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Research Article

A Class of Semilocal E-Preinvex Functions and Its Applications in Nonlinear Programming

Hehua Jiao,[1,2] Sanyang Liu,[1] and Xinying Pai[1,3]

[1] *College of Science, Xidian University, Xi'an, Shanxi 710071, China*
[2] *College of Mathematics and Computer Science, Yangtze Normal University, Fuling, Chongqing 408100, China*
[3] *Department of Mathematics, China University of Petroleum, Qingdao, Shandong 266555, China*

Correspondence should be addressed to Hehua Jiao, jiaohh361@126.com

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A kind of generalized convex set, called as local star-shaped E-invex set with respect to η, is presented, and some of its important characterizations are derived. Based on this concept, a new class of functions, named as semilocal E-preinvex functions, which is a generalization of semi-E-preinvex functions and semilocal E-convex functions, is introduced. Simultaneously, some of its basic properties are discussed. Furthermore, as its applications, some optimality conditions and duality results are established for a nonlinear programming.

1. Introduction

It is well known that convexity and generalized convexity have been playing a key role in many aspects of optimization, such as duality theorems, optimality conditions, and convergence of optimization algorithms. Therefore, the research on characterizations and generalizations of convexity is one of the most important aspects in mathematical programming and optimization theory in [1, 2]. During the past several decades, many significant generalizations of convexity have been proposed.

In 1977, Ewing [3] presented a generalized convexity known as semilocal convexity, where the concept is applied to provide sufficient optimality conditions in variational and control problems. Generalizations of semilocal convex functions and their properties have been studied by Kaul and Kaur [4, 5] and Kaur [6]. In [7], optimality conditions and duality results were established for nonlinear programming involving semilocal preinvex and related functions. These results are extended in [8] for a multiple-objective programming problems. In [9, 10], Lyall et al. investigated the optimality conditions and duality results for fractional single- (multiple-) objective programming involving semilocal preinvex and related functions, respectively.

On the other hand, in 1999, Youness [11] introduced the concepts of E-convex sets, E-convex functions, and E-convex programming, discussed some of their basic properties, and obtained some optimality results on E-convex programming. In 2002, Chen [12] brought forward a class of semi-E-convex functions and also discussed its basic properties. In 2007, by combining the concept of semi-E-convexity and that of semilocal convexity, Hu et al. [13] put forward the concept of generalized convexity called as semilocal E-convexity, studied some of its characterizations, and obtained some optimality conditions and duality results for semilocal E-convex programming. In [14], optimality and duality were further studied for a fractional multiple-objective programming involving semilocal E-convexity. In 2009, Fulga and Preda [15] extended the E-convexity to E-preinvexity and local E-preinvexity and discussed some of their properties and an application. In 2011, Luo and Jian [16] introduced semi-E-preinvex maps in Banach spaces and studied some of their properties.

Motivated by research work of [13–16] and references therein, in this paper, we present the concept of semilocal E-preinvexity and discuss its some important properties. Furthermore, as applications of semilocal E-preinvexity, we establish the optimality conditions and duality results for a nonlinear programming. The concept of semilocal E-preinvexity unifies the concepts of semilocal E-convexity and semi-E-preinvexity. Thus, we extend the work of [10, 12, 13] and generalize the results obtained in the literatures on this topic.

2. Preliminaries

Throughout the paper, let R^n denote the n-dimensional Euclidean space, and let $E : R^n \to R^n$ and $\eta : R^n \times R^n \to R^n$ be two fixed mappings. In this section, we review some related definitions and some results which will be used in this paper.

Definition 2.1 (see [11]). A set $K \subset R^n$ is said to be E-convex if there is a map E such that

$$\lambda E(x) + (1-\lambda)E(y) \in K, \quad \forall x, y \in K,\ 0 \le \lambda \le 1. \tag{2.1}$$

Definition 2.2 (see [11]). A function $f : R^n \to R$ is said to be E-convex on a set $K \subset R^n$ if there is a map E such that K is an E-convex set and

$$f(\lambda E(x) + (1-\lambda)E(y)) \le \lambda f(E(x)) + (1-\lambda) f(E(y)), \quad \forall x, y \in K,\ 0 \le \lambda \le 1. \tag{2.2}$$

Definition 2.3 (see [12]). A function $f : R^n \to R$ is said to be semi-E-convex on a set $K \subset R^n$ if there is a map E such that K is an E-convex set and

$$f(\lambda E(x) + (1-\lambda)E(y)) \le \lambda f(x) + (1-\lambda) f(y), \quad \forall x, y \in K,\ 0 \le \lambda \le 1. \tag{2.3}$$

Definition 2.4 (see [15]). A set $K \subset R^n$ is said to be E-invex with respect to η if

$$E(y) + \lambda \eta(E(x), E(y)) \in K, \quad \forall x, y \in K,\ 0 \le \lambda \le 1. \tag{2.4}$$

Definition 2.5 (see [15]). Let $K \subset R^n$ be an E-invex set with respect to η. A function $f : R^n \to R$ is said to be E-preinvex on K with respect to η if

$$f(E(y) + \lambda\eta(E(x), E(y))) \leq \lambda f(E(x)) + (1-\lambda) f(E(y)), \quad \forall x, y \in K,\ 0 \leq \lambda \leq 1. \tag{2.5}$$

Definition 2.6 (see [16]). Let $K \subset R^n$ be an E-invex set with respect to η. A function $f : R^n \to R$ is said to be semi-E-preinvex on K with respect to η if

$$f(E(y) + \lambda\eta(E(x), E(y))) \leq \lambda f(x) + (1-\lambda) f(y), \quad \forall x, y \in K,\ 0 \leq \lambda \leq 1. \tag{2.6}$$

Definition 2.7 (see [17]). A set $K \subset R^n$ is said to be local star-shaped invex with respect to η if for any $x, y \in K$, there is a maximal positive number $a(x, y) \leq 1$ satisfying

$$y + \lambda\eta(x, y) \in K, \quad \forall\lambda \in [0, a(x, y)]. \tag{2.7}$$

Definition 2.8 (see [13]). A set $K \subset R^n$ is said to be local star-shaped E-convex if there is a map E such that corresponding to each pair of points $x, y \in K$, and there is a maximal positive number $a(x, y) \leq 1$ satisfying

$$\lambda E(x) + (1-\lambda) E(y) \in K, \quad \forall\lambda \in [0, a(x, y)]. \tag{2.8}$$

Definition 2.9 (see [13]). A function $f : R^n \to R$ is said to be semilocal E-convex on a local star-shaped E-convex set $K \subset R^n$ if for each pair of $x, y \in K$(with a maximal positive number $a(x, y) \leq 1$ satisfying (2.8)), there exists a positive number $b(x, y) \leq a(x, y)$ satisfying

$$f(\lambda(E(x)) + (1-\lambda) E(y)) \leq \lambda f(x) + (1-\lambda) f(y), \quad \forall\lambda \in [0, b(x, y)]. \tag{2.9}$$

Definition 2.10 (see [18]). A vector function $f : X_0 \to R^k$ is said to be a convex-like function if for any $x, y \in X_0 \subset R^n$ and $0 \leq \lambda \leq 1$, there is $z \in X_0$ such that

$$f(z) \leq \lambda f(x) + (1-\lambda) f(y), \tag{2.10}$$

where the inequalities are taken component wise.

Lemma 2.11 (see [19]). *Let S be a nonempty set in R^n, and let $\psi : S \to R^k$ be a convexlike function then either $\psi(x) < 0$ has a solution $x \in S$ or $\lambda^T\psi(x) \geq 0$ for all $x \in S$ and some $\lambda \in R^k$, $\lambda \geq 0$, and $\lambda \neq 0$, but both alternatives are never true.*

3. Local Star-Shaped E-Invex Set

In this section, we introduce the local star-shaped E-invex set with respect to a given mapping η and discuss some of its basic characterizations.

Definition 3.1. A set $K \subset R^n$ is said to be local star-shaped E-invex with respect to a given mapping η if there is a map E such that corresponding to each pair of points $x, y \in K$, and there is a maximal positive number $a(x,y) \leq 1$ satisfying

$$E(y) + \lambda\eta(E(x), E(y)) \in K, \quad 0 \leq \lambda \leq a(x,y). \tag{3.1}$$

Remark 3.2. Every E-convex set is a local star-shaped E-invex set with respect to η, where $\eta(x,y) = x - y, a(x,y) = 1$, for all $x, y \in R^n$. Every local star-shaped E-convex set is a local star-shaped E-invex set with respect to η, where $\eta(x,y) = x - y$, for all $x, y \in R^n$. Every E-invex set with respect to η is a local star-shaped E-invex set with respect to η, where $a(x,y) = 1$, for all $x, y \in R^n$. But their converses are not necessarily true.

The following example shows that local star-shaped E-invex set is more general than E-convex set, E-invex set, and local star-shaped E-convex set.

Example 3.3. Let $K = [-4, -1) \bigcup [1, 4]$,

$$E(x) = \begin{cases} x^2 & \text{if } |x| \leq 2, \\ -1 & \text{if } |x| > 2, \end{cases}$$
$$\eta(x,y) = \begin{cases} x - y & \text{if } x \geq 0, y \geq 0, \text{ or } x \leq 0, \ y \leq 0, \\ -1 - y & \text{if } x > 0, \ y \leq 0, \text{ or } x \geq 0, \ y < 0, \\ 1 - y & \text{if } x < 0, \ y \geq 0, \text{ or } x \leq 0, \ y > 0. \end{cases} \tag{3.2}$$

We can testify that K is a local star-shaped E-invex set with respect to η.

However, when $x_0 = 1$, $y_0 = 3$, there exists a $\lambda_1 \in [0,1]$ such that $\lambda_1 E(x_0) + (1 - \lambda_1)E(y_0) = -1 + 2\lambda_1 \notin K$, namely, K is not an E-convex set.

Also, there is a $\lambda_2 \in [0,1]$ such that $E(y_0) + \lambda_2\eta(E(x_0), E(y_0)) = -1 \notin K$, that is, K is not an E-invex set with respect to η.

Similarly, for any positive number $a \leq 1$, there exists a sufficiently small positive number $\lambda_3 \leq a$ satisfying $\lambda_3 E(x_0) + (1 - \lambda_3)E(y_0) = -1 + 2\lambda_3 \notin K$, that is, K is not a local star-shaped E-convex set.

Proposition 3.4. *If a set $K \subset R^n$ is local star-shaped E-invex with respect to η, then $E(K) \subset K$.*

Proof. Since K is local star-shaped E-invex, then for any $x, y \in K$, there exists a maximal positive number $a(x,y) \leq 1$ satisfying $E(y) + \lambda\eta(E(x), E(y)) \in K$, for all $\lambda \in [0, a(x,y)]$.

Thus, for $\lambda = 0$, $E(y) \in K$.

Hence, $E(K) \subset K$. □

Proposition 3.5. *Let $E(K)$ be local star-shaped invex with respect to η, $E(K) \subset K$, then K is local star-shaped E-invex with respect to the same η.*

Proof. Assume that $x, y \in K$, then $E(x), E(y) \in E(K)$. Since $E(K)$ is local star-shaped invex with respect to η, thus for $E(x), E(y) \in E(K)$, there exists a positive number $a(E(x), E(y)) \leq 1$ satisfying

$$E(y) + \lambda\eta(E(x), E(y)) \in E(K) \subset K, \quad \forall\lambda \in [0, a(E(x), E(y))]. \tag{3.3}$$

Hence, K is local star-shaped E-invex with respect to η. □

Remark 3.6. Every local star-shaped invex set with respect to η is local star-shaped E-invex set, where E is an identity map, but its converse is not necessarily true. See the following example.

Example 3.7. Let $K_1 = \{(x,y) \in R^2 : (x,y) = \lambda_1(0,0) + \lambda_2(2,3) + \lambda_3(0,2)\}$, $K_2 = \{(x,y) \in R^2 : (x,y) = \lambda_1(0,0) + \lambda_2(-2,-3) + \lambda_3(0,-4)\}$, and $K = K_1 \cup K_2$, where $\lambda_1, \lambda_2, \lambda_3 \geq 0$ and $\sum_{i=1}^{3} \lambda_i = 1$. Let $\eta : R^2 \times R^2 \to R^2$ be defined as $\eta(x,y) = x - y$, and let $E : R^2 \to R^2$ be defined as

$$E(x,y) = \begin{cases} (0,3) & \text{if } (x,y) = (2,3), \\ (0,y-1), & \text{if } (x,y) \in K_1 \setminus \{(0,0),(2,3)\}, \\ (0,y) & \text{if } (x,y) \in K_2. \end{cases} \tag{3.4}$$

It is not difficult to prove that K is a local star-shaped E-invex set with respect to η. However, by taking $x = (2,3)$, $y = (0,-4)$, we know that there exists no maximal positive number $a(x,y) \leq 1$ such that $y + \lambda\eta(x,y) \in K$, for all $\lambda \in [0, a(x,y)]$.

That is, K is not a local star-shaped invex set with respect to η.

Proposition 3.8. *Let $K_i \subset R^n$ $(i = 1,2,\ldots,m)$ be a collection of local star-shaped E-invex sets with the same map η, then $\bigcap_{i=1}^{m} K_i$ is a local star-shaped E-invex set with respect to η.*

Proof. For all $x, y \in \bigcap_{i=1}^{m} K_i$, we have $x, y \in K_i$ $(i = 1,2,\ldots,m)$.

Since K_i $(i = 1,2,\ldots,m)$ are all local star-shaped E-invex sets, then there exist positive numbers $a_i(x,y) \leq 1$ $(i = 1,2,\ldots,m)$ such that

$$E(y) + \lambda\eta(E(x),E(y)) \in K_i, \quad \forall\lambda \in [0, a_i(x,y)],\ i = 1,2,\ldots,m. \tag{3.5}$$

Taking $a(x,y) = \min a_i(x,y)$, $i = 1,2,\ldots,m$, we can get

$$E(y) + \lambda\eta(E(x),E(y)) \in \bigcap_{i=1}^{m} K_i, \quad \forall\lambda \in [0, a(x,y)]. \tag{3.6}$$

Therefore, the proposition is proved. □

Remark 3.9. Even if K_1, K_2 are all local star-shaped E-invex set with respect to η, $K_1 \cup K_2$ is not necessarily a local star-shaped E-invex set. See the following example.

Example 3.10. Let the map $\eta : R^2 \times R^2 \to R^2$ be defined as $\eta(x,y) = x - y$, and the map $E : R^2 \to R^2$ be defined as $E(x,y) = (x/2, y/2)$. Consider the two sets

$$\begin{gathered} K_1 = \left\{(x,y) \in R^2 : (x,y) = \lambda_1(0,0) + \lambda_2(2,0) + \lambda_3(0,2)\right\}, \\ K_2 = \left\{(x,y) \in R^2 : (x,y) = \lambda_1(0,0) + \lambda_2(-2,0) + \lambda_3(0,-2)\right\}, \end{gathered} \tag{3.7}$$

where $\lambda_i \geq 0$, $i = 1,2,3$, and $\sum_{i=1}^{3} \lambda_i = 1$.

We can easily prove that the two sets K_1, K_2 are all local star-shaped E-invex sets with respect to η. However, when $x = (2,0)$, $y = (0,-2)$, there is not a positive number $a(x,y) \le 1$ such that

$$E(y) + \lambda(E(x), E(y)) = (\lambda, -1+\lambda) \in K_1 \cup K_2, \quad \forall \lambda \in (0, a(x,y)]. \tag{3.8}$$

Thus, $K_1 \cup K_2$ is not a local star-shaped E-invex set with respect to η.

Proposition 3.11. *Let $K \subset R^n$ be a local star-shaped E_1 and E_2-invex set with respect to the same η, then K is a local star-shaped $(E_1 \circ E_2)$ and $(E_2 \circ E_1)$-invex set with respect to the same η.*

Proof. By contradiction, assume that for a pair of $x, y \in K$, for all $a(x,y) \in (0,1]$, there exists a $\lambda_0 \in (0, a(x,y)]$ such that $E_1 \circ E_2(y) + \lambda_0 \eta((E_1 \circ E_2(x), E_1 \circ E_2(y)) \notin K$, that is, $E_1(E_2 y) + \lambda_0 \eta((E_1(E_2 x), E_1(E_2 y)) \notin K$.

Since, from Proposition 3.4, $E_2(x), E_2(y) \in K$, then $E_1(E_2 y) + \lambda_0 \eta(E_1(E_2 x), E_1(E_2 y)) \notin K$ contradicts the local star-shaped E_1-invexity of K.

Hence, K is a local star-shaped $(E_1 \circ E_2)$-invex set.

Similarly, K is a local star-shaped $(E_2 \circ E_1)$-invex set. □

4. Semilocal E-Preinvex Functions

In the section, we present the concept of semilocal E-preinvex function and study some of its properties. We first recall a relevant definition.

Definition 4.1 (see [15]). A function $f : R^n \to R$ is said to be local E-preinvex on $k \subset R^n$ with respect to η if for any $x, y \in K$ (with a maximal positive number $a(x,y) \le 1$ satisfying (3.1)), there exists $0 < b(x,y) \le a(x,y)$ such that K is a local star-shaped E-invex set and

$$f(E(y) + \lambda\eta(E(x), E(y))) \le \lambda f(E(x)) + (1-\lambda) f(E(y)), \quad \forall \lambda \in [0, b(x,y)]. \tag{4.1}$$

Definition 4.2. A function $f : R^n \to R$ is said to be semilocal E-preinvex on $k \subset R^n$ with respect to η if for any $x, y \in K$ (with a maximal positive number $a(x,y) \le 1$ satisfying (3.1)), there exists $0 < b(x,y) \le a(x,y)$ such that K is a local star-shaped E-invex set and

$$f(E(y) + \lambda\eta(E(x), E(y))) \le \lambda f(x) + (1-\lambda) f(y), \quad \forall \lambda \in [0, b(x,y)]. \tag{4.2}$$

If the inequality sign above is strict for any $x, y \in K$ and $x \ne y$, then f is called a strict semilocal E-preinvex function.

A vector function $f : R^n \to R^k$ is said to be semilocal E-preinvex on a local star-shaped E-invex set $K \subset R^n$ with respect to η if for each pair of points $x, y \in K$(with a maximal positive number $a(x,y) \le 1$ satisfying (3.1)), there exists a positive number $b(x,y) \le a(x,y)$ satisfying

$$f(E(y) + \lambda\eta(E(x), E(y))) \le \lambda f(x) + (1-\lambda) f(y), \quad \forall \lambda \in [0, b(x,y)], \tag{4.3}$$

where the inequalities are taken component wise.

The definition of strict semilocal E-preinvex of a vector function $f : R^n \to R^k$ is similar to the one for a vector semilocal E-preinvex function.

Remark 4.3. Every semilocal E-convex function on a local star-shaped set K is a semilocal E-preinvex function, where $\eta(x, y) = x - y$, for all $x, y \in R^n$. Every semi-E-preinvex function with respect to η is a semilocal E-preinvex function, where $a(x, y) = b(x, y) = 1$, for all $x, y \in R^n$. But their converses are not necessarily true.

We give below an example of semilocal E-preinvex function, which is neither a semilocal E-convex function nor a semi-E-preinvex function.

Example 4.4. Let the map $E : R \to R$ be defined as

$$E(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } 1 < x \leq 2, \\ x & \text{if } 0 \leq x \leq 1 \text{ or } x > 2, \end{cases} \tag{4.4}$$

and the map $\eta : R \times R \to R$ be defined as

$$\eta(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 - x & \text{if } x \neq y. \end{cases} \tag{4.5}$$

Obviously, R is a local star-shaped E-convex set and a local star-shaped E-invex set with respect to η. Let $f : R \to R$ be defined as

$$f(x) = \begin{cases} 0 & \text{if } 1 < x \leq 2, \\ 1 & \text{if } x > 2, \\ -x + 1 & \text{if } 0 \leq x \leq 1, \\ -x + 2 & \text{if } x < 0. \end{cases} \tag{4.6}$$

We can prove that f is semilocal E-preinvex on R with respect to η. However, when $x_0 = 2$, $y_0 = 3$, and for any $b \in (0, 1]$, there exists a sufficiently small $\lambda_0 \in (0, b]$ satisfying

$$f(\lambda_0 E(x_0) + (1 - \lambda_0) E(y_0)) = f(3 - 2\lambda_0) = 1 > 1 - \lambda_0 = \lambda_0 f(x_0) + (1 - \lambda_0) f(y_0). \tag{4.7}$$

That is, $f(x)$ is not a semilocal E-convex function on R.

Similarly, taking $x_1 = 1$, $y_1 = 4$, we have

$$f(E(y_1) + \lambda_1 \eta(E(x_1), E(y_1))) = f(4) = 1 > 1 - \lambda_1 = \lambda_1 f(x_1) + (1 - \lambda_1) f(y_1), \tag{4.8}$$

for some $\lambda_1 \in [0, 1]$.

Thus, $f(x)$ is not a semi-E-preinvex function on R with respect to η.

Theorem 4.5. *Let $f : K \subset R^n \to R$ be a local E-preinvex function on a local star-shaped E-invex set K with respect to η, then f is a semilocal E-preinvex function if and only if $f(E(x)) \leq f(x)$, for all $x \in K$.*

Proof. Suppose that f is a semilocal E-preinvex function on set K with respect to η, then for each pair of points $x, y \in K$ (with a maximal positive number $a(x,y) \leq 1$ satisfying (3.1)), there exists a positive number $b(x,y) \leq a(x,y)$ satisfying

$$f(E(x) + \lambda\eta(E(y)), E(x)) \leq \lambda f(y) + (1-\lambda)f(x), \quad \lambda \in [0, b(x,y)]. \tag{4.9}$$

By letting $\lambda = 0$, we have $f(E(x)) \leq f(x)$, for all $x \in K$.

Conversely, assume that f is a local E-preinvex function on a local star-shaped E-invex set K, then for any $x, y \in K$, there exist $a(x,y) \in (0,1]$ satisfying (3.1) and $b(x,y) \in (0, a(x,y)]$ such that

$$f(E(y) + \lambda\eta(E(x), E(y))) \leq \lambda f(E(x)) + (1-\lambda)f(E(y)), \quad \forall\lambda \in [0, b(x,y)]. \tag{4.10}$$

Since $f(E(x)) \leq f(x)$, for all $x \in K$, then

$$f(E(y) + \lambda\eta(E(x), E(y))) \leq \lambda f(x) + (1-\lambda)f(y), \quad \forall\lambda \in [0, b(x,y)]. \tag{4.11}$$

The proof is completed. □

Remark 4.6. A local E-preinvex function on a local star-shaped E-invex set with respect to η is not necessarily a semilocal E-preinvex function.

Example 4.7. Let K, E, and η be the same as the ones of Example 3.3 and $f : R \to R$ be defined by $f(x) = x^2$, then f is local E-preinvex on K with respect to η.

Since $f(E(2)) = 16 > f(2) = 4$, from Theorem 4.5, it follows that f is not a semilocal E-preinvex function.

Theorem 4.8. *Let $f : R^n \to R$ be a semilocal E-preinvex function on a local star-shaped E-invex set $K \subset R^n$ with respect to η, and let $\varphi : R \to R$ be a nondecreasing and convex function, then $\varphi(f(x))$ is semilocal E-preinvex on K with respect to η.*

The proof is easy and is omitted.

Theorem 4.9. *If the functions $f_i : R^n \to R$ $(i = 1, 2, \ldots, m)$ are all semilocal E-preinvex on a local star-shaped E-invex set $K \subset R^n$ with respect to the same η, then the function $f(x) = \sum_{i=1}^{m} a_i f_i(x)$ is semilocal E-preinvex on K with respect to η for all $a_i \geq 0$, $i = 1, 2, \ldots, m$.*

Proof. Since K is a local star-shaped E-invex set with respect to η, then for all $x, y \in K$, there exists a positive number $a(x,y) \leq 1$ such that

$$E(y) + \lambda\eta(E(x), E(y)) \in K, \quad \forall\lambda \in [0, a(x,y)]. \tag{4.12}$$

On the other hand, f_i $(i = 1, 2, \dots, m)$ are all semilocal E-preinvex on K with respect to the same η; thus, there exist positive numbers $b_i(x, y) \le a(x, y)$ such that

$$f_i(E(y) + \lambda\eta(E(x), E(y))) \le \lambda f_i(x) + (1 - \lambda) f_i(y), \quad \forall\lambda \in [0, b_i(x, y)],\ i = 1, 2, \dots, m. \tag{4.13}$$

Now, letting $b(x, y) = \min b_i(x, y),\ i = 1, 2, \dots, m$, we have

$$\sum_{i=1}^{m} a_i f_i(E(y) + \lambda\eta(E(x), E(y))) \le \lambda \sum_{i=1}^{m} a_i f_i(x) + (1 - \lambda) \sum_{i=1}^{m} a_i f_i(y), \quad \forall\lambda \in [0, b(x, y)]. \tag{4.14}$$

That is, $f(x)$ is semilocal E-preinvex on $K$with respect to η. □

Definition 4.10. The set $G = \{(x, \alpha) : x \in K \subset R^n, \alpha \in R\}$ is said to be a local star-shaped E-invex set with respect to η corresponding to R^n if there are two maps η, E and a maximal positive number $a((x, \alpha_1), (y, \alpha_2)) \le 1$, for each $(x, \alpha_1), (y, \alpha_2) \in G$ such that

$$(E(y) + \lambda\eta(E(x), E(y)), \lambda\alpha_1 + (1 - \lambda)\alpha_2) \in G, \quad \forall\lambda \in [0, a((x, \alpha_1), (y, \alpha_2))]. \tag{4.15}$$

Theorem 4.11. *Let $K \subset R^n$ be a local star-shaped E-invex set with respect to η, then f is a semilocal E-preinvex function on K with respect to η if and only if its epigraph $G_f = \{(x, \alpha) : x \in K, f(x) \le \alpha, \alpha \in R\}$ is a local star-shaped E-invex set with respect to η corresponding to R^n.*

Proof. Assume that f is semilocal E-preinvex on K with respect to η and $(x, \alpha_1), (y, \alpha_2) \in G_f$, then $x, y \in K$, and $f(x) \le \alpha_1$, $f(y) \le \alpha_2$. Since K is a local star-shaped E-invex set, there is a maximal positive number $a(x, y) \le 1$ such that

$$E(\mathrm{y}) + \lambda\eta(E(x), E(y)) \in K, \quad \forall\lambda \in [0, a(x, y)]. \tag{4.16}$$

In addition, in view of f being a semilocal E-preinvex function on K with respect to η, there is a positive number $b(x, y) \le a(x, y)$ such that

$$f(E(y) + \lambda\eta(E(x), E(y))) \le \lambda f(x) + (1 - \lambda) f(y) \le \lambda\alpha_1 + (1 - \lambda)\alpha_2, \quad \forall\lambda \in [0, b(x, y)]. \tag{4.17}$$

That is, $(E(y) + \lambda\eta(E(x), E(y)), \lambda\alpha_1 + (1 - \lambda)\alpha_2) \in G_f$, for all $\lambda \in [0, b(x, y)]$.

Therefore, $G_f = \{(x, \alpha) : x \in K, f(x) \le \alpha, \alpha \in R\}$ is a local star-shaped E-invex set with respect to η corresponding to R^n.

Conversely, if G_f is a local star-shaped E-invex set with respect to η corresponding to R^n, then for any points $(x, f(x)), (y, f(y)) \in G_f$, there is a maximal positive number $a((x, f(x)), (y, f(y))) \le 1$ such that

$$(E(y) + \lambda\eta(E(x), E(y)), \lambda f(x) + (1 - \lambda) f(y)) \in G_f, \quad \forall\lambda \in [0, a((x, f(x)), (y, f(y)))]. \tag{4.18}$$

That is, $E(y)+\lambda\eta(E(x),E(y))\in K$,

$$f\big(E(y)+\lambda\eta(E(x),E(y))\big)\leq\lambda f(x)+(1-\lambda)f(y),\quad\forall\lambda\in\big[0,a\big((x,f(x)),(y,f(y))\big)\big].\tag{4.19}$$

Thus, K is a local star-shaped E-invex set, and f is a semilocal E-preinvex function on K with respect to η. □

Theorem 4.12. *If f is a semilocal E-preinvex function on a local star-shaped E-invex set $K\subset R^n$ with respect to η, then the level set $S_\alpha=\{x\in K: f(x)\leq\alpha\}$ is a local star-shaped E-invex set for any $\alpha\in R$.*

Proof. For any $\alpha\in R$ and $x,y\in S_\alpha$, then $x,y\in K$ and $f(x)\leq\alpha$, $f(y)\leq\alpha$. Since K is a local star-shaped E-invex set, there is a maximal positive number $a(x,y)\leq 1$ such that

$$E(y)+\lambda\eta(E(x),E(y))\in K,\quad\forall\lambda\in\big[0,a(x,y)\big].\tag{4.20}$$

In addition, due to the semilocal E-preinvexity of f, there is a positive number $b(x,y)\leq a(x,y)$ such that

$$f\big(E(y)+\lambda\eta(E(x),E(y))\big)\leq\lambda f(x)+(1-\lambda)f(y)\leq\lambda\alpha+(1-\lambda)\alpha=\alpha,\quad\forall\lambda\in\big[0,b(x,y)\big].\tag{4.21}$$

That is, $E(y)+\lambda\eta(E(x),E(y))\in S_\alpha$, for all $\lambda\in[0,b(x,y)]$.

Therefore, S_α is a local star-shaped E-invex set with respect to η for any $\alpha\in R$. □

Theorem 4.13. *Let f be a real-valued function defined on a local star-shaped E-invex set $K\subset R^n$, then f is a semilocal E-preinvex function with respect to η if and only if for each pair of points $x,y\in K$ (with a maximal positive number $a(x,y)\leq 1$ satisfying* (3.1)*), there exists a positive number $b(x,y)\leq a(x,y)$ such that*

$$f\big(E(y)+\lambda\eta(E(x),E(y))\big)<\lambda\alpha+(1-\lambda)\beta,\quad\forall\lambda\in\big[0,b(x,y)\big],\tag{4.22}$$

whenever $f(x)<\alpha$, $f(y)<\beta$.

Proof. Let $x,y\in K$ and $\alpha,\beta\in R$ such that $f(x)<\alpha$, $f(y)<\beta$. Due to the local star-shaped E-invexity of K, there is a maximal positive number $a(x,y)\leq 1$ such that

$$E(y)+\lambda\eta(E(x),E(y))\in K,\quad\forall\lambda\in\big[0,a(x,y)\big].\tag{4.23}$$

In addition, owing to the semilocal E-preinvexity of f, there is a positive number $b(x,y)\leq a(x,y)$ such that

$$f\big(E(y)+\lambda\eta(E(x),E(y))\big)\leq\lambda f(x)+(1-\lambda)f(y)<\lambda\alpha+(1-\lambda)\beta,\quad\forall\lambda\in\big[0,b(x,y)\big].\tag{4.24}$$

Conversely, let $(x,\alpha) \in G_f$, $(y,\beta) \in G_f$ (see epigraph G_f in Theorem 4.11), then $x, y \in K$, $f(x) \leq \alpha$, and $f(y) \leq \beta$. Hence, $f(x) < \alpha + \epsilon$ and $f(y) < \beta + \epsilon$ hold for any $\epsilon > 0$. According to the hypothesis, for $x, y \in K$ (with a positive number $a(x,y) \leq 1$ satisfying (3.1)), there exists a positive number $b(x,y) \leq a(x,y)$ such that

$$f(E(y) + \lambda\eta(E(x), E(y))) < \lambda\alpha + (1-\lambda)\beta + \epsilon, \quad \forall\lambda \in [0, b(x,y)]. \tag{4.25}$$

Let $\epsilon \to 0^+$, then

$$f(E(y) + \lambda\eta(E(x), E(y))) \leq \lambda\alpha + (1-\lambda)\beta, \quad \forall\lambda \in [0, b(x,y)]. \tag{4.26}$$

That is, $(E(y) + \lambda\eta(E(x), E(y)), \lambda\alpha + (1-\lambda)\beta) \in G_f$, for all $\lambda \in [0, b(x,y)]$.

Therefore, G_f is a local star-shaped E-invex set corresponding to R^n.

From Theorem 4.11, it follows that f is semilocal E-preinvex on K with respect to η. □

5. Nonlinear Programming

In this section, we discuss the optimality conditions and Mond-Weir type duality for nonlinear programming involving semilocal E-preinvex and related functions.

We firstly consider the nonlinear programming problem without constraint as follows:

$$(\mathrm{P}) \min f(x), \quad x \in K, \tag{5.1}$$

where K is a local star-shaped E-invex set and the objective function f is a semilocal E-preinvex function on K with respect to η.

Theorem 5.1. *The following statements hold for programming (P).*

(i) *The optimal solution set ω for (P) is a local star-shaped E-invex set with respect to η.*

(ii) *If x_0 is a local minimum for (P) and $E(x_0) = x_0$, then x_0 is a global minimum for (P).*

(iii) *If the real-valued function f is a strict semilocal E-preinvex function on K with respect to η, then the global optimal solution for (P) is unique.*

Proof. (i) Assume that $x, y \in \omega$, then $x, y \in K$ and $f(x) = f(y)$. On account of K being a local star-shaped E-invex set with respect to η, there is a maximal positive number $a(x,y) \leq 1$ such that

$$E(y) + \lambda\eta(E(x), E(y)) \in K, \quad \forall\lambda \in [0, a(x,y)]. \tag{5.2}$$

Besides, due to semilocal E-preinvexity of f, there is a positive number $b(x,y) \leq a(x,y)$ such that

$$f(E(y) + \lambda\eta(\mathrm{E}(x), E(y))) \leq \lambda f(x) + (1-\lambda) f(y) = f(x), \quad \forall\lambda \in [0, b(x,y)]. \tag{5.3}$$

From the optimality of x, we have $f(E(y) + \lambda\eta(E(x), E(y))) \geq f(x)$.

Hence, $f(E(y)+\lambda\eta(E(x),E(y)))=f(x)$, that is, $E(y)+\lambda\eta(E(x),E(y))\in\omega$, for all $\lambda\in[0,b(x,y)]$.

This shows that ω is a local star-shaped E-invex set with respect to η.

(ii) Assume that $N_\epsilon(x_0)$ is a neighbourhood of x_0 with radius $\epsilon>0$, and f attains its local minimum at $x_0\in N_\epsilon(x_0)\cap K$. For $x,x_0\in K$, there is a maximal positive number $a(x,x_0)\le 1$ such that $E(x_0)+\lambda\eta(E(x),E(x_0))\in K$, for all $\lambda\in[0,a(x,x_0)]$.

Moreover, there is a positive number $b(x,x_0)\le a(x,x_0)$ such that

$$f\big(E(x_0)+\lambda\eta(E(x),E(x_0))\big)\le\lambda f(x)+(1-\lambda)f(x_0),\quad\forall\lambda\in[0,b(x,x_0)].\tag{5.4}$$

Owing to $E(x_0)=x_0$, so for sufficiently small $\lambda>0$,

$$E(x_0)+\lambda\eta(E(x),E(x_0))\in N_\epsilon(x_0)\cap K.\tag{5.5}$$

Thus, $f(x_0)\le f(E(x_0)+\lambda\eta(E(x),E(x_0)))\le\lambda f(x)+(1-\lambda)f(x_0)$, namely, $f(x_0)\le f(x)$.

This means that x_0 is a global minimum for (P).

(iii) By contradiction, assume that $x_1,x_2\in K$ are two global optimal solutions for (P) and $x_1\ne x_2$. For $x_1,x_2\in K$, there is a maximal positive number $a(x_1,x_2)\le 1$ such that

$$E(x_2)+\lambda\eta(E(x_1),E(x_2))\in K,\quad\forall\lambda\in[0,a(x_1,x_2)].\tag{5.6}$$

Furthermore, there is a positive number $b(x_1,x_2)\le a(x_1,x_2)$ such that

$$f\big(E(x_2)+\lambda\eta(E(x_1),E(x_2))\big)<\lambda f(x_1)+(1-\lambda)f(x_2)=f(x_1),\quad\forall\lambda\in[0,b(x_1,x_2)].\tag{5.7}$$

This contradicts the fact that x_1 is a global optimal solution for (P).

Therefore, the global optimal solution for (P) is unique. □

Theorem 5.2. *Let $u\in K$ and $E(u)=u$. If f is differentiable on K, then u is a minimum for programming (P) if and only if u satisfies the inequality $\nabla f(u)^T\eta(E(v),u)\ge 0$, for all $v\in K$.*

Proof. Assume that u is a minimum for (P). Since K is a local star-shaped E-invex set with respect to η, for any $v\in K$, there is a maximal positive number $a(u,v)\le 1$ such that

$$E(u)+\lambda\eta(E(v),E(u))\in K,\quad\forall\lambda\in(0,a(u,v)].\tag{5.8}$$

From the differentiability of f and $E(u)=u$, we get

$$f(u)\le f\big(E(u)+\lambda\eta(E(\mathrm{v}),E(u))\big)=f(u)+\lambda\nabla f(u)^T\eta(E(v),u)+o(\lambda).\tag{5.9}$$

Dividing the inequality above by λ and letting $\lambda\to 0^+$, we have

$$\nabla f(u)^T\eta(E(v),u)\ge 0.\tag{5.10}$$

Conversely, if $\nabla f(u)^T \eta(E(v), u) \geq 0$, owing to semilocal E-preinvexity of f on K, there is a positive number $b(u,v) \leq a(u,v)$ such that

$$f(E(u) + \lambda\eta(E(v), E(u))) \leq \lambda f(v) + (1-\lambda) f(u), \quad \forall \lambda \in (0, b(u,v)], \tag{5.11}$$

which together with $E(u) = u$ implies

$$f(v) - f(u) \geq \frac{f(u + \lambda\eta(E(v), u)) - f(u)}{\lambda}. \tag{5.12}$$

Letting $\lambda \to 0^+$ in the inequality above, we obtain

$$f(v) - f(u) \geq \nabla f(u)^T \eta(E(v), u) \geq 0. \tag{5.13}$$

This follows that u is a minimum for (P). □

Next, we consider the optimization problem with inequality constraint

$$(\text{NP}) \begin{cases} \min f(x) \\ \text{s.t.} \quad g_i(x) \leq 0, \quad i \in I, \\ x \in K. \end{cases} \tag{5.14}$$

Denote the feasible set of (NP) by $K_0 = \{x \in K, g_i(x) \leq 0, i \in I\}$, where $I = \{1, 2, \ldots, m\}$, and $K \subset R^n$ is an open local star-shaped E-invex set with respect to η.

If the constraint functions $g_i(x)$ $(i \in I)$ are all semilocal E-preinvex on K with respect to the same map η, then, from Theorem 4.12 and Proposition 3.8, we can conclude that the feasible set K_0 is a local star-shaped E-invex set with respect to η. Moreover, from Theorem 5.1, we can obtain the following theorem easily.

Theorem 5.3. *Assume that f, g_i $(i \in I)$ are all semilocal E-preinvex on K with respect to η, then*

(i) *K_0 is a local star-shaped E-invex set with respect to η;*

(ii) *the optimal solution set ω for (NP) is a local star-shaped E-invex set with respect to η;*

(iii) *if x_0 is a local minimum for (NP) and $E(x_0) = x_0$, then x_0 is a global minimum for (NP);*

(iv) *if the real-valued function f is a strict semilocal E-preinvex function on K with respect to η, then the global optimal solution for (NP) is unique.*

For convenience of discussion, we give the following notation:

$$g(x) = (g_i(x), i \in I), \qquad g_J(x) = (g_j(x), j \in J), \quad \text{where } J \subset I. \tag{5.15}$$

For $x^* \in K_0$, denote $I(x^*) = \{i \in I : g_i(x^*) = 0\}$, $\overline{I}(x^*) = I \setminus I(x^*)$.

To discuss the necessary optimality conditions for the corresponding programming, we first give a lemma as follows.

Lemma 5.4. *Let x^* be a local optimal solution for (NP). Assume that g_j is continuous at x^* for any $j \in \overline{I}(x^*)$, and f, $g_{I(x^*)}$ possess the directional derivatives at x^* along the direction $\eta(E(x), x^*)$ for each $x \in K$, then the system:*

$$\begin{gathered} f'(x^*; \eta(E(x), x^*)) < 0, \\ g'_{I(x^*)}(x^*; \eta(E(x), x^*)) < 0 \end{gathered} \tag{5.16}$$

has no solution in K, where $f'(x^*; d)$ denotes the directional derivative of f at x^* along the direction d and $g'_{I(x^*)}(x^*; d) = (g'_i(x^*; d), i \in I(x^*))$.

The proof of this lemma is similar to the one of [10, Lemma 13].

Theorem 5.5. *Let x^* be a local optimal solution for (NP). Assume that functions g_j are continuous at x^* for any $j \in \overline{I}(x^*)$, and f, g possess the directional derivatives with respect to $\eta(E(x), x^*)$ at x^* for each $x \in K$. If $(f'(x^*; \eta(E(x), x^*)), g'_{I(x^*)}(x^*; \eta(E(x), x^*)))$ is a convex-like function and $E(x^*) = x^*$, then there are $\lambda_0^* \in R$, $\overline{\lambda}^* \in R^m$ such that*

$$\lambda_0^* f'(x^*; \eta(E(x), x^*)) + \overline{\lambda}^{*T} g'(x^*; \eta(E(x), x^*)) \geq 0, \quad \forall x \in K, \tag{5.17}$$

$$\overline{\lambda}^{*T} g(x^*) = 0, \quad g(x^*) \leq 0, 0 \neq \left(\lambda_0^*, \overline{\lambda}^*\right) \geq 0. \tag{5.18}$$

Additionally, if g is a semilocal E-preinvex function on K with respect to η and there is $\widehat{x} \in K$ such that $g(\widehat{x}) < 0$, then there exists $\lambda^* \in R^m$ such that

$$f'(x^*; \eta(E(x), x^*)) + \lambda^{*T} g'(x^*; \eta(E(x), x^*)) \geq 0, \quad \forall x \in K, \tag{5.19}$$

$$\lambda^{*T} g(x^*) = 0, \quad g(x^*) \leq 0, \lambda^* \geq 0. \tag{5.20}$$

Proof. Define vector function $\varphi(x) = (f'(x^*; \eta(E(x), x^*)), g'_{I(x^*)}(x^*; \eta(E(x), x^*)))$. Then $\varphi(x)$ is a convexlike function. By Lemma 5.4, the system $\varphi(x) < 0$ has no solution in K. Thus, from Lemma 2.11, there are $\lambda_0^* \in R, \overline{\lambda}^*_{I(x^*)} \in R^{|I(x^*)|}$ such that

$$\lambda_0^* f'(x^*; \eta(E(x), x^*)) + \overline{\lambda}_{I(x^*)}^{*T} g'_{I(x^*)}(x^*; \eta(E(x), x^*)) \geq 0, \quad \forall x \in K, 0 \neq \left(\lambda_0^*, \overline{\lambda}^*_{I(x^*)}\right) \geq 0. \tag{5.21}$$

Hence, by letting $\overline{\lambda}^* = (\overline{\lambda}^*_{I(x^*)}, 0_{\overline{I}(x^*)})$, we further have

$$\begin{gathered} \lambda_0^* f'(x^*; \eta(E(x), x^*)) + \overline{\lambda}^{*T} g'(x^*; \eta(E(x), x^*)) \geq 0, \quad \forall x \in K, \\ \overline{\lambda}^{*T} g(x^*) = 0, \quad g(x^*) \leq 0, \ 0 \neq \left(\lambda_0^*, \overline{\lambda}^*\right) \geq 0. \end{gathered} \tag{5.22}$$

Subsequently, we testify that $\lambda_0^* \neq 0$, and if this is not true, we get from above

$$\overline{\lambda}^{*T} g'(x^*; \eta(E(x), x^*)) \geq 0, \quad \forall x \in K, \qquad \overline{\lambda}^{*T} g(x^*) = 0, \quad g(x^*) \leq 0, \ 0 \neq \overline{\lambda}^* \geq 0. \tag{5.23}$$

On account of g being semilocal E-preinvex on K with respect to η and $E(x^*) = x^*$, from Proposition 5.9(i) below, we have $g(\widehat{x}) - g(x^*) \geq g'(x^*; \eta(E(\widehat{x}), x^*))$.

So $\overline{\lambda}^{*T} g(\widehat{x}) \geq 0$. But this contradicts the fact that $g(\widehat{x}) < 0$ and $\overline{\lambda}^* > 0$.

Thus, $\lambda_0^* > 0$. Dividing (5.17) and the first equality of (5.18) by λ_0^* and letting $\lambda^* = \overline{\lambda}^* \backslash \lambda_0^*$, we know that (5.19) and (5.20) hold.

Consequently, the whole proof is finished. □

To discuss the sufficient optimality conditions for (NP), we further generalize the concept of semilocal E-preinvex function as follows.

Definition 5.6. A real-valued function f defined on a local star-shaped E-invex set $k \subset R^n$ is said to be quasisemilocal E-preinvex (with respect to η) if for all $x, y \in K$ (with a maximal positive number $a(x,y) \leq 1$ satisfying (3.1)) satisfying $f(x) \leq f(y)$, there is a positive number $b(x,y) \leq a(x,y)$ such that

$$f(E(y) + \lambda \eta(E(x), E(y))) \leq f(y), \quad \forall \lambda \in [0, b(x,y)]. \tag{5.24}$$

The definition of quasi-semilocal E-preinvex of a vector function $f : R^n \to R^k$ is similar to the one for a vector semilocal E-preinvex function.

Definition 5.7. A real-valued function f defined on a local star-shaped E-invex set $K \subset R^n$ is said to be pseudosemilocal E-preinvex (with respect to η) if for all $x, y \in K$ (with a maximal positive number $a(x,y) \leq 1$ satisfying (3.1)) satisfying $f(x) < f(y)$, there are a positive number $b(x,y) \leq a(x,y)$ and a positive number $c(x,y)$ such that

$$f(E(y) + \lambda \eta(E(x), E(y))) \leq f(y) - \lambda c(x,y), \quad \forall \lambda \in [0, b(x,y)]. \tag{5.25}$$

The definition of pseudo-semilocal E-preinvex of a vector function $f : R^n \to R^k$ is similar to the one for a vector semilocal E-preinvex function.

Remark 5.8. Every semilocal E-preinvex function on a local star-shaped E-invex set K with respect to η is both a quasi-semilocal E-preinvex function and a pseudo-semilocal E-preinvex function.

We now present one of their elementary properties.

Proposition 5.9. *Let f be a real-valued function on a local star-shaped E-invex set $K \subset R^n$, and f possesses directional derivative with respect to the direction $\eta(E(x), y)$ at y for all $x, y \in K$. If $E(y) = y$, then the following statements hold true:*

(i) *if f is semilocal E-preinvex on K with respect to η, then $f'(y; \eta(E(x), y)) \leq f(x) - f(y)$,*

(ii) *if f is quasi-semilocal E-preinvex on K with respect to η, then $f(x) \leq f(y)$ implies that $f'(y; \eta(E(x), y)) \leq 0$,*

(iii) *if f is pseudo-semilocal E-preinvex on K with respect to η, then $f(x) < f(y)$ implies that $f'(y;\eta(E(x),y)) < 0$.*

The proof is obvious by using the related definitions and is omitted.

Theorem 5.10. *Let $x^* \in K_0$ and $E(x^*) = x^*$. Suppose that f, g possess directional derivatives with respect to the direction $\eta(E(x),x^*)$ at x^* for any $x \in K$, and assume that there is $\lambda^* \in R^m$ such that* (5.19) *and* (5.20) *hold. If f is pseudo-semilocal E-preinvex on K and $g_{I(x^*)}$ is quasi-semilocal E-preinvex on K with respect to η, then x^* is an optimal solution for (NP).*

Proof. Due to $g_{I(x^*)}(x) \le g_{I(x^*)}(x^*) = 0$, for all $x \in K_0$, it follows from Proposition 5.9(ii) that $g'_{I(x^*)}(x^*,\eta(E(x),x^*)) \le 0$, for all $x \in K_0$, which together with $\lambda^*_{I(x^*)} \ge 0$ implies

$$\lambda^{*T}_{I(x^*)} g'_{I(x^*)}(x^*,\eta(E(x),x^*)) \le 0, \quad \forall x \in K_0. \tag{5.26}$$

Moreover, using $\lambda^{*T} g(x^*) = 0$, $g(x^*) \le 0$, and $\lambda^* \ge 0$, we get

$$\lambda^{*T} g'(x^*;\eta(E(x),x^*)) \le 0, \quad \forall x \in K_0. \tag{5.27}$$

Therefore, from (5.19), we have $f'(x^*;\eta(E(x),x^*)) \ge 0$, for all $x \in K_0$.

Thus, from Proposition 5.9(iii), this implies $f(x) \ge f(x^*)$, for all $x \in K_0$.

That is, x^* is an optimal solution for (NP). □

Theorem 5.11. *Let $x^* \in K_0$ and $E(x^*) = x^*$. Suppose that f, g possess directional derivatives with respect to the direction $\eta(E(x),x^*)$ at x^* for any $x \in K$, and there is a $\lambda^* \in R^m$ such that* (5.19) *and* (5.20) *hold. If $(f + \lambda^{*T}_{I(x^*)} g_{I(x^*)})$ is pseudo-semilocal E-preinvex on K with respect to η, then x^* is an optimal solution for (NP).*

Proof. Considering $\lambda^{*T} g(x^*) = 0$, $g(x^*) \le 0$, $\lambda^* \ge 0$, and the given conditions, we have $(f + \lambda^{*T}_{I(x^*)} g_{I(x^*)})'(x^*;\eta(E(x),x^*)) = f'(x^*;\eta(E(x),x^*)) + \lambda^{*T} g'(x^*;\eta(E(x),x^*)) \ge 0$, for all $x \in K_0$. Hence, from Proposition 5.9(iii), we get

$$\left(f + \lambda^{*T}_{I(x^*)} g_{I(x^*)}\right)(x) \ge \left(f + \lambda^{*T}_{I(x^*)} g_{I(x^*)}\right)(x^*), \quad \forall x \in K_0. \tag{5.28}$$

The inequality above together with $\lambda^{*T}_{I(x^*)} g_{I(x^*)}(x^*) = 0$ follows:

$$f(x) + \lambda^{*T}_{I(x^*)} g_{I(x^*)}(x) \ge f(x^*), \quad \forall x \in K_0. \tag{5.29}$$

On account of $\lambda^*_{I(x^*)} \ge 0$ and $g_{I(x^*)}(x) \le 0$, we obtain

$$f(x) \ge f(x^*), \quad \forall x \in K_0. \tag{5.30}$$

Therefore, x^* is an optimal solution for (NP). □

The following conclusion is a direct corollary of Theorem 5.10 or Theorem 5.11.

Corollary 5.12. *Let $x^* \in K_0$ and $E(x^*) = x^*$. Suppose that f, g possess directional derivatives with respect to the direction $\eta(E(x), x^*)$ at x^* for any $x \in K$, and assume that there is a $\lambda^* \in R^m$ such that* (5.19) *and* (5.20) *hold. If f and $g_{I(x^*)}$ are semilocal E-preinvex functions on K with respect to η, then x^* is an optimal solution for (NP).*

Finally, we consider the following Mond-Weir type dual problem of (NP):

$$(\mathrm{DP})\begin{cases}\max f(u)\\ \text{s.t.}\quad f'(u;\eta(E(x),u)) + \lambda^T g'(u;\eta(E(x),u)) \geq 0, & \forall x \in K,\\ \lambda^T g(u) \geq 0,\\ \lambda \in R^m, & \lambda \geq 0,\\ u \in K.\end{cases} \tag{5.31}$$

Theorem 5.13 (weak duality). *Let x and (u, λ) be arbitrary feasible solutions of (NP) and (DP), respectively. If f and g are all semilocal E-preinvex functions on K with respect to η, and they possess directional derivatives with respect to the direction $\eta(E(x), u)$ at u, where $E(u) = u$, $x \in K$, then $f(x) \geq f(u)$.*

Proof. Considering f and g being semilocal E-preinvex on K with respect to η and $E(u) = u$, we get from Proposition 5.9(i)

$$f(x) \geq f(u) + f'(u;\eta(E(x),u)), \qquad g(x) \geq g(u) + g'(u;\eta(E(x),u)). \tag{5.32}$$

Combining the first constraint condition of (DP) and the inequalities above, we have

$$f(x) \geq f(u) - \lambda^T g'(u;\eta(E(x),u)) \geq f(u) + \lambda^T (g(u) - g(x)). \tag{5.33}$$

Hence, on account of $\lambda \geq 0$, $g(x) \leq 0$, and $\lambda^T g(u) \geq 0$, we obtain $f(x) \geq f(u)$. □

Theorem 5.14 (strong duality). *Assume that x^* is an optimal solution for (NP), $E(x^*) = x^*$, and $E(u) = u$ for any feasible point (u, λ) of (DP). Suppose that f and g are semilocal E-preinvex on K with respect to η and g_j is continuous at x^* for any $j \in \overline{I}(x^*)$, and they possess directional derivatives with respect to the direction $\eta(E(x), x^*)$ at x^* and the direction $\eta(x^*, u)$ at u, respectively, where $x \in K$. Further, assume that there is $\hat{x} \in K$ such that $g(\hat{x}) < 0$. If $f'(x^*;\eta(E(x), x^*)), g'_{I(x^*)}(x^*;\eta(E(x), x^*))$ is a convex-like function, then there is a $\lambda^* \in R^m$ such that (x^*, λ^*) is an optimal solution for (DP).*

Proof. From the assumptions and Theorem 5.5, we can conclude that there is $\lambda^* \geq 0$ such that (x^*, λ^*) is a feasible point for (DP). Assume that (u, λ) is a feasible solution of (DP). On account of f and g being semilocal E-preinvex on K with respect to η and $E(u) = u$, we get from Proposition 5.9(i)

$$f(x^*) - f(u) \geq f'(u;\eta(x^*,u)), \qquad g(x^*) - g(u) \geq g'(u;\eta(x^*,u)). \tag{5.34}$$

Combining the first constraint condition of (DP) and the relationships above, we have

$$f(x^*) - f(u) \geq -\lambda^T g'(u; \eta(x^*, u)) \geq \lambda^T (g(u) - g(x^*)). \tag{5.35}$$

Noticing that $\lambda \geq 0$, $g(x^*) \leq 0$, and $\lambda^T g(u) \geq 0$, we know $f(x^*) \geq f(u)$.

Therefore, (x^*, λ^*) is an optimal solution for (DP). □

Theorem 5.15 (converse duality). *Suppose that $x^* \in K_0$ and $(\overline{u}, \overline{\lambda})$ is a feasible point for (DP). Further, suppose that f and g are semilocal E-preinvex on K with respect to η, and f, g possess directional derivatives with respect to the direction $\eta(E(x), \overline{u})$ at $\overline{u}$ for any $x \in K$. If $f(x^*) = f(\overline{u})$ and $E(\overline{u}) = \overline{u}$, then x^* is an optimal solution for (NP).*

Proof. Since f and g are semilocal E-preinvex on K with respect to η and $E(\overline{u}) = \overline{u}$, we have from Proposition 5.9(i)

$$f(x) - f(\overline{u}) \geq f'(\overline{u}; \eta(E(x), \overline{u})), \quad g(x) - g(\overline{u}) \geq g'(\overline{u}; \eta(E(x), \overline{u})), \quad \forall x \in K_0. \tag{5.36}$$

On account of $(\overline{u}, \overline{\lambda})$ being a feasible point for (DP), we get from the first constraint inequality of (DP) and the two relationships above

$$f(x) - f(\overline{u}) \geq -\overline{\lambda}^T g'(\overline{u}; \eta(E(x), \overline{u})) \geq \overline{\lambda}^T (g(\overline{u}) - g(x)). \tag{5.37}$$

This together with $\overline{\lambda} \geq 0$, $g(x) \leq 0$, $f(x^*) = f(\overline{u})$, and $\overline{\lambda}^T g(\overline{u}) \geq 0$ follows that

$$f(x) \geq f(x^*). \tag{5.38}$$

Thus, x^* is an optimal solution for (NP). □

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