^*)q(x^*, y^*) + l(x^*)q(x^*, x^*) + r(x^*)q(x^*, y^*) \\
&= k(x^*)q(x^*, y^*) + k(x^*)q(x^*, y^*) + r(x^*)q(x^*, y^*) \\
&\preceq k(x^*)q(x^*, y^*) + k(x^*)q(x^*, y^*) + l(x^*)q(x^*, y^*) + r(x^*)q(x^*, y^*) \\
&= (2k + l + r)(x^*)q(x^*, y^*).
\end{aligned} \tag{3.20}$$

Since $(2k+l+r)(x^*) < 1$, Lemma 2.3(1) shows that $q(x^*, y^*) = \theta$ and also we have $q(x^*, x^*) = \theta$, hence by Lemma 2.6(1), $x^* = y^*$. Therefore, the fixed point is unique. □

In Theorem 3.5, if $k(x), l(x)$, and $r(x)$ are constants, then we have the following corollary.

Corollary 3.6 ([21, Theorem 3.5]). *Let (X, d) be a complete cone metric space and q is a c-distance on X. Suppose the mapping $f : X \to X$ satisfies the following contractive condition:*

$$(1 - r)q(fx, fy) \preceq kq(x, fy) + lq(x, fx), \tag{3.21}$$

for all $x, y \in X$, where k, l, r are none negative real numbers such that $2k + l + r < 1$. Then f has a fixed point $x^* \in X$ and for any $x \in X$, iterative sequence $\{f^n x\}$ converges to the fixed point. If $v = fv$, then $q(v, v) = \theta$. The fixed point is unique.

Example 3.7. Let $E = \mathbb{R}$ and $P = \{x \in E : x \geq 0\}$. Let $X = [0,1]$ and define a mapping $d : X \times X \to E$ by $d(x, y) = |x - y|$ for all $x, y \in X$. Then (X, d) is a complete cone metric space. Define a mapping $q : X \times X \to E$ by $q(x, y) = y$ for all $x, y \in X$. Then q is a c-distance on X. Define the mapping $f : X \to X$ by $fx = x^2/4$ for all $x \in X$. Take $k(x) = (x + 2)/4$, $x \in X$. Observe that

(a) $k(fx) = ((x^2/4) + 2)/4 = (1/4)((x^2/4) + 2) \leq (1/4)(x + 2) = k(x)$ for all $x \in X$.

(b) For all $x \in X$, we have

$$\begin{aligned}
q(fx, fy) &= fy \\
&= \frac{y^2}{4} \\
&\preceq \frac{y}{2} \\
&\preceq \left(\frac{x}{4} + \frac{1}{2}\right)y
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{4}(x+2)y \\
&= k(x)y \\
&= k(x)q(x,y).
\end{aligned} \tag{3.22}$$

Therefore, the conditions of Theorem 3.1 are satisfied. Hence f has a unique fixed point $x = 0$ with $q(0,0) = 0$.

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References

[1] S. Janković, Z. Kadelburg, and S. Radenović, "On Cone Metric Spaces: a survey," *Nonlinear Analysis*, vol. 74, no. 7, pp. 2591–2601, 2011.

[2] Z. Kadelburg, S. Radenović, and V. Rakočević, "Topological vector space-valued Cone Metric Spaces and fixed point theorems," *Fixed Point Theory and Applications*, vol. 2010, Article ID 170253, 17 pages, 2010.

[3] Z. Kadelburg and S. Radenović, "Coupled fixed point results under tvs-Cone Metric Spaces and w-cone-distance," *Advances Fixed Point Theory*, vol. 2, no. 1, 2012.

[4] L. Ćirić, H. Lakzian, and V. Rakočević, "Fixed point theorems for w-cone distance contraction mappings in tvs-Cone Metric Spaces," *Fixed Point Theory and Applications*, vol. 2012, article 3, 2012.

[5] Z. Kadelburg and S. Radenović, "Some common fixed point results in non-normal Cone Metric Spaces," *Journal of Nonlinear Science and its Applications*, vol. 3, no. 3, pp. 193–202, 2010.

[6] Z. Kadelburg, S. Radenović, and V. Rakočević, "A note on the equivalence of some metric and cone metric fixed point results," *Applied Mathematics Letters*, vol. 24, no. 3, pp. 370–374, 2011.

[7] X. Zhang, "Fixed point theorem of generalized quasi-contractive mapping in cone metric space," *Computers & Mathematics with Applications*, vol. 62, no. 4, pp. 1627–1633, 2011.

[8] D. Đorić, Z. Kadelburg, and S. Radenović, "Coupled fixed point for mappings without mixed monotone property," *Applied Mathematics Letters*, vol. 25, no. 11, pp. 1803–1808, 2012.

[9] L.-G. Huang and X. Zhang, "Cone Metric Spaces and fixed point theorems of contractive mappings," *Journal of Mathematical Analysis and Applications*, vol. 332, no. 2, pp. 1468–1476, 2007.

[10] M. Abbas and G. Jungck, "Common fixed point results for noncommuting mappings without continuity in Cone Metric Spaces," *Journal of Mathematical Analysis and Applications*, vol. 341, no. 1, pp. 416–420, 2008.

[11] M. Abbas and B. E. Rhoades, "Fixed and periodic point results in Cone Metric Spaces," *Applied Mathematics Letters*, vol. 22, no. 4, pp. 511–515, 2009.

[12] A. Azam and M. Arshad, "Common fixed points of generalized contractive maps in Cone Metric Spaces," *Iranian Mathematical Society. Bulletin*, vol. 35, no. 2, pp. 255–264, 2009.

[13] D. Ilić and V. Rakočević, "Common fixed point for maps on cone metric space," *Journal of Mathematical Analysis and Applications*, vol. 341, pp. 876–882, 2008.

[14] D. Ilić and V. Rakočević, "Quasi-contraction on a cone metric space," *Applied Mathematics Letters*, vol. 22, no. 5, pp. 728–731, 2009.

[15] D. Wardowski, "Endpoints and fixed points of set-valued contractions in Cone Metric Spaces," *Nonlinear Analysis*, vol. 71, no. 1-2, pp. 512–516, 2009.

[16] Y. J. Cho, R. Saadati, and S. Wang, "Common fixed point theorems on generalized distance in ordered Cone Metric Spaces," *Computers & Mathematics with Applications*, vol. 61, no. 4, pp. 1254–1260, 2011.
[17] S. Wang and B. Guo, "Distance in Cone Metric Spaces and common fixed point theorems," *Applied Mathematics Letters*, vol. 24, no. 10, pp. 1735–1739, 2011.
[18] W. Sintunavarat, Y. J. Cho, and P. Kumam, "Common fixed point theorems for *c*-distance in ordered Cone Metric Spaces," *Computers & Mathematics with Applications*, vol. 62, no. 4, pp. 1969–1978, 2011.
[19] M. Đorđević, D. Đorić, Z. Kadelburg, S. Radenović, and D. Spasić, "Fixed point results under *c*-distance in tvs-Cone Metric Spaces," *Fixed Point Theory and Applications*, vol. 2011, article 29, 2011.
[20] Y. J. Cho, Z. Kadelburg, R. Saadati, and W. Shatanawi, "Coupled fixed point theorems under weak contractions," *Discrete Dynamics in Nature and Society*, vol. 2012, Article ID 184534, 9 pages, 2012.
[21] Z. M. Fadail, A. G. B. Ahmad, and Z. Golubović, "Fixed Point Theorems of single-valued mapping for *c*-distance in Cone Metric Spaces," *Abstract and Applied Analysis*, vol. 2012, Article ID 826815, 11 pages, 2012.
[22] Z. M. Fadail and A. G. B. Ahmad, "Coupled fixed point theorems of single-valued mapping for *c*-distance in Cone Metric Spaces," *Journal of Applied Mathematics*, vol. 2012, Article ID 246516, 20 pages, 2012.
[23] Z. M. Fadail and A. G. B. Ahmad, "Common coupled fixed point theorems of single-valued mapping for *c*-distance in Cone Metric Spaces," *Abstract and Applied Analysis*, vol. 2012, Article ID 901792, 24 pages, 2012.
[24] W. Shatanawi, E. Karapinar, and H. Aydi, "Coupled coincidence points in partially ordered cone metric spaces with *c*-distance," *Journal of Applied Mathematics*, vol. 2012, Article ID 312078, 15 pages, 2012.
[25] G. Jungck, S. Radenović, S. Radojević, and V. Rakočević, "Common fixed point theorems for weakly compatible pairs on Cone Metric Spaces," *Fixed Point Theory and Applications*, vol. 2009, Article ID 643840, 13 pages, 2009.
[26] O. Kada, T. Suzuki, and W. Takahashi, "Nonconvex minimization theorems and fixed point theorems in complete metric spaces," *Mathematica Japonica*, vol. 44, no. 2, pp. 381–391, 1996.