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Research Article

Iterative Methods for the Sum of Two Monotone Operators

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We introduce an iterative for finding the zeros point of the sum of two monotone operators. We prove that the suggested method converges strongly to the zeros point of the sum of two monotone operators.

1. Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \to H$ be a single-valued nonlinear mapping and let $B : H \to 2^H$ be a multivalued mapping. The "so-called" quasi-variational inclusion problem is to find a $u \in 2^H$ such that

$$0 \in Ax + Bx. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $(A+B)^{-1}(0)$. A number of problems arising in structural analysis, mechanics, and economics can be studied in the framework of this kind of variational inclusions; see, for instance, [1–4]. The problem (1.1) includes many problems as special cases.

(1) If $B = \partial\phi : H \to 2^H$, where $\phi : H \to R \cup +\infty$ is a proper convex lower semicontinuous function and $\partial\phi$ is the subdif and if onlyerential of ϕ, then the variational inclusion problem (1.1) is equivalent to find $u \in H$ such that

$$\langle Au, y-u \rangle + \phi(y) - \phi(u) \geq 0, \quad \forall y \in H, \tag{1.2}$$

which is called the mixed quasi-variational inequality (see, Noor [5]).

(2) If $B = \partial\delta_C$, where C is a nonempty closed convex subset of H and $\delta_C : H \to [0, \infty]$ is the indicator function of C, that is,

$$\delta_C = \begin{cases} 0, & x \in C, \\ +\infty, & x \notin C, \end{cases} \tag{1.3}$$

then the variational inclusion problem (1.1) is equivalent to find $u \in C$ such that

$$\langle Au, v - u \rangle \geq 0, \quad \forall v \in C. \tag{1.4}$$

This problem is called Hartman-Stampacchia variational inequality (see, e.g., [6]).

Recently, Zhang et al. [7] introduced a new iterative scheme for finding a common element of the set of solutions to the inclusion problem, and the set of fixed points of nonexpansive mappings in Hilbert spaces. Peng et al. [8] introduced another iterative scheme by the viscosity approximate method for finding a common element of the set of solutions of a variational inclusion with set-valued maximal monotone mapping and inverse strongly monotone mappings, the set of solutions of an equilibrium problem, and the set of fixed points of a nonexpansive mapping. For some related works, please see [9–27] and the references therein.

Inspired and motivated by the works in the literature, in this paper, we introduce an iterative for solving the problem (1.1). We prove that the suggested method converges strongly to the zeros point of the sum of two monotone operators $A + B$.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. Let C be a nonempty closed convex subset of H. Recall that a mapping $A : C \to H$ is said to be α-inverse strongly-monotone if and if only

$$\langle Ax - Ay, x - y \rangle \geq \alpha \| Ax - Ay \|^2 \tag{2.1}$$

for some $\alpha > 0$ and for all $x, y \in C$. It is known that if A is α-inverse strongly monotone, then

$$\| Ax - Ay \| \leq \frac{1}{\alpha} \| x - y \| \tag{2.2}$$

for all $x, y \in C$.

Let B be a mapping of H into 2^H. The effective domain of B is denoted by $\mathrm{dom}(B)$, that is,

$$\mathrm{dom}(B) = \{ x \in H : Bx \neq \emptyset \}. \tag{2.3}$$

A multivalued mapping B is said to be a monotone operator on H if and if only

$$\langle x - y, u - v \rangle \geq 0 \tag{2.4}$$

for all $x, y \in \mathrm{dom}(B)$, $u \in Bx$, and $v \in By$. A monotone operator B on H is said to be maximal if and only if its graph is not strictly contained in the graph of any other monotone operator on H. Let B be a maximal monotone operator on H and let $B^{-1}0 = \{x \in H : 0 \in Bx\}$.

For a maximal monotone operator B on H and $\lambda > 0$, we may define a single-valued operator:

$$J_\lambda^B = (I + \lambda B)^{-1} : H \longrightarrow \mathrm{dom}(B), \tag{2.5}$$

which is called the resolvent of B for λ. It is known that the resolvent J_λ^B is firmly nonexpansive, that is,

$$\left\| J_\lambda^B x - J_\lambda^B y \right\|^2 \leq \left\langle J_\lambda^B x - J_\lambda^B y, x - y \right\rangle \tag{2.6}$$

for all $x, y \in C$ and $B^{-1}0 = F(J_\lambda^B)$ for all $\lambda > 0$.

The following resolvent identity is well known: for $\lambda > 0$ and $\mu > 0$, there holds the following identity:

$$J_\lambda^B x = J_\mu^B \left(\frac{\mu}{\lambda} x + \left(1 - \frac{\mu}{\lambda} \right) J_\lambda^B x \right), \quad x \in H. \tag{2.7}$$

We use the following notation:

(i) $x_n \rightharpoonup x$ stands for the weak convergence of (x_n) to x;

(ii) $x_n \longrightarrow x$ stands for the strong convergence of (x_n) to x.

We need the following lemmas for the next section.

Lemma 2.1 (see [28]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let the mapping $A : C \longrightarrow H$ be α-inverse strongly monotone and let $\lambda > 0$ be a constant. Then, one has*

$$\|(I - \lambda A)x - (I - \lambda A)y\|^2 \leq \|x - y\|^2 + \lambda(\lambda - 2\alpha)\|Ax - Ay\|^2, \quad \forall x, y \in C. \tag{2.8}$$

In particular, if $0 \leq \lambda \leq 2\alpha$, then $I - \lambda A$ is nonexpansive.

Lemma 2.2 (see [29]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with*

$$0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1. \tag{2.9}$$

Suppose that

$$x_{n+1} = (1 - \beta_n) y_n + \beta_n x_n \tag{2.10}$$

for all $n \geq 0$ and

$$\limsup_{n \to \infty} \left(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\| \right) \leq 0. \tag{2.11}$$

Then, $\lim_{n \to \infty} \|y_n - x_n\| = 0$.

Lemma 2.3 (see [30]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\gamma_n)a_n + \delta_n\gamma_n, \tag{2.12}$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(1) $\sum_{n=1}^{\infty}\gamma_n = \infty$;

(2) $\limsup_{n\to\infty}\delta_n \leq 0$ *or* $\sum_{n=1}^{\infty}|\delta_n\gamma_n| < \infty$.

Then $\lim_{n\to\infty}a_n = 0$.

3. Main Results

In this section, we will prove our main result.

Theorem 3.1. *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let A be an α-inverse strongly monotone mapping of C into H and let B be a maximal monotone operator on H, such that the domain of B is included in C. Let $J_\lambda^B = (I+\lambda B)^{-1}$ be the resolvent of B for $\lambda > 0$. Suppose that $(A+B)^{-1}0 \neq \emptyset$. For $u \in C$ and given $x_0 \in C$, let $\{x_n\} \subset C$ be a sequence generated by*

$$x_{n+1} = \beta_n x_n + (1-\beta_n)J_{\lambda_n}^B(\alpha_n u + (1-\alpha_n)(x_n - \lambda_n A x_n)) \tag{3.1}$$

for all $n \geq 0$, where $\{\lambda_n\} \subset (0,2\alpha)$, $\{\alpha_n\} \subset (0,1)$, and $\{\beta_n\} \subset (0,1)$ satisfy

(i) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_n \alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$;

(iii) $a \leq \lambda_n \leq b$ *where* $[a,b] \subset (0,2\alpha)$ *and* $\lim_{n\to\infty}(\lambda_{n+1}-\lambda_n) = 0$.

Then $\{x_n\}$ generated by (3.1) *converges strongly to* $\widetilde{x} = P_{(A+B)^{-1}0}(u)$.

Proof. First, we choose any $z \in (A+B)^{-1}0$. Note that

$$z = J_{\lambda_n}^B(z - \lambda_n(1-\alpha_n)Az) = J_{\lambda_n}^B(\alpha_n z + (1-\alpha_n)(z-\lambda_n Az)) \tag{3.2}$$

for all $n \geq 0$. Since J_λ^B is nonexpansive for all $\lambda > 0$, we have

$$\begin{aligned}
&\left\|J_{\lambda_n}^B(\alpha_n u + (1-\alpha_n)(x_n - \lambda_n A x_n)) - z\right\|^2 \\
&\quad = \left\|J_{\lambda_n}^B(\alpha_n u + (1-\alpha_n)(x_n - \lambda_n A x_n)) - J_{\lambda_n}^B(\alpha_n z + (1-\alpha_n)(z-\lambda_n Az))\right\|^2 \\
&\quad \leq \|(\alpha_n u + (1-\alpha_n)(x_n - \lambda_n A x_n)) - (\alpha_n z + (1-\alpha_n)(z-\lambda_n Az))\|^2 \\
&\quad = \|(1-\alpha_n)((x_n - \lambda_n A x_n) - (z - \lambda_n Az)) + \alpha_n(u-z)\|^2.
\end{aligned} \tag{3.3}$$

Since A is α-inverse strongly monotone, we get

$$\begin{aligned}
&\|(1-\alpha_n)((x_n-\lambda_n Ax_n)-(z-\lambda_n Az))+\alpha_n(u-z)\|^2\\
&\quad\le(1-\alpha_n)\|(x_n-\lambda_n Ax_n)-(z-\lambda_n Az)\|^2+\alpha_n\|u-z\|^2\\
&\quad=(1-\alpha_n)\|(x_n-z)-\lambda_n(Ax_n-Az)\|^2+\alpha_n\|u-z\|^2\\
&\quad=(1-\alpha_n)\left(\|x_n-z\|^2-2\lambda_n\langle Ax_n-Az,x_n-z\rangle+\lambda_n^2\|Ax_n-Az\|^2\right)+\alpha_n\|u-z\|^2\\
&\quad\le(1-\alpha_n)\left(\|x_n-z\|^2-2\alpha\lambda_n\|Ax_n-Az\|^2+\lambda_n^2\|Ax_n-Az\|^2\right)+\alpha_n\|u-z\|^2\\
&\quad=(1-\alpha_n)\left(\|x_n-z\|^2+\lambda_n(\lambda_n-2\alpha)\|Ax_n-Az\|^2\right)+\alpha_n\|u-z\|^2.
\end{aligned}\tag{3.4}$$

By (3.3) and (3.4), we obtain

$$\begin{aligned}
&\left\|J_{\lambda_n}^B(\alpha_n u+(1-\alpha_n)(x_n-\lambda_n Ax_n))-z\right\|^2\\
&\quad\le(1-\alpha_n)\left(\|x_n-z\|^2+\lambda_n(\lambda_n-2\alpha)\|Ax_n-Az\|^2\right)+\alpha_n\|u-z\|^2\\
&\quad\le(1-\alpha_n)\|x_n-z\|^2+\alpha_n\|u-z\|^2.
\end{aligned}\tag{3.5}$$

It follows from (3.1) and (3.5) that

$$\begin{aligned}
\|x_{n+1}-z\|^2&=\left\|\beta_n(x_n-z)+(1-\beta_n)\left(J_{\lambda_n}^B(\alpha_n u+(1-\alpha_n)(x_n-\lambda_n Ax_n))-z\right)\right\|^2\\
&\le\beta_n\|x_n-z\|^2+(1-\beta_n)\left\|J_{\lambda_n}^B(\alpha_n u+(1-\alpha_n)(x_n-\lambda_n Ax_n))-z\right\|^2\\
&\le\beta_n\|x_n-z\|^2+(1-\beta_n)\left((1-\alpha_n)\|x_n-z\|^2+\alpha_n\|u-z\|^2\right)\\
&=\left[1-(1-\beta_n)\alpha_n\right]\|x_n-z\|^2+(1-\beta_n)\alpha_n\|u-z\|^2\\
&\le\max\left\{\|x_n-z\|^2,\|u-z\|^2\right\}.
\end{aligned}\tag{3.6}$$

By induction, we have

$$\|x_{n+1}-z\|\le\max\{\|x_0-z\|,\|u-z\|\}.\tag{3.7}$$

Therefore, $\{x_n\}$ is bounded. We deduce immediately that $\{Ax_n\}$ is also bounded. Set $u_n=\alpha_n u+(1-\alpha_n)(x_n-\lambda_n Ax_n)$ for all n. Then $\{u_n\}$ and $\{J_{\lambda_n}^B u_n\}$ are bounded.

Next, we estimate $\|J^B_{\lambda_{n+1}}u_{n+1} - J^B_{\lambda_n}u_n\|$. In fact, we have

$$\begin{aligned}
&\left\|J^B_{\lambda_{n+1}}u_{n+1} - J^B_{\lambda_n}u_n\right\| \\
&\quad= \left\|J^B_{\lambda_{n+1}}(\alpha_{n+1}u + (1-\alpha_{n+1})(x_{n+1} - \lambda_{n+1}Ax_{n+1})) - J^B_{\lambda_n}(\alpha_n u + (1-\alpha_n)(x_n - \lambda_n Ax_n))\right\| \\
&\quad\le \left\|J^B_{\lambda_{n+1}}(\alpha_{n+1}u + (1-\alpha_{n+1})(x_{n+1} - \lambda_{n+1}Ax_{n+1})) - J^B_{\lambda_{n+1}}(\alpha_n u + (1-\alpha_n)(x_n - \lambda_n Ax_n))\right\| \\
&\qquad+ \left\|J^B_{\lambda_{n+1}}(\alpha_n u + (1-\alpha_n)(x_n - \lambda_n Ax_n)) - J^B_{\lambda_n}(\alpha_n u + (1-\alpha_n)(x_n - \lambda_n Ax_n))\right\| \\
&\quad\le \|(\alpha_{n+1}u + (1-\alpha_{n+1})(x_{n+1} - \lambda_{n+1}Ax_{n+1})) - (\alpha_n u + (1-\alpha_n)(x_n - \lambda_n Ax_n))\| \\
&\qquad+ \left\|J^B_{\lambda_{n+1}}u_n - J^B_{\lambda_n}u_n\right\| \\
&\quad\le \|(I - \lambda_{n+1}A)x_{n+1} - (I - \lambda_{n+1}A)x_n\| + |\lambda_{n+1} - \lambda_n|\|Ax_n\| \\
&\qquad+ \alpha_{n+1}(\|u\| + \|x_{n+1}\| + \lambda_{n+1}\|Ax_{n+1}\|) + \alpha_n(\|u\| + \|x_n\| + \lambda_n\|Ax_n\|) + \left\|J^B_{\lambda_{n+1}}u_n - J^B_{\lambda_n}u_n\right\|.
\end{aligned} \tag{3.8}$$

Since $I - \lambda_{n+1}A$ is nonexpansive for $\lambda_{n+1} \in (0, 2\alpha)$, we have $\|(I - \lambda_{n+1}A)x_{n+1} - (I - \lambda_{n+1}A)x_n\| \le \|x_{n+1} - x_n\|$. By the resolvent identity (2.7), we have

$$J^B_{\lambda_{n+1}}u_n = J^B_{\lambda_n}\left(\frac{\lambda_n}{\lambda_{n+1}}u_n + \left(1 - \frac{\lambda_n}{\lambda_{n+1}}\right)J^B_{\lambda_{n+1}}u_n\right). \tag{3.9}$$

It follows that

$$\begin{aligned}
\left\|J^B_{\lambda_{n+1}}u_n - J^B_{\lambda_n}u_n\right\| &= \left\|J^B_{\lambda_n}\left(\frac{\lambda_n}{\lambda_{n+1}}u_n + \left(1 - \frac{\lambda_n}{\lambda_{n+1}}\right)J^B_{\lambda_{n+1}}u_n\right) - J^B_{\lambda_n}u_n\right\| \\
&\le \left\|\left(\frac{\lambda_n}{\lambda_{n+1}}u_n + \left(1 - \frac{\lambda_n}{\lambda_{n+1}}\right)J^B_{\lambda_{n+1}}u_n\right) - u_n\right\| \\
&\le \frac{|\lambda_{n+1} - \lambda_n|}{\lambda_{n+1}}\left\|u_n - J^B_{\lambda_{n+1}}u_n\right\|.
\end{aligned} \tag{3.10}$$

So,

$$\begin{aligned}
\left\|J^B_{\lambda_{n+1}}u_{n+1} - J^B_{\lambda_n}u_n\right\| \le{}& \|x_{n+1} - x_n\| + |\lambda_{n+1} - \lambda_n|\|Ax_n\| + \alpha_{n+1}(\|u\| + \|x_{n+1}\| + \lambda_{n+1}\|Ax_{n+1}\|) \\
&+ \alpha_n(\|u\| + \|x_n\| + \lambda_n\|Ax_n\|) + \frac{|\lambda_{n+1} - \lambda_n|}{\lambda_{n+1}}\left\|u_n - J^B_{\lambda_{n+1}}u_n\right\|.
\end{aligned} \tag{3.11}$$

Thus,

$$\limsup_{n\to\infty}\left(\left\|J^B_{\lambda_{n+1}}u_{n+1} - J^B_{\lambda_n}u_n\right\| - \|x_{n+1} - x_n\|\right) \le 0. \tag{3.12}$$

From Lemma 2.2, we get

$$\lim_{n\to\infty}\left\|J_{\lambda_n}^B u_n - x_n\right\| = 0. \tag{3.13}$$

Consequently, we obtain

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = \lim_{n\to\infty}(1-\beta_n)\left\|J_{\lambda_n}^B u_n - x_n\right\| = 0. \tag{3.14}$$

From (3.5) and (3.6), we have

$$\begin{aligned}
\|x_{n+1}-z\|^2 &\le \beta_n\|x_n-z\|^2 + (1-\beta_n)\left\|J_{\lambda_n}^B(\alpha_n u + (1-\alpha_n)(x_n-\lambda_n Ax_n)) - z\right\|^2 \\
&\le (1-\beta_n)\left\{(1-\alpha_n)\left(\|x_n-z\|^2 + \lambda_n(\lambda_n-2\alpha)\|Ax_n-Az\|^2\right) + \alpha_n\|u-z\|^2\right\} \\
&\quad + \beta_n\|x_n-z\|^2 \\
&= \left[1-(1-\beta_n)\alpha_n\right]\|x_n-z\|^2 + (1-\beta_n)\lambda_n(\lambda_n-2\alpha)\|Ax_n-Az\|^2 \\
&\quad + (1-\beta_n)\alpha_n\|u-z\|^2 \\
&\le \|x_n-z\|^2 + (1-\beta_n)\lambda_n(\lambda_n-2\alpha)\|Ax_n-Az\|^2 + (1-\beta_n)\alpha_n\|u-z\|^2.
\end{aligned} \tag{3.15}$$

It follows that

$$\begin{aligned}
&(1-\beta_n)\lambda_n(2\alpha-\lambda_n)\|Ax_n-Az\|^2 \\
&\quad \le \|x_n-z\|^2 - \|x_{n+1}-z\|^2 + (1-\beta_n)\alpha_n\|u-z\|^2 \\
&\quad \le (\|x_n-z\| - \|x_{n+1}-z\|)\|x_{n+1}-x_n\| + (1-\beta_n)\alpha_n\|u-z\|^2.
\end{aligned} \tag{3.16}$$

Since $\lim_{n\to\infty}\alpha_n = 0$, $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$, and $\liminf_{n\to\infty}(1-\beta_n)\lambda_n(2\alpha-\lambda_n) > 0$, we have

$$\lim_{n\to\infty}\|Ax_n - Az\| = 0. \tag{3.17}$$

Put $\widetilde{x} = P_{(A+B)^{-1}0}(u)$. Set $v_n = x_n - (\lambda_n/(1-\alpha_n))(Ax_n - A\widetilde{x})$ for all n. Take $z = \widetilde{x}$ in (3.17) to get $\|Ax_n - A\widetilde{x}\| \to 0$. First, we prove $\limsup_{n\to\infty}\langle u-\widetilde{x}, v_n-\widetilde{x}\rangle \le 0$. We take a subsequence $\{v_{n_i}\}$ of $\{v_n\}$ such that

$$\limsup_{n\to\infty}\langle u-\widetilde{x}, v_n-\widetilde{x}\rangle = \lim_{i\to\infty}\langle u-\widetilde{x}, v_{n_i}-\widetilde{x}\rangle. \tag{3.18}$$

It is clear that $\{v_{n_i}\}$ is bounded due to the boundedness of $\{x_n\}$ and $\|Ax_n - A\widetilde{x}\| \to 0$. Then, there exists a subsequence $\{v_{n_{i_j}}\}$ of $\{v_{n_i}\}$ which converges weakly to some point $w \in C$.

Hence, $\{x_{n_{i_j}}\}$ also converges weakly to w because of $\|v_{n_{i_j}} - x_{n_{i_j}}\| \to 0$. By the similar argument as that in [31], we can show that $w \in (A+B)^{-1}0$. This implies that

$$\limsup_{n\to\infty}\langle u-\tilde{x}, v_n-\tilde{x}\rangle = \lim_{j\to\infty}\left\langle u-\tilde{x}, v_{n_{i_j}}-\tilde{x}\right\rangle = \langle u-\tilde{x}, w-\tilde{x}\rangle. \tag{3.19}$$

Note that $\tilde{x} = P_{(A+B)^{-1}0}(u)$. Then, $\langle u-\tilde{x}, w-\tilde{x}\rangle \le 0, w \in (A+B)^{-1}0$. Therefore,

$$\limsup_{n\to\infty}\langle u-\tilde{x}, v_n-\tilde{x}\rangle \le 0. \tag{3.20}$$

Finally, we prove that $x_n \to \tilde{x}$. From (3.1), we have

$$\begin{aligned}
\|x_{n+1}-\tilde{x}\|^2 &\le \beta_n\|x_n-\tilde{x}\|^2 + (1-\beta_n)\left\|J_{\lambda_n}^B u_n - \tilde{x}\right\|^2 \\
&= \beta_n\|x_n-\tilde{x}\|^2 + (1-\beta_n)\left\|J_{\lambda_n}^B u_n - J_{\lambda_n}^B(\tilde{x}-(1-\alpha_n)\lambda_n A\tilde{x})\right\|^2 \\
&\le \beta_n\|x_n-\tilde{x}\|^2 + (1-\beta_n)\|u_n-(\tilde{x}-(1-\alpha_n)\lambda_n A\tilde{x})\|^2 \\
&= \beta_n\|x_n-\tilde{x}\|^2 + (1-\beta_n)\|\alpha_n u + (1-\alpha_n)(x_n-\lambda_n Ax_n) - (\tilde{x}-(1-\alpha_n)\lambda_n A\tilde{x})\|^2 \\
&= \beta_n\|x_n-\tilde{x}\|^2 + (1-\beta_n)\|(1-\alpha_n)((x_n-\lambda_n Ax_n)-(\tilde{x}-\lambda_n A\tilde{x})) + \alpha_n(u-\tilde{x})\|^2 \\
&= \beta_n\|x_n-\tilde{x}\|^2 + (1-\beta_n) \\
&\quad\times\Big((1-\alpha_n)^2\|(x_n-\lambda_n Ax_n)-(\tilde{x}-\lambda_n A\tilde{x})\|^2 \\
&\qquad +2\alpha_n(1-\alpha_n)\langle u-\tilde{x}, (x_n-\lambda_n Ax_n)-(\tilde{x}-\lambda_n A\tilde{x})\rangle + \alpha_n^2\|u-\tilde{x}\|^2\Big) \\
&\le \beta_n\|x_n-\tilde{x}\|^2 + (1-\beta_n) \\
&\quad\times\Big((1-\alpha_n)\|x_n-\tilde{x}\|^2 + 2\alpha_n(1-\alpha_n)\langle u-\tilde{x}, x_n-\lambda_n(Ax_n-A\tilde{x})-\tilde{x}\rangle \\
&\qquad +\alpha_n^2\|u-\tilde{x}\|^2\Big) \\
&\le [1-(1-\beta_n)\alpha_n]\|x_n-\tilde{x}\|^2 + (1-\beta_n)\alpha_n\Big\{2(1-\alpha_n)\langle u-\tilde{x}, v_n-\tilde{x}\rangle + \alpha_n\|u-\tilde{x}\|^2\Big\}.
\end{aligned} \tag{3.21}$$

It is clear that $\sum_n(1-\beta_n)\alpha_n = \infty$ and $\limsup_{n\to\infty}(2(1-\alpha_n)\langle u-\tilde{x}, v_n-\tilde{x}\rangle + \alpha_n\|u-\tilde{x}\|^2) \le 0$. We can therefore apply Lemma 2.3 to conclude that $x_n \to \tilde{x}$. This completes the proof. □

4. Applications

Next, we consider the problem for finding the minimum norm solution of a mathematical model related to equilibrium problems. Let C be a nonempty, closed, and convex subset of a Hilbert space and let $G : C \times C \to R$ be a bifunction satisfying the following conditions:

(E1) $G(x, x) = 0$ for all $x \in C$;

(E2) G is monotone, that is, $G(x, y) + G(y, x) \leq 0$ for all $x, y \in C$;

(E3) for all $x, y, z \in C$, $\limsup_{t\downarrow 0} G(tz + (1-t)x, y) \leq G(x, y)$;

(E4) for all $x \in C$, $G(x, \cdot)$ is convex and lower semicontinuous.

Then, the mathematical model related to equilibrium problems (with respect to C) is to find $\widetilde{x} \in C$ such that

$$G(\widetilde{x}, y) \geq 0 \tag{4.1}$$

for all $y \in C$. The set of such solutions $\widetilde{x}$ is denoted by $EP(G)$. The following lemma appears implicitly in Blum and Oettli [32].

Lemma 4.1. *Let C be a nonempty, closed, and convex subset of H and let G be a bifunction of $C \times C$ into R satisfying (E1)–(E4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$G(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C. \tag{4.2}$$

The following lemma was given by Combettes and Hirstoaga [33].

Lemma 4.2. *Assume that $G : C \times C \to R$ satisfies (E1)–(E4). For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \left\{ z \in C : G(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0,\ \forall y \in C \right\} \tag{4.3}$$

for all $x \in H$. Then, the following holds:

(1) *T_r is single valued;*

(2) *T_r is a firmly nonexpansive mapping, that is, for all $x, y \in H$,*

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle; \tag{4.4}$$

(3) *$F(T_r) = EP(G)$;*

(4) *$EP(G)$ is closed and convex.*

We call such T_r the resolvent of G for $r > 0$. Using Lemmas 4.1 and 4.2, we have the following lemma. See [34] for a more general result.

Lemma 4.3. *Let H be a Hilbert space and let C be a nonempty, closed, and convex subset of H. Let $G : C \times C \to R$ satisfy (E1)–(E4). Let A_G be a multivalued mapping of H into itself defined by*

$$A_G x = \begin{cases} \{z \in H : G(x, y) \geq \langle y - x, z\rangle,\ \forall y \in C\}, & x \in C, \\ \emptyset, & x \notin C. \end{cases} \tag{4.5}$$

Then, $EP(G) = A_G^{-1}(0)$ and A_G is a maximal monotone operator with $\mathrm{dom}(A_G) \subset C$. Further, for any $x \in H$ and $r > 0$, the resolvent T_r of G coincides with the resolvent of A_G; that is,

$$T_r x = (I + rA_G)^{-1}x. \tag{4.6}$$

Form Lemma 4.3 and Theorems 3.1, we have the following result.

Theorem 4.4. *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let G be a bifunction from $C \times C \to R$ satisfying (E1)–(E4) and let T_λ be the resolvent of G for $\lambda > 0$. Suppose $EP(G) \neq \emptyset$. For $u \in C$ and given $x_0 \in C$, let $\{x_n\} \subset C$ be a sequence generated by*

$$x_{n+1} = \beta_n x_n + (1 - \beta_n) T_{\lambda_n}(\alpha_n u + (1 - \alpha_n)x_n) \tag{4.7}$$

for all $n \geq 0$, where $\{\lambda_n\} \subset (0, \infty)$, $\{\alpha_n\} \subset (0, 1)$, and $\{\beta_n\} \subset (0, 1)$ satisfy

(i) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_n \alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$;

(iii) $a \leq \lambda_n \leq b$ *where* $[a, b] \subset (0, \infty)$ *and* $\lim_{n\to\infty}(\lambda_{n+1} - \lambda_n) = 0$.

Then $\{x_n\}$ converges strongly to a point $\tilde{x} = P_{EP(G)}(u)$.

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References

[1] M. A. Noor and K. I. Noor, "Sensitivity analysis for quasi-variational inclusions," *Journal of Mathematical Analysis and Applications*, vol. 236, no. 2, pp. 290–299, 1999.

[2] S. S. Chang, "Set-valued variational inclusions in Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 248, no. 2, pp. 438–454, 2000.

[3] S. S. Chang, "Existence and approximation of solutions for set-valued variational inclusions in Banach space," *Nonlinear Analysis, Theory, Methods and Applications*, vol. 47, no. 1, pp. 583–594, 2001.

[4] V. F. Demyanov, G. E. Stavroulakis, L. N. Polyakova, and P. D. Panagiotopoulos, *Quasidifferentiability and Nonsmooth Modelling in Mechanics, Engineering and Economics*, vol. 10, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1996.

[5] M. A. Noor, "Generalized set-valued variational inclusions and resolvent equations," *Journal of Mathematical Analysis and Applications*, vol. 228, no. 1, pp. 206–220, 1998.

[6] P. Hartman and G. Stampacchia, "On some non-linear elliptic differential-functional equations," *Acta Mathematica*, vol. 115, pp. 271–310, 1966.

[7] S.-S. Zhang, J. H. W. Lee, and C. K. Chan, "Algorithms of common solutions to quasi variational inclusion and fixed point problems," *Applied Mathematics and Mechanics*, vol. 29, no. 5, pp. 571–581, 2008.

[8] J.-W. Peng, Y. Wang, D. S. Shyu, and J.-C. Yao, "Common solutions of an iterative scheme for variational inclusions, equilibrium problems, and fixed point problems," *Journal of Inequalities and Applications*, vol. 2008, Article ID 720371, 15 pages, 2008.

[9] S. Takahashi and W. Takahashi, "Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces," *Journal of Mathematical Analysis and Applications*, vol. 331, no. 1, pp. 506–515, 2007.

[10] S. Takahashi and W. Takahashi, "Strong convergence theorem for a generalized equilibrium problem and a nonexpansive mapping in a Hilbert space," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 69, no. 3, pp. 1025–1033, 2008.
[11] Y. Yao, Y.-C. Liou, and J.-C. Yao, "Convergence theorem for equilibrium problems and fixed point problems of infinite family of nonexpansive mappings," *Fixed Point Theory and Applications*, vol. 2007, Article ID 64363, 12 pages, 2007.
[12] Y. Yao and J.-C. Yao, "On modified iterative method for nonexpansive mappings and monotone mappings," *Applied Mathematics and Computation*, vol. 186, no. 2, pp. 1551–1558, 2007.
[13] L.-C. Ceng and J.-C. Yao, "A hybrid iterative scheme for mixed equilibrium problems and fixed point problems," *Journal of Computational and Applied Mathematics*, vol. 214, no. 1, pp. 186–201, 2008.
[14] S. Adly, "Perturbed algorithms and sensitivity analysis for a general class of variational inclusions," *Journal of Mathematical Analysis and Applications*, vol. 201, no. 2, pp. 609–630, 1996.
[15] Y. Yao, Y.-C. Liou, and J.-C. Yao, "An iterative algorithm for approximating convex minimization problem," *Applied Mathematics and Computation*, vol. 188, no. 1, pp. 648–656, 2007.
[16] Y. Yao, R. Chen, and J.-C. Yao, "Strong convergence and certain control conditions for modified Mann iteration," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 68, no. 6, pp. 1687–1693, 2008.
[17] X. P. Ding, "Perturbed Ishikawa type iterative algorithm for generalized quasivariational inclusions," *Applied Mathematics and Computation*, vol. 141, no. 2-3, pp. 359–373, 2003.
[18] N.-J. Huang, "Mann and Ishikawa type perturbed iterative algorithms for generalized nonlinear implicit quasi-variational inclusions," *Computers & Mathematics with Applications*, vol. 35, no. 10, pp. 1–7, 1998.
[19] Y.-P. Fang and N.-J. Huang, "H-monotone operator and resolvent operator technique for variational inclusions," *Applied Mathematics and Computation*, vol. 145, no. 2-3, pp. 795–803, 2003.
[20] L.-J. Lin, "Variational inclusions problems with applications to Ekeland's variational principle, fixed point and optimization problems," *Journal of Global Optimization*, vol. 39, no. 4, pp. 509–527, 2007.
[21] R. U. Verma, "General system of (A, η)-monotone variational inclusion problems based on generalized hybrid iterative algorithm," *Nonlinear Analysis. Hybrid Systems*, vol. 1, no. 3, pp. 326–335, 2007.
[22] J.-W. Peng and J.-C. Yao, "A new hybrid-extragradient method for generalized mixed equilibrium problems, fixed point problems and variational inequality problems," *Taiwanese Journal of Mathematics*, vol. 12, no. 6, pp. 1401–1432, 2008.
[23] Y. Yao and N. Shahzad, "Strong convergence of a proximal point algorithm with general errors," *Optimization Letters*. In press.
[24] Y. Yao and N. Shahzad, "New methods with perturbations for non-expansive mappings in Hilbert spaces," *Fixed Point Theory and Applications*. In press.
[25] Y. Yao, R. Chen, and H.-K. Xu, "Schemes for finding minimum-norm solutions of variational inequalities," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 72, no. 7-8, pp. 3447–3456, 2010.
[26] Y. Yao, R. Chen, and Y. C. Liou, "A unified implicit algorithm for solving the triplehierarchical constrained optimization problem," *Mathematical & Computer Modelling*.
[27] Y. Yao, Y. C. Liou, and S. M. Kang, "Two-step projection methods for a system of variational inequality problems in Banach spaces," *Journal of Global Optimization*. In press.
[28] W. Takahashi and M. Toyoda, "Weak convergence theorems for nonexpansive mappings and monotone mappings," *Journal of Optimization Theory and Applications*, vol. 118, no. 2, pp. 417–428, 2003.
[29] T. Suzuki, "Strong convergence theorems for infinite families of nonexpansive mappings in general Banach spaces," *Fixed Point Theory and Applications*, vol. 2005, no. 1, pp. 103–123, 2005.
[30] H.-K. Xu, "Iterative algorithms for nonlinear operators," *Journal of the London Mathematical Society*, vol. 66, no. 1, pp. 240–256, 2002.
[31] S. Takahashi, W. Takahashi, and M. Toyoda, "Strong convergence theorems for maximal monotone operators with nonlinear mappings in Hilbert spaces," *Journal of Optimization Theory and Applications*, vol. 147, no. 1, pp. 27–41, 2010.
[32] E. Blum and W. Oettli, "From optimization and variational inequalities to equilibrium problems," *The Mathematics Student*, vol. 63, no. 1–4, pp. 123–145, 1994.
[33] P. L. Combettes and S. A. Hirstoaga, "Equilibrium programming in Hilbert spaces," *Journal of Nonlinear and Convex Analysis*, vol. 6, no. 1, pp. 117–136, 2005.
[34] K. Aoyama, Y. Kimura, W. Takahashi, and M. Toyoda, "On a strongly nonexpansive sequence in Hilbert spaces," *Journal of Nonlinear and Convex Analysis*, vol. 8, no. 3, pp. 471–489, 2007.