

## Research Article

# Type-2 Fuzzy Soft Sets and Their Applications in Decision Making

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Molodtsov introduced the theory of soft sets, which can be used as a general mathematical tool for dealing with uncertainty. This paper aims to introduce the concept of the type-2 fuzzy soft set by integrating the type-2 fuzzy set theory and the soft set theory. Some operations on the type-2 fuzzy soft sets are given. Furthermore, we investigate the decision making based on type-2 fuzzy soft sets. By means of level soft sets, we propose an adjustable approach to type-2 fuzzy-soft-set based decision making and give some illustrative examples. Moreover, we also introduce the weighted type-2 fuzzy soft set and examine its application to decision making.

## 1. Introduction

Soft set theory [1], firstly proposed by Molodtsov, is a general mathematical tool for dealing with uncertainty. Compared with some traditional mathematical tools for dealing with uncertainties, such as the theory of probability, the theory of fuzzy sets [2], and the theory of rough sets [3], the advantage of soft set theory is that it is free from the inadequacy of the parametrization tools of those theories. It has been demonstrated that soft set theory brings about a rich potential for applications in many fields like functions smoothness, Riemann integration, decision making, measurement theory, game theory, and so forth [1].

Soft set theory has received much attention since its introduction by Molodtsov. Maji et al. [4] first defined some operations on soft sets and introduced the soft set into the decision making problems [5]. Chen et al. [6] proposed a new definition of soft set parameterization reduction and compared it with attributes reduction in rough set theory [3]. Kong et al. [7] introduced the definition of normal parameter reduction into soft sets. Ali et al. [8] gave some new operations in soft set theory. Zou and Xiao [9] presented some data analysis approaches of soft sets under incomplete information. Çağman and Enginoğlu [10] defined soft matrices which were a matrix representation of the soft sets. Moreover, they

[11] redefined the operations of soft sets and constructed a uni-int decision making method. Herawan and Deris [12] presented an alternative approach for mining regular association rules and maximal association rules from transactional datasets using soft set theory. Gong et al. [13] proposed the concept of bijective soft set and defined some operations on it. The algebraic structure of soft set theories has been investigated in recent years. In [14], Aktaş and Çağman gave a definition of soft groups and studied their basic properties. Jun [15] introduced the notion of soft BCK/BCI-algebras and soft subalgebras. Jun and Park [16] examined the algebraic structure of BCK/BCI-algebras. Feng et al. [17] initiated the study of soft semirings by using the soft set theory and investigated several related properties. Acar et al. [18] defined soft rings and introduced their initial basic properties such as soft ideals and soft homomorphisms. Yamak et al. [19] studied soft hypergroupoids. Anvariye et al. [20] investigated the algebraic hyperstructures of soft sets associated to semihypergroups.

It should be noted that all of above works are based on the classical soft set theory. The soft set model, however, can also be combined with other mathematical models. Maji et al. [21] first introduced the concept of fuzzy soft sets by combining the soft sets and fuzzy sets. Majumdar and Samanta [22] defined generalised fuzzy soft sets and discussed application of generalised fuzzy soft sets in decision making problem and medical diagnosis. By combining the vague set and soft set models, Xu et al. [23] introduced the notion of vague soft set. Yang et al. [24] introduced the concept of the interval-valued fuzzy soft set which is a combination of the soft set and the interval-valued fuzzy set. Feng et al. [25] focused on a tentative approach to soft sets combined with fuzzy sets and rough sets and proposed three different types of hybrid models, which are called rough soft sets, soft rough sets, and soft-rough fuzzy sets, respectively. Bhattacharya and Davvaz [26] introduced the concepts of intuitionistic fuzzy lower soft rough approximation and IF upper soft rough approximation space. Maji et al. [27, 28] proposed the notion of intuitionistic fuzzy soft sets by integrating the soft sets and intuitionistic fuzzy sets [29]. By combining the interval-valued intuitionistic fuzzy sets and soft sets, Jiang et al. [30] obtained a new soft set model: interval-valued intuitionistic fuzzy soft set theory. Aygünöglu and Aygün [31] focused on fuzzy soft groups, homomorphism of fuzzy soft groups, and normal fuzzy soft groups.

According to Mendel [32], there exist at least four sources of uncertainties in type-1 fuzzy logic systems (T1 FLS), which are as follows: (1) meanings of the words that are used in the antecedents and consequents of rules can be uncertain (words mean different things to different people); (2) consequents may have a histogram of values associated with them, especially when knowledge is extracted from a group of experts, all of whom do not collectively agree; (3) measurements that activate a T1 FLS may be noisy and therefore uncertain; (4) the data that are used to tune the parameters of a T1 FLS may also be noisy. All these uncertainties lead to uncertain fuzzy-set membership functions. Ordinary type-1 fuzzy sets cannot model such uncertainties directly, because they are characterized by crisp membership functions. Type-2 fuzzy sets are capable of modeling the four uncertainties. The concept of type-2 fuzzy sets, first proposed by Zadeh [33], is an extension of a type-1 fuzzy set in which its membership function falls into a fuzzy set in the interval  $[0, 1]$ . Because type-2 fuzzy sets can improve certain kinds of inference better than do fuzzy sets with increasing imprecision, uncertainty, and fuzziness in information, type-2 fuzzy sets are gaining more and more in popularity. The basic concepts of type-2 fuzzy set theory and its extensions, as well as some practical applications, can be found in [34–40].

However, in the practical applications, we are often faced with the situation in which the evaluation of parameters is a fuzzy concept. For instance, when we are going to buy a car, we need to consider the safety of car. We can provide some linguistic terms, such as good,

medium, and bad, as the evaluation about the safety of car. Here, good, medium, and bad are fuzzy concepts and they can be represented by fuzzy sets rather than exact numerical values, interval numbers, intuitionistic fuzzy numbers, and interval-valued intuitionistic fuzzy numbers. Obviously, it is very difficult for the classical soft set and its existing extensions to deal with the above case because the evaluation of parameters of the object is a fuzzy concept rather than an exact numerical value, an interval number, an intuitionistic fuzzy number, and an interval-valued intuitionistic fuzzy number. Hence, it is necessary to extend soft set theory to accommodate the situations in which the evaluation of parameters is a fuzzy concept. As mentioned above, type-2 fuzzy set can be used to represent the fuzziness of the above evaluation of parameters directly. Thus, it is very necessary to extend soft set theory using type-2 fuzzy set. The purpose of this paper is to further extend the concept of soft set theory by combining type-2 fuzzy set and soft set, from which we can obtain a new soft set model: type-2 fuzzy soft sets. We present the concept of type-2 fuzzy soft sets and define some operations on type-2 fuzzy soft sets. Moreover, we also investigate the applications of type-2 fuzzy soft sets and weighted type-2 fuzzy soft sets in decision making problems.

The remainder of this paper is organized as follows. After recalling some preliminaries, Section 3 presents the concept of the type-2 fuzzy soft set and some operations on type-2 fuzzy soft sets. In the sequel, applications of type-2 fuzzy soft sets and weighted type-2 fuzzy soft sets in decision making problems are, respectively, shown in Sections 4 and 5. Finally, conclusions are given in Section 6.

## 2. Preliminaries

In this section, we will review the concepts of soft set, type-2 fuzzy set, and interval type-2 fuzzy set.

### 2.1. Soft Sets

Let  $U$  be an initial universe of objects and  $E_U$  ( $E$ , for short) the set of parameters in relation to objects in  $U$ . Parameters are often attributes, characteristics, or properties of objects. Let  $P(U)$  denote the power set of  $U$  and  $A \subseteq E$ . Molodtsov [1] defined a soft set as follows.

*Definition 2.1* (see [1]). A pair  $(F, A)$  is called a soft set over  $U$ , where  $F$  is a mapping given by

$$F : A \longrightarrow P(U). \quad (2.1)$$

In other words, a soft set over  $U$  is a parameterized family of subsets of the universe  $U$ . For  $\varepsilon \in A$ ,  $F(\varepsilon)$  is regarded as the set of  $\varepsilon$ -approximate elements of the soft set  $(F, A)$ . Clearly, a soft set is not a set. For illustration, Molodtsov considered several examples in [1].

*Definition 2.2* (see [5]). Let  $(F, A)$  be a soft set over  $U$ . The choice value of an object  $h_i \in U$  is  $c_i$ , given by

$$c_i = \sum_{\varepsilon \in A} h_{i\varepsilon}, \quad (2.2)$$

where if  $h_i \in F(\varepsilon)$  then  $h_{i\varepsilon} = 1$ , otherwise  $h_{i\varepsilon} = 0$ .

**Table 1:** Tabular representation of  $(F, A)$ .

$U$	$\varepsilon_1$	$\varepsilon_2$	$\varepsilon_3$	$\varepsilon_4$	$\varepsilon_5$
$h_1$	0	1	0	1	0
$h_2$	1	0	0	1	1
$h_3$	0	0	1	1	0
$h_4$	0	1	1	0	0
$h_5$	0	1	1	1	0
$h_6$	1	0	1	0	0

As an illustration, let us consider the following example originally introduced by Molodtsov [1].

*Example 2.3* (A house purchase problem). Suppose the following. The universe  $U = \{h_1, h_2, h_3, h_4, h_5, h_6\}$  is the set of six houses under consideration.  $A$  is the set of parameters that Mr. X is interested in buying a house.  $A = \{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5\}$ , where  $\varepsilon_i$  ( $i = 1, 2, 3, 4, 5$ ) stands for the parameters in a word of “expensive,” “beautiful”, “wooden”, “in the green surroundings”, and “convenient traffic”, respectively. That means, out of available houses in  $U$ , Mr. X is to select that house which qualifies with all (or with maximum number of) parameters of the set  $A$ . In this case, to define a soft set means to point out expensive houses, beautiful houses, and so on. The soft set  $(F, A)$  describes the “attractiveness of the houses” which Mr. X (say) is going to buy.

Suppose that

$$\begin{aligned} F(\varepsilon_1) &= \{h_2, h_6\}, & F(\varepsilon_2) &= \{h_1, h_4, h_5\}, & F(\varepsilon_3) &= \{h_3, h_4, h_5, h_6\}, \\ F(\varepsilon_4) &= \{h_1, h_2, h_3, h_5\}, & F(\varepsilon_5) &= \{h_2\}. \end{aligned} \quad (2.3)$$

The soft set  $(F, A)$  is a parametrized family  $\{F(\varepsilon_i), i = 1, 2, 3, 4, 5\}$  of subsets of the set  $U$  and gives us a collection of approximate descriptions of an object. Consider the mapping  $F$  which is “houses ( $\cdot$ )” where dot ( $\cdot$ ) is to be filled up by a parameter  $\varepsilon \in A$ . For instance,  $F(\varepsilon_1)$  means “houses (expensive)” whose functional value is the set  $\{h \in U, h \text{ is an expensive house}\} = \{h_2, h_6\}$ . Thus, we can view the soft set  $(F, A)$  as a collection of approximations as below:

$$\begin{aligned} (F, A) &= \{\text{expensive houses} = \{h_2, h_6\}, \text{beautiful houses} = \{h_1, h_4, h_5\}, \\ &\text{wooden houses} = \{h_3, h_4, h_5, h_6\}, \\ &\text{houses in the green surroundings} = \{h_1, h_2, h_3, h_5\}, \\ &\text{convenient traffic houses} = \{h_2\}\}. \end{aligned} \quad (2.4)$$

In order to store a soft set in a computer, we could represent a soft set in the form of a 0-1 two-dimensional table. Table 1 is the tabular representation of the soft set  $(F, A)$ . If  $h_i \in F(\varepsilon_j)$ , then  $h_{ij} = 1$ , otherwise  $h_{ij} = 0$ , where  $h_{ij}$  are the entries in Table 1.

## 2.2. Type-2 Fuzzy Sets and Interval Type-2 Fuzzy Sets

In the current subsection, we recall the notions of type-2 fuzzy sets and interval type-2 fuzzy sets from [32, 35, 39].

*Definition 2.4.* Let  $U$  be a finite and nonempty set, which is referred to as the universe. A type-2 fuzzy set, denoted by  $\tilde{A}$ , is characterized by a type-2 membership function  $\mu_{\tilde{A}}(x, u) : U \times I \rightarrow I$ , where  $x \in U$ ,  $I = [0, 1]$  and  $u \in J_x \subseteq I$ , that is,

$$\tilde{A} = \{((x, u), \mu_{\tilde{A}}(x, u)) \mid x \in U, u \in J_x \subseteq I\}, \quad (2.5)$$

where  $0 \leq \mu_{\tilde{A}}(x, u) \leq 1$ .  $\tilde{A}$  can also be expressed as

$$\tilde{A} = \int_{x \in U} \int_{u \in J_x} \frac{\mu_{\tilde{A}}(x, u)}{(x, u)} = \int_{x \in U} \frac{\left[ \int_{u \in J_x} f_x(u)/u \right]}{x}, \quad J_x \subseteq I, \quad (2.6)$$

where  $f_x(u) = \mu_{\tilde{A}}(x, u)$ .

The class of all type-2 fuzzy sets of the universe  $U$  is denoted by  $F_{T2}(U)$ .

*Definition 2.5.* At each value of  $x$ , say  $x = x'$ , the 2D plane whose axes are  $u$  and  $\mu_{\tilde{A}}(x', u)$  is called the vertical slice of  $\mu_{\tilde{A}}(x, u)$ . A secondary membership function is a vertical slice of  $\mu_{\tilde{A}}(x, u)$ . It is  $\mu_{\tilde{A}}(x = x', u)$  for  $x' \in U$  and for all  $u \in J_{x'} \subseteq I$ , that is,

$$\mu_{\tilde{A}}(x = x', u) = \mu_{\tilde{A}}(x') = \int_{u \in J_{x'}} \frac{f_{x'}(u)}{u}, \quad \forall u \in J_{x'} \subseteq I, \quad (2.7)$$

where  $0 \leq f_{x'}(u) \leq 1$ . The amplitude of a secondary membership function is called a secondary grade. In Definition 2.4,  $f_x(u)$  and  $\mu_{\tilde{A}}(x, u)$  are all secondary grades.

*Definition 2.6.* The domain of a secondary membership function is called the primary membership of  $x$ . In Definition 2.5,  $J_{x'}$  is the primary membership of  $x'$ .

*Definition 2.7.* If all the secondary grades of a type-2 fuzzy set  $\tilde{A}$  are equal to 1, that is,  $\mu_{\tilde{A}}(x, u) = 1$ , for all  $x \in U$  and for all  $u \in J_x \subseteq I$ , then  $\tilde{A}$  is defined as an interval type-2 fuzzy set.

### 2.3. Operations of Type-2 Fuzzy Sets

Let  $U$  be a nonempty universe,  $\tilde{A}, \tilde{B} \in F_{T2}(U)$ :

$$\begin{aligned}\tilde{A} &= \int_{x \in U} \frac{\mu_{\tilde{A}}(x)}{x} = \int_{x \in U} \frac{\left[ \int_{u \in J_x^u} f_x(u)/u \right]}{x}, \quad J_x^u \subseteq I, \\ \tilde{B} &= \int_{x \in U} \frac{\mu_{\tilde{B}}(x)}{x} = \int_{x \in U} \frac{\left[ \int_{w \in J_x^w} g_x(w)/w \right]}{x}, \quad J_x^w \subseteq I.\end{aligned}\tag{2.8}$$

The union, intersection, and complement for type-2 fuzzy sets are defined as follows.

(1) Union of two type-2 fuzzy sets  $\tilde{A} \cup \tilde{B}$ :

$$\mu_{\tilde{A} \cup \tilde{B}}(x) = \int_{u \in J_x^u} \int_{w \in J_x^w} \frac{[f_x(u) \wedge g_x(w)]}{(u \vee w)} = \mu_{\tilde{A}}(x) \sqcup \mu_{\tilde{B}}(x), \quad x \in U,\tag{2.9}$$

where  $\wedge$  is the minimum operation,  $\vee$  is the maximum operation,  $\sqcup$  is called the join operation,  $\mu_{\tilde{A} \cup \tilde{B}}(x)$ ,  $\mu_{\tilde{A}}(x)$ , and  $\mu_{\tilde{B}}(x)$  are the secondary membership functions, and all are type-1 fuzzy sets.

(2) Intersection of two type-2 fuzzy sets  $\tilde{A} \cap \tilde{B}$ :

$$\mu_{\tilde{A} \cap \tilde{B}}(x) = \int_{u \in J_x^u} \int_{w \in J_x^w} \frac{[f_x(u) \wedge g_x(w)]}{(u \wedge w)} = \mu_{\tilde{A}}(x) \sqcap \mu_{\tilde{B}}(x), \quad x \in U,\tag{2.10}$$

where  $\wedge$  is the minimum operation,  $\sqcap$  is called the meet operation,  $\mu_{\tilde{A} \cap \tilde{B}}(x)$ ,  $\mu_{\tilde{A}}(x)$ , and  $\mu_{\tilde{B}}(x)$  are the secondary membership functions, and all are type-1 fuzzy sets.

(3) Complement of a type-2 fuzzy set  $\sim \tilde{A}$ :

$$\mu_{\sim \tilde{A}}(x) = \neg \mu_{\tilde{A}}(x) = \int_{u \in J_x^u} \frac{f_x(u)}{(1-u)}.\tag{2.11}$$

*Example 2.8.* Let  $U = \{h_1, h_2, h_3, h_4, h_5, h_6\}$  a nonempty universe, and let  $\tilde{A}$  and  $\tilde{B}$  be two type-2 fuzzy sets over the same universe  $U$ .

Suppose that

$$\begin{aligned}\tilde{A} &= \frac{0.3/0.1 + 1/0.5}{h_1} + \frac{1/0.5 + 0.3/0.6}{h_2} + \frac{1/0.8}{h_3} \\ &\quad + \frac{0.7/0.5 + 0.2/0.6}{h_4} + \frac{0.5/0.9}{h_5} + \frac{0.3/0.2 + 1/0.6}{h_6},\end{aligned}$$

$$\begin{aligned} \tilde{B} = & \frac{0.7/0.1 + 1/0.2}{h_1} + \frac{1/0.6}{h_2} + \frac{0.6/0.5 + 1/0.9}{h_3} \\ & + \frac{0.4/0.4 + 1.0/0.5}{h_4} + \frac{0.6/0.9}{h_5} + \frac{1/0.6 + 0.5/0.8}{h_6}. \end{aligned} \quad (2.12)$$

Then, we have that

$$\begin{aligned} \tilde{A} \cap \tilde{B} = & \frac{0.7/0.1 + 1.0/0.2}{h_1} + \frac{1.0/0.5 + 0.3/0.6}{h_2} + \frac{0.6/0.5 + 1/0.8}{h_3} \\ & + \frac{0.4/0.4 + 0.7/0.5}{h_4} + \frac{0.5/0.9}{h_5} + \frac{0.3/0.2 + 1.0/0.6}{h_6}, \\ \tilde{A} \cup \tilde{B} = & \frac{0.3/0.1 + 0.3/0.2 + 1.0/0.5}{h_1} + \frac{1.0/0.6}{h_2} + \frac{0.6/0.8 + 1/0.9}{h_3} \\ & + \frac{0.7/0.5 + 0.2/0.6}{h_4} + \frac{0.5/0.9}{h_5} + \frac{1/0.6 + 0.5/0.8}{h_6}, \\ \sim \tilde{A} = & \frac{0.3/0.9 + 1/0.5}{h_1} + \frac{1/0.5 + 0.3/0.4}{h_2} + \frac{1/0.2}{h_3} \\ & + \frac{0.7/0.5 + 0.2/0.4}{h_4} + \frac{0.5/0.1}{h_5} + \frac{0.3/0.8 + 1/0.4}{h_6}. \end{aligned} \quad (2.13)$$

#### 2.4. Cut Sets of Type-2 Fuzzy Sets

*Definition 2.9* (see [41, 42]). Let  $\tilde{A}$  be a type-2 fuzzy set on the universe  $U$ ,  $\mu_{\tilde{A}}(x)$  the secondary membership function of  $\tilde{A}$ , and  $\mu_{\tilde{A}}(x) = \int_{u \in J_x} f_x(u)/u$ ,  $x \in U$ ,  $J_x \subseteq I$ . Then the secondary  $\alpha$ -cut set  $\tilde{A}^\alpha$  of  $\tilde{A}$  is defined by

$$\tilde{A}^\alpha = \int_{x \in U} \frac{\left[ \int_{u \in J_x^\alpha} 1/u \right]}{x}, \quad (2.14)$$

where  $J_x^\alpha = \{u \mid f_x(u) \geq \alpha, u \in J_x\} \subseteq [0, 1]$  and  $\alpha \in [0, 1]$ .

By Definitions 2.7 and 2.9, we can see that the secondary  $\alpha$ -cut set  $\tilde{A}^\alpha$  of  $\tilde{A}$  is an interval type-2 fuzzy set.

*Definition 2.10* (see [41, 42]). Let  $\tilde{A}$  be an interval type-2 fuzzy set on the universe  $U$ ,  $\mu_{\tilde{A}}(x)$  the secondary membership function of  $\tilde{A}$ , and  $\mu_{\tilde{A}}(x) = \int_{u \in J_x} 1/u$ ,  $x \in U$ ,  $J_x \subseteq I$ . Then the primary  $\lambda$ -cut set of  $\tilde{A}$  is defined by

$$\tilde{A}_\lambda = \{x \mid u(x) \geq \lambda, \forall u(x) \in J_x\}, \quad (2.15)$$

where  $J_x \subseteq [0, 1]$  and  $\lambda \in [0, 1]$ .

By Definition 2.10, the primary  $\lambda$ -cut set  $\tilde{A}_\lambda^\alpha$  of the secondary  $\alpha$ -cut set  $\tilde{A}^\alpha$  of the type-2 fuzzy set  $\tilde{A}$  is defined as follows.

*Definition 2.11* (see [41, 42]). Let  $\tilde{A}^\alpha$  be the secondary  $\alpha$ -cut set of the type-2 fuzzy set  $\tilde{A}$ . Then the primary  $\lambda$ -cut set of  $\tilde{A}^\alpha$  is defined by

$$\tilde{A}_\lambda^\alpha = \{x \mid u(x) \geq \lambda, \forall u(x) \in J_x^\alpha\}, \quad (2.16)$$

where  $J_x^\alpha \subseteq J_x \subseteq [0, 1]$ ,  $\alpha \in [0, 1]$ , and  $\lambda \in [0, 1]$ .

The cut set of the type-2 fuzzy set contains the secondary cut set and the primary cut set, and it is the primary  $\lambda$ -cut set of the secondary  $\alpha$ -cut set of the type-2 fuzzy set. Therefore, when we compute the cut set of the type-2 fuzzy set  $\tilde{A}$ , the first step is to compute the secondary  $\alpha$ -cut set  $\tilde{A}^\alpha$  of  $\tilde{A}$ , and the second step is to compute the primary  $\lambda$ -cut set  $\tilde{A}_\lambda^\alpha$  of  $\tilde{A}^\alpha$ .

### 3. Type-2 Fuzzy Soft Sets

In this section, we will initiate the study on hybrid structures involving both type-2 fuzzy sets and soft sets. In Section 3.1, we introduce the concept of the type-2 fuzzy soft set which is an extension of the soft set [32]. Next, in Section 3.2, we discuss some operations on type-2 fuzzy soft sets.

#### 3.1. Concept of Type-2 Fuzzy Soft Sets

*Definition 3.1.* Let  $U$  be an initial universe and  $A \subseteq E$  a set of parameters; a pair  $(\mathcal{F}, A)$  is called a type-2 fuzzy soft set over  $U$ , where  $\mathcal{F}$  is a mapping given by

$$\mathcal{F} : A \longrightarrow F_{T2}(U). \quad (3.1)$$

In other words, a type-2 fuzzy soft set is a parameterized family of type-2 fuzzy subsets of  $U$ . For any  $\varepsilon \in A$ ,  $\mathcal{F}(\varepsilon)$  is referred as the set of  $\varepsilon$ -approximate elements of the type-2 fuzzy soft set  $(\mathcal{F}, A)$ , it is actually a type-2 fuzzy set on  $U$ , and it can be written as.

$$\mathcal{F}(\varepsilon) = \int_{x \in U} \int_{u \in J_x} \frac{\mu_{\mathcal{F}(\varepsilon)}(x, u)}{(x, u)} = \int_{x \in U} \frac{\left[ \int_{u \in J_x} f_x(u)/u \right]}{x}, \quad J_x \subseteq I. \quad (3.2)$$

Here,  $u$ ,  $\mu_{\mathcal{F}(\varepsilon)}(x, u)$  are, respectively, the primary membership degree and secondary membership degree that object  $x$  holds on parameter  $\varepsilon$ .

To illustrate the idea, let us consider the following example (adapted from Mendel and Wu [43]).

*Example 3.2* (see [43]). Consider a type-2 fuzzy soft set  $(\mathcal{F}, A)$  over  $U$ , where  $U$  is a set of six houses under the consideration of a decision maker to purchase, which is denoted by  $U = \{h_1, h_2, h_3, h_4, h_5, h_6\}$ , and  $A$  is a parameter set, where  $A = \{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5\} = \{\text{expensive; beautiful; wooden; in the green surroundings; convenient traffic}\}$ . The type-2 fuzzy soft set  $(\mathcal{F}, A)$  describes the ‘‘attractiveness of the houses’’ to this decision maker.



Suppose that

$$\begin{aligned}
\mathcal{F}(\varepsilon_1) &= \frac{\text{in expensive}}{h_1} + \frac{\text{moderately inexpensive}}{h_2} + \frac{\text{very expensive}}{h_3} \\
&\quad + \frac{\text{just right}}{h_4} + \frac{\text{moderately expensive}}{h_5} + \frac{\text{expensive}}{h_6}, \\
\mathcal{F}(\varepsilon_2) &= \frac{\text{moderately beautiful}}{h_1} + \frac{\text{just right}}{h_2} + \frac{\text{very beautiful}}{h_3} \\
&\quad + \frac{\text{moderately not beautiful}}{h_4} + \frac{\text{beautiful}}{h_5} + \frac{\text{not beautiful}}{h_6}, \\
\mathcal{F}(\varepsilon_3) &= \frac{\text{wooden}}{h_1} + \frac{\text{moderately not wooden}}{h_2} + \frac{\text{not wooden}}{h_3} \\
&\quad + \frac{\text{just right}}{h_4} + \frac{\text{moderately wooden}}{h_5} + \frac{\text{very wooden}}{h_6}, \\
\mathcal{F}(\varepsilon_4) &= \frac{\text{in the moderately green surroundings}}{h_1} + \frac{\text{in the green surroundings}}{h_2} \\
&\quad + \frac{\text{in the very green surroundings}}{h_3} + \frac{\text{just right}}{h_4} \\
&\quad + \frac{\text{not in the green surroundings}}{h_5} + \frac{\text{moderately not in the green surroundings}}{h_6}, \\
\mathcal{F}(\varepsilon_5) &= \frac{\text{very convenient traffic}}{h_1} + \frac{\text{inconvenient traffic}}{h_2} + \frac{\text{moderately convenient traffic}}{h_3} \\
&\quad + \frac{\text{convenient traffic}}{h_4} + \frac{\text{moderately inconvenient traffic}}{h_5} + \frac{\text{just right}}{h_6}.
\end{aligned} \tag{3.3}$$

We can convert the above linguistic terms into the corresponding fuzzy sets and obtain the following results:

$$\begin{aligned}
\mathcal{F}(\varepsilon_1) &= \frac{0.3/0.1 + 1/0.2 + 0.7/0.3}{h_1} + \frac{0.4/0.3 + 1/0.5 + 0.3/0.6}{h_2} \\
&\quad + \frac{0.6/0.5 + 1/0.9}{h_3} + \frac{0.3/0.4 + 0.8/0.5 + 0.2/0.6}{h_4} + \frac{0.5/0.7}{h_5} \\
&\quad + \frac{0.3/0.1 + 1/0.6 + 0.5/0.7 + 0.2/0.9}{h_6},
\end{aligned}$$

$$\begin{aligned}
\mathcal{F}(\varepsilon_2) &= \frac{0.3/0.1 + 0.6/0.4 + 0.5/0.7 + 0.5/0.8}{h_1} + \frac{0.1/0.1 + 0.6/0.2 + 0.9/0.6}{h_2} \\
&\quad + \frac{0.7/0.8}{h_3} + \frac{0.9/0.4}{h_4} + \frac{0.6/0.5 + 1/0.6}{h_5} + \frac{0.6/0.1 + 0.7/0.4}{h_6}, \\
\mathcal{F}(\varepsilon_3) &= \frac{0.2/0.6 + 0.8/0.8 + 0.6/0.9}{h_1} + \frac{0.6/0.4 + 0.8/0.7}{h_2} + \frac{0.5/0.3 + 0.9/0.4 + 0.4/0.6}{h_3} \\
&\quad + \frac{0.3/0.5 + 0.9/0.6 + 0.4/0.8}{h_4} + \frac{0.6/0.5 + 1/0.6}{h_5} + \frac{1/0.9}{h_6}, \\
\mathcal{F}(\varepsilon_4) &= \frac{0.5/0.7}{h_1} + \frac{0.3/0.1 + 0.8/0.5 + 0.5/0.8}{h_2} + \frac{0.7/0.4 + 0.7/0.8}{h_3} \\
&\quad + \frac{0.2/0.4 + 0.7/0.5 + 0.5/0.6}{h_4} + \frac{0.7/0.1 + 0.6/0.2 + 0.3/0.5 + 0.5/0.6}{h_5} \\
&\quad + \frac{0.4/0.5 + 0.3/0.7}{h_6}, \\
\mathcal{F}(\varepsilon_5) &= \frac{0.9/0.5 + 1/0.8}{h_1} + \frac{0.1/0.1 + 0.9/0.5 + 0.6/0.6}{h_2} \\
&\quad + \frac{0.3/0.2 + 0.7/0.4 + 0.8/0.5 + 0.9/0.7}{h_3} + \frac{0.7/0.5 + 1/0.8}{h_4} \\
&\quad + \frac{0.9/0.5}{h_5} + \frac{0.2/0.6 + 0.5/0.7 + 0.4/0.9}{h_6}.
\end{aligned} \tag{3.4}$$

The type-2 fuzzy soft set  $(\mathcal{F}, A)$  is a parameterized family  $\{\mathcal{F}(\varepsilon_i), i = 1, 2, 3, 4, 5\}$  of type-2 fuzzy sets on  $U$ , and

$$\begin{aligned}
(\mathcal{F}, A) &= \left\{ \begin{array}{l} \text{expensive houses} \\ = \frac{0.3/0.1 + 1/0.2 + 0.7/0.3}{h_1} + \frac{0.4/0.3 + 1/0.5 + 0.3/0.6}{h_2} + \frac{0.6/0.5 + 1/0.9}{h_3} \\ \quad + \frac{0.3/0.4 + 0.8/0.5 + 0.2/0.6}{h_4} + \frac{0.5/0.7}{h_5} + \frac{0.3/0.1 + 1/0.6 + 0.5/0.7 + 0.2/0.9}{h_6}, \\ \text{beautiful houses} \\ = \frac{0.3/0.1 + 0.6/0.4 + 0.5/0.7 + 0.5/0.8}{h_1} + \frac{0.1/0.1 + 0.6/0.2 + 0.9/0.6}{h_2} + \frac{0.7/0.8}{h_3} \\ \quad + \frac{0.9/0.4}{h_4} + \frac{0.6/0.5 + 1/0.6}{h_5} + \frac{0.6/0.1 + 0.7/0.4}{h_6}, \end{array} \right.
\end{aligned}$$

wooden houses

$$= \frac{0.2/0.6 + 0.8/0.8 + 0.6/0.9}{h_1} + \frac{0.6/0.4 + 0.8/0.7}{h_2} + \frac{0.5/0.3 + 0.9/0.4 + 0.4/0.6}{h_3} \\ + \frac{0.3/0.5 + 0.9/0.6 + 0.4/0.8}{h_4} + \frac{0.6/0.5 + 1/0.6}{h_5} + \frac{1/0.9}{h_6},$$

in the green surroundings houses

$$= \frac{0.5/0.7}{h_1} + \frac{0.3/0.1 + 0.8/0.5 + 0.5/0.8}{h_2} + \frac{0.7/0.4 + 0.7/0.8}{h_3} \\ + \frac{0.2/0.4 + 0.7/0.5 + 0.5/0.6}{h_4} + \frac{0.7/0.1 + 0.6/0.2 + 0.3/0.5 + 0.5/0.6}{h_5} \\ + \frac{0.4/0.5 + 0.3/0.7}{h_6},$$

convenient traffic houses

$$= \frac{0.9/0.5 + 1/0.8}{h_1} + \frac{0.1/0.1 + 0.9/0.5 + 0.6/0.6}{h_2} \\ + \frac{0.3/0.2 + 0.7/0.4 + 0.8/0.5 + 0.9/0.7}{h_3} + \frac{0.7/0.5 + 1/0.8}{h_4} + \frac{0.9/0.5}{h_5} \\ + \frac{0.2/0.6 + 0.5/0.7 + 0.4/0.9}{h_6} \Bigg\}.$$

(3.5)

Table 2 gives the tabular representation of the type-2 fuzzy soft set  $(\mathcal{F}, A)$ . We can see that the precise evaluation for each object on each parameter is unknown. For example, we cannot present the precise degree of how expensive house  $h_1$  is; however, we can say that house  $h_1$  is inexpensive.

*Definition 3.3.* For two type-2 fuzzy soft sets  $(\mathcal{F}, A)$  and  $(\mathcal{G}, B)$  over  $U$ , one says that  $(\mathcal{F}, A)$  is a type-2 fuzzy soft subset of  $(\mathcal{G}, B)$  if and only if  $A \subseteq B$  and for all  $\varepsilon \in A$ ,  $\mathcal{F}(\varepsilon) \subseteq \mathcal{G}(\varepsilon)$ . One denotes this relationship by  $(\mathcal{F}, A) \tilde{\subseteq} (\mathcal{G}, B)$ .  $(\mathcal{F}, A)$  is said to be a type-2 fuzzy soft super set of  $(\mathcal{G}, B)$ , if  $(\mathcal{G}, B)$  is a type-2 fuzzy soft subset of  $(\mathcal{F}, A)$ . One denotes it by  $(\mathcal{F}, A) \tilde{\supseteq} (\mathcal{G}, B)$ .

*Example 3.4.* Let  $(\mathcal{F}, A)$  and  $(\mathcal{G}, B)$  be two type-2 fuzzy soft sets over the same universe  $U$  as follows:

$$\mathcal{F}(\varepsilon_1) = \frac{0.3/0.1 + 1/0.2 + 0.7/0.3}{h_1} + \frac{1/0.5 + 0.3/0.6}{h_2} + \frac{0.6/0.5 + 1/0.9}{h_3} \\ + \frac{0.3/0.4 + 0.8/0.5 + 0.2/0.6}{h_4} + \frac{0.5/0.7}{h_5} + \frac{0.3/0.1 + 1/0.6 + 0.2/0.9}{h_6},$$

$$\begin{aligned}
\mathcal{F}(\varepsilon_2) &= \frac{0.3/0.6 + 0.9/0.8 + 0.7/0.9}{h_1} + \frac{0.6/0.4 + 0.9/0.7}{h_2} + \frac{0.5/0.3 + 1.0/0.4 + 0.4/0.6}{h_3} \\
&\quad + \frac{0.3/0.5 + 0.9/0.6 + 0.4/0.8}{h_4} + \frac{0.6/0.5 + 1/0.6}{h_5} + \frac{1/0.9}{h_6}, \\
\mathcal{G}(\varepsilon_1) &= \frac{0.4/0.1 + 1/0.2 + 0.8/0.3}{h_1} + \frac{0.6/0.2 + 1/0.5 + 0.5/0.6}{h_2} + \frac{0.6/0.5 + 1/0.9}{h_3} \\
&\quad + \frac{0.4/0.4 + 1.0/0.5 + 0.2/0.6}{h_4} + \frac{0.6/0.7}{h_5} + \frac{0.3/0.1 + 1/0.6 + 0.5/0.9}{h_6}, \\
\mathcal{G}(\varepsilon_2) &= \frac{0.4/0.6 + 1.0/0.8 + 0.8/0.9}{h_1} + \frac{1.0/0.4 + 0.9/0.7}{h_2} + \frac{0.5/0.3 + 1.0/0.4 + 0.4/0.6}{h_3} \\
&\quad + \frac{0.3/0.5 + 1.0/0.6 + 0.5/0.8}{h_4} + \frac{0.7/0.5 + 1/0.6}{h_5} + \frac{0.4/0.6 + 1/0.9}{h_6}, \\
\mathcal{G}(\varepsilon_3) &= \frac{1/0.8}{h_1} + \frac{0.1/0.1 + 0.8/0.5 + 0.6/0.6}{h_2} + \frac{0.3/0.2 + 0.8/0.5 + 0.9/0.7}{h_3} \\
&\quad + \frac{0.7/0.6 + 1/0.8}{h_4} + \frac{0.9/0.7}{h_5} + \frac{0.3/0.6 + 1.0/0.7 + 0.6/0.9}{h_6},
\end{aligned} \tag{3.6}$$

where  $U = \{h_1, h_2, h_3, h_4, h_5, h_6\}$  is the set of houses,  $A = \{\varepsilon_1, \varepsilon_2\} = \{\text{expensive; wooden}\}$ , and  $B = \{\varepsilon_1, \varepsilon_2, \varepsilon_3\} = \{\text{expensive; wooden; beautiful}\}$ .

Clearly, by Definition 3.3, we have  $(\mathcal{F}, A) \tilde{\subseteq} (\mathcal{G}, B)$ .

*Definition 3.5.* Two type-2 fuzzy soft sets  $(\mathcal{F}, A)$  and  $(\mathcal{G}, B)$  over a common universe  $U$  are said to be type-2 fuzzy soft equal if  $(\mathcal{F}, A)$  is a type-2 fuzzy soft subset of  $(\mathcal{G}, B)$  and  $(\mathcal{G}, B)$  is a type-2 fuzzy soft subset of  $(\mathcal{F}, A)$ , which can be denoted by  $(\mathcal{F}, A) \tilde{=} (\mathcal{G}, B)$ .

### 3.2. Operations on Type-2 Fuzzy Soft Sets

*Definition 3.6* (see [4]). Let  $E = \{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n\}$  be a parameter set. The not set of  $E$ , denoted by  $\neg E$ , is defined by  $\neg E = \{\neg\varepsilon_1, \neg\varepsilon_2, \dots, \neg\varepsilon_n\}$ , where  $\neg\varepsilon_i = \text{not } \varepsilon_i$ .

*Definition 3.7.* The complement of a type-2 fuzzy soft set  $(\mathcal{F}, A)$  is denoted by  $(\mathcal{F}, A)^c$ , and it is defined by

$$(\mathcal{F}, A)^c = (\mathcal{F}^c, \neg A), \tag{3.7}$$

where  $\mathcal{F}^c : \neg A \rightarrow F_{T_2}(U)$  is a mapping given by  $\mathcal{F}^c(\varepsilon) = \sim (\mathcal{F}(\neg\varepsilon))$ , for all  $\varepsilon \in \neg A$ .

**Table 2:** A type-2 fuzzy soft set  $(\mathcal{F}, A)$ .

$U$	$\varepsilon_1$	$\varepsilon_2$	$\varepsilon_3$	$\varepsilon_4$	$\varepsilon_5$
$h_1$	$\frac{0.3}{0.1} + \frac{1}{0.2} + \frac{0.7}{0.3}$	$\frac{0.3}{0.1} + \frac{0.6}{0.4} + \frac{0.5}{0.7} + \frac{0.5}{0.8}$	$\frac{0.2}{0.6} + \frac{0.8}{0.8} + \frac{0.6}{0.9}$	$\frac{0.5}{0.7}$	$\frac{0.9}{0.5} + \frac{1}{0.8}$
$h_2$	$\frac{0.4}{0.3} + \frac{1}{0.5} + \frac{0.3}{0.6}$	$\frac{0.1}{0.1} + \frac{0.6}{0.2} + \frac{0.9}{0.6}$	$\frac{0.6}{0.4} + \frac{0.8}{0.7}$	$\frac{0.3}{0.1} + \frac{0.8}{0.5} + \frac{0.5}{0.8}$	$\frac{0.1}{0.1} + \frac{0.9}{0.5} + \frac{0.6}{0.6}$
$h_3$	$\frac{0.6}{0.5} + \frac{1}{0.9}$	$\frac{0.7}{0.8}$	$\frac{0.5}{0.3} + \frac{0.9}{0.4} + \frac{0.4}{0.6}$	$\frac{0.7}{0.4} + \frac{0.7}{0.8}$	$\frac{0.3}{0.2} + \frac{0.7}{0.4} + \frac{0.8}{0.5} + \frac{0.9}{0.7}$
$h_4$	$\frac{0.3}{0.4} + \frac{0.8}{0.5} + \frac{0.2}{0.6}$	$\frac{0.9}{0.4}$	$\frac{0.3}{0.5} + \frac{0.9}{0.6} + \frac{0.4}{0.8}$	$\frac{0.2}{0.4} + \frac{0.7}{0.5} + \frac{0.5}{0.6}$	$\frac{0.7}{0.5} + \frac{1}{0.8}$
$h_5$	$\frac{0.5}{0.7}$	$\frac{0.6}{0.5} + \frac{1}{0.6}$	$\frac{0.6}{0.5} + \frac{1}{0.6}$	$\frac{0.7}{0.1} + \frac{0.6}{0.2} + \frac{0.3}{0.5} + \frac{0.5}{0.6}$	$\frac{0.9}{0.5}$
$h_6$	$\frac{0.3}{0.1} + \frac{1}{0.6} + \frac{0.5}{0.7} + \frac{0.2}{0.9}$	$\frac{0.6}{0.1} + \frac{0.7}{0.4}$	$1$	$\frac{0.4}{0.5} + \frac{0.3}{0.7}$	$\frac{0.2}{0.6} + \frac{0.5}{0.7} + \frac{0.4}{0.9}$

Example 3.8. Following Example 3.2, the complement  $(\mathcal{F}, A)^c = (\mathcal{F}^c, \neg A)$  of the type-2 fuzzy soft set  $(\mathcal{F}, A)$  is given below:

$\mathcal{F}^c$ (not expensive)

$=\sim (\mathcal{F}(\text{expensive}))$

$$\begin{aligned} &= \sim \left( \frac{0.3/0.1 + 1/0.2 + 0.7/0.3}{h_1} + \frac{0.4/0.3 + 1/0.5 + 0.3/0.6}{h_2} + \frac{0.6/0.5 + 1/0.9}{h_3} \right. \\ &\quad \left. + \frac{0.3/0.4 + 0.8/0.5 + 0.2/0.6}{h_4} + \frac{0.5/0.7}{h_5} + \frac{0.3/0.1 + 1/0.6 + 0.5/0.7 + 0.2/0.9}{h_6} \right) \\ &= \frac{0.3/0.9 + 1/0.8 + 0.7/0.7}{h_1} + \frac{0.4/0.7 + 1/0.5 + 0.3/0.4}{h_2} + \frac{0.6/0.5 + 1/0.1}{h_3} \\ &\quad + \frac{0.3/0.6 + 0.8/0.5 + 0.2/0.4}{h_4} + \frac{0.5/0.3}{h_5} + \frac{0.3/0.9 + 1/0.4 + 0.5/0.3 + 0.2/0.1}{h_6}, \end{aligned}$$

$\mathcal{F}^c$ (not beautiful)

$=\sim (\mathcal{F}(\text{beautiful}))$

$$\begin{aligned} &= \sim \left( \frac{0.3/0.1 + 0.6/0.4 + 0.5/0.7 + 0.5/0.8}{h_1} + \frac{0.1/0.1 + 0.6/0.2 + 0.9/0.6}{h_2} + \frac{0.7/0.8}{h_3} \right. \\ &\quad \left. + \frac{0.9/0.4}{h_4} + \frac{0.6/0.5 + 1/0.6}{h_5} + \frac{0.6/0.1 + 0.7/0.4}{h_6} \right) \\ &= \frac{0.3/0.9 + 0.6/0.6 + 0.5/0.3 + 0.5/0.2}{h_1} + \frac{0.1/0.9 + 0.6/0.8 + 0.9/0.4}{h_2} + \frac{0.7/0.2}{h_3} \\ &\quad + \frac{0.9/0.6}{h_4} + \frac{0.6/0.5 + 1/0.4}{h_5} + \frac{0.6/0.9 + 0.7/0.6}{h_6}, \end{aligned}$$

$\mathcal{F}^c$ (not wooden)

$=\sim (\mathcal{F}(\text{wooden}))$

$$\begin{aligned} &= \sim \left( \frac{0.2/0.6 + 0.8/0.8 + 0.6/0.9}{h_1} + \frac{0.6/0.4 + 0.8/0.7}{h_2} + \frac{0.5/0.3 + 0.9/0.4 + 0.4/0.6}{h_3} \right. \\ &\quad \left. + \frac{0.3/0.5 + 0.9/0.6 + 0.4/0.8}{h_4} + \frac{0.6/0.5 + 1/0.6}{h_5} + \frac{1/0.9}{h_6} \right) \\ &= \frac{0.2/0.4 + 0.8/0.2 + 0.6/0.1}{h_1} + \frac{0.6/0.6 + 0.8/0.3}{h_2} + \frac{0.5/0.7 + 0.9/0.6 + 0.4/0.4}{h_3} \\ &\quad + \frac{0.3/0.5 + 0.9/0.4 + 0.4/0.2}{h_4} + \frac{0.6/0.5 + 1/0.4}{h_5} + \frac{1/0.1}{h_6}, \end{aligned}$$

$$\begin{aligned}
& \mathcal{F}^c(\text{not in the green surroundings}) \\
& \sim \mathcal{F}(\text{in the green surroundings}) \\
& \sim \mathcal{F}\left(\frac{0.5/0.7}{h_1} + \frac{0.3/0.1 + 0.8/0.5 + 0.5/0.8}{h_2} \right. \\
& \quad + \frac{0.7/0.4 + 0.7/0.8}{h_3} + \frac{0.2/0.4 + 0.7/0.5 + 0.5/0.6}{h_4} \\
& \quad + \frac{0.7/0.1 + 0.6/0.2 + 0.3/0.5 + 0.5/0.6}{h_5} \\
& \quad \left. + \frac{0.4/0.5 + 0.3/0.7}{h_6}\right) \\
& = \frac{0.5/0.3}{h_1} + \frac{0.3/0.9 + 0.8/0.5 + 0.5/0.2}{h_2} + \frac{0.7/0.6 + 0.7/0.2}{h_3} \\
& \quad + \frac{0.2/0.6 + 0.7/0.5 + 0.5/0.4}{h_4} + \frac{0.7/0.9 + 0.6/0.8 + 0.3/0.5 + 0.5/0.4}{h_5} + \frac{0.4/0.5 + 0.3/0.3}{h_6}, \\
& \mathcal{F}^c(\text{not convenient traffic}) \\
& \sim \mathcal{F}(\text{convenient traffic}) \\
& \sim \mathcal{F}\left(\frac{0.9/0.5 + 1/0.8}{h_1} + \frac{0.1/0.1 + 0.9/0.5 + 0.6/0.6}{h_2} \right. \\
& \quad + \frac{0.3/0.2 + 0.7/0.4 + 0.8/0.5 + 0.9/0.7}{h_3} + \frac{0.7/0.5 + 1/0.8}{h_4} \\
& \quad \left. + \frac{0.9/0.5}{h_5} + \frac{0.2/0.6 + 0.5/0.7 + 0.4/0.9}{h_6}\right) \\
& = \frac{0.9/0.5 + 1/0.2}{h_1} + \frac{0.1/0.9 + 0.9/0.5 + 0.6/0.4}{h_2} + \frac{0.3/0.8 + 0.7/0.6 + 0.8/0.5 + 0.9/0.3}{h_3} \\
& \quad + \frac{0.7/0.5 + 1/0.2}{h_4} + \frac{0.9/0.5}{h_5} + \frac{0.2/0.4 + 0.5/0.3 + 0.4/0.1}{h_6}.
\end{aligned} \tag{3.8}$$

*Definition 3.9.* Let  $(\mathcal{F}, A)$  and  $(\mathcal{G}, B)$  be two type-2 fuzzy soft sets over  $U$ . Then “ $(\mathcal{F}, A)$  AND  $(\mathcal{G}, B)$ ” is defined by  $(\mathcal{F}, A) \wedge (\mathcal{G}, B) = (\mathcal{L}, A \times B)$ , where  $\mathcal{L}(\alpha, \beta) = \mathcal{F}(\alpha) \cap \mathcal{G}(\beta)$ , for all  $(\alpha, \beta) \in A \times B$ .

*Definition 3.10.* Let  $(\mathcal{F}, A)$  and  $(\mathcal{G}, B)$  be two type-2 fuzzy soft sets over  $U$ . Then “ $(\mathcal{F}, A)$  OR  $(\mathcal{G}, B)$ ” is defined by  $(\mathcal{F}, A) \vee (\mathcal{G}, B) = (\mathcal{K}, A \times B)$ , where  $\mathcal{K}(\alpha, \beta) = \mathcal{F}(\alpha) \cup \mathcal{G}(\beta)$ , for all  $(\alpha, \beta) \in A \times B$ .

*Example 3.11.* Let  $(\mathcal{F}, A)$  and  $(\mathcal{G}, B)$  be two type-2 fuzzy soft sets over the same universe  $U$ . Here  $U = \{h_1, h_2, h_3, h_4, h_5, h_6\}$  is the set of houses,  $A = \{\varepsilon_1, \varepsilon_2\} = \{\text{expensive; wooden}\}$ , and  $B = \{e_1, e_2\} = \{\text{beautiful; convenient traffic}\}$ .

Suppose that

$$\begin{aligned}
 \mathcal{F}(\varepsilon_1) &= \frac{0.3/0.1 + 1/0.5}{h_1} + \frac{1/0.5 + 0.3/0.6}{h_2} + \frac{1/0.8}{h_3} \\
 &\quad + \frac{0.7/0.5 + 0.2/0.6}{h_4} + \frac{0.5/0.9}{h_5} + \frac{0.3/0.2 + 1/0.6}{h_6}, \\
 \mathcal{F}(\varepsilon_2) &= \frac{0.3/0.6 + 0.9/0.8}{h_1} + \frac{0.6/0.4 + 0.9/0.7}{h_2} + \frac{1.0/0.5}{h_3} \\
 &\quad + \frac{0.9/0.6 + 0.4/0.8}{h_4} + \frac{0.6/0.5 + 1/0.6}{h_5} + \frac{1/0.9}{h_6}, \\
 \mathcal{G}(e_1) &= \frac{0.7/0.1 + 1/0.2}{h_1} + \frac{1/0.6}{h_2} + \frac{0.6/0.5 + 1/0.9}{h_3} \\
 &\quad + \frac{0.4/0.4 + 1.0/0.5}{h_4} + \frac{0.6/0.9}{h_5} + \frac{1/0.6 + 0.5/0.8}{h_6}, \\
 \mathcal{G}(e_2) &= \frac{0.5/0.6 + 1.0/0.8}{h_1} + \frac{1.0/0.4 + 0.9/0.7}{h_2} + \frac{0.5/0.3 + 1.0/0.4}{h_3} \\
 &\quad + \frac{1.0/0.2}{h_4} + \frac{0.7/0.5 + 1/0.6}{h_5} + \frac{0.8/0.6}{h_6}.
 \end{aligned} \tag{3.9}$$

Compute the results of the “AND” operation and “OR” operation on  $(\mathcal{F}, A)$  and  $(\mathcal{G}, B)$ , respectively. Let  $(\mathcal{F}, A) \wedge (\mathcal{G}, B) = (\mathcal{H}, A \times B)$ . Then, by Definition 3.9, we have that

$$\begin{aligned}
 \mathcal{H}(\varepsilon_1, e_1) &= \mathcal{F}(\varepsilon_1) \cap \mathcal{G}(e_1) \\
 &= \frac{0.7/0.1 + 1.0/0.2}{h_1} + \frac{1.0/0.5 + 0.3/0.6}{h_2} \\
 &\quad + \frac{0.6/0.5 + 1/0.8}{h_3} + \frac{0.4/0.4 + 0.7/0.5}{h_4} + \frac{0.5/0.9}{h_5} + \frac{0.3/0.2 + 1.0/0.6}{h_6}, \\
 \mathcal{H}(\varepsilon_1, e_2) &= \mathcal{F}(\varepsilon_1) \cap \mathcal{G}(e_2) \\
 &= \frac{0.3/0.1 + 1.0/0.5}{h_1} + \frac{1.0/0.4 + 0.9/0.5 + 0.3/0.6}{h_2} \\
 &\quad + \frac{0.5/0.3 + 1.0/0.4}{h_3} + \frac{0.7/0.2}{h_4} + \frac{0.5/0.5 + 0.5/0.6}{h_5} + \frac{0.3/0.2 + 0.8/0.6}{h_6}, \\
 \mathcal{H}(\varepsilon_2, e_1) &= \mathcal{F}(\varepsilon_2) \cap \mathcal{G}(e_1) \\
 &= \frac{0.7/0.1 + 0.9/0.2}{h_1} + \frac{0.6/0.4 + 0.9/0.6}{h_2} + \frac{1.0/0.5}{h_3} \\
 &\quad + \frac{0.4/0.4 + 0.9/0.5}{h_4} + \frac{0.6/0.5 + 0.6/0.6}{h_5} + \frac{1.0/0.6 + 0.5/0.8}{h_6},
 \end{aligned}$$



$$\begin{aligned}
\mathcal{H}(\varepsilon_2, e_2) &= \mathcal{F}(\varepsilon_2) \cap \mathcal{G}(e_2) \\
&= \frac{0.5/0.6 + 0.9/0.8}{h_1} + \frac{0.9/0.4 + 0.9/0.7}{h_2} \\
&\quad + \frac{0.5/0.3 + 1.0/0.4}{h_3} + \frac{0.9/0.2}{h_4} + \frac{0.7/0.5 + 1.0/0.6}{h_5} + \frac{0.8/0.6}{h_6},
\end{aligned} \tag{3.10}$$

That is,

$$\begin{aligned}
(\mathcal{F}, A) \wedge (\mathcal{G}, B) &= \left\{ \begin{array}{l} \text{expensive and beautiful houses} \\ \\ \text{expensive and convenient traffic houses} \\ \\ \text{wooden and beautiful houses} \\ \\ \text{wooden and convenient traffic houses} \end{array} \right. \\
&= \frac{0.7/0.1 + 1.0/0.2}{h_1} + \frac{1.0/0.5 + 0.3/0.6}{h_2} + \frac{0.6/0.5 + 1/0.8}{h_3} \\
&\quad + \frac{0.4/0.4 + 0.7/0.5}{h_4} + \frac{0.5/0.9}{h_5} + \frac{0.3/0.2 + 1.0/0.6}{h_6}, \\
&= \frac{0.3/0.1 + 1.0/0.5}{h_1} + \frac{1.0/0.4 + 0.9/0.5 + 0.3/0.6}{h_2} + \frac{0.5/0.3 + 1.0/0.4}{h_3} \\
&\quad + \frac{0.7/0.2}{h_4} + \frac{0.5/0.5 + 0.5/0.6}{h_5} + \frac{0.3/0.2 + 0.8/0.6}{h_6}, \\
&= \frac{0.7/0.1 + 0.9/0.2}{h_1} + \frac{0.6/0.4 + 0.9/0.6}{h_2} + \frac{1.0/0.5}{h_3} + \frac{0.4/0.4 + 0.9/0.5}{h_4} \\
&\quad + \frac{0.6/0.5 + 0.6/0.6}{h_5} + \frac{1.0/0.6 + 0.5/0.8}{h_6}, \\
&= \frac{0.5/0.6 + 0.9/0.8}{h_1} + \frac{0.9/0.4 + 0.9/0.7}{h_2} + \frac{0.5/0.3 + 1.0/0.4}{h_3} \\
&\quad + \frac{0.9/0.2}{h_4} + \frac{0.7/0.5 + 1.0/0.6}{h_5} + \frac{0.8/0.6}{h_6} \left. \vphantom{\frac{0.5/0.6 + 0.9/0.8}{h_1}} \right\}.
\end{aligned} \tag{3.11}$$

Let  $(\mathcal{F}, A) \vee (\mathcal{G}, B) = (\mathcal{K}, A \times B)$ . Then, by Definition 3.10, we have that

$$\begin{aligned}
\mathcal{K}(\varepsilon_1, e_1) &= \mathcal{F}(\varepsilon_1) \cup \mathcal{G}(e_1) \\
&= \frac{0.3/0.1 + 0.3/0.2 + 1.0/0.5}{h_1} + \frac{1.0/0.6}{h_2} \\
&\quad + \frac{0.6/0.8 + 1/0.9}{h_3} + \frac{0.7/0.5 + 0.2/0.6}{h_4} + \frac{0.5/0.9}{h_5} + \frac{1/0.6 + 0.5/0.8}{h_6},
\end{aligned}$$

$$\begin{aligned}
\mathcal{K}(\varepsilon_1, e_2) &= \mathcal{F}(\varepsilon_1) \cup \mathcal{G}(e_2) \\
&= \frac{0.5/0.6 + 1.0/0.8}{h_1} + \frac{1.0/0.5 + 0.3/0.6 + 0.9/0.7}{h_2} \\
&\quad + \frac{1.0/0.8}{h_3} + \frac{0.7/0.5 + 0.2/0.6}{h_4} + \frac{0.5/0.9}{h_5} + \frac{0.8/0.6}{h_6}, \\
\mathcal{K}(\varepsilon_2, e_1) &= \mathcal{F}(\varepsilon_2) \cup \mathcal{G}(e_1) \\
&= \frac{0.3/0.6 + 0.9/0.8}{h_1} + \frac{0.6/0.6 + 0.9/0.7}{h_2} \\
&\quad + \frac{0.6/0.5 + 1.0/0.9}{h_3} + \frac{0.9/0.6 + 0.4/0.8}{h_4} + \frac{0.6/0.9}{h_5} + \frac{1/0.9}{h_6}, \\
\mathcal{K}(\varepsilon_2, e_2) &= \mathcal{F}(\varepsilon_2) \cup \mathcal{G}(e_2) \\
&= \frac{0.3/0.6 + 0.9/0.8}{h_1} + \frac{0.6/0.4 + 0.9/0.7}{h_2} + \frac{1.0/0.5}{h_3} \\
&\quad + \frac{0.9/0.6 + 0.4/0.8}{h_4} + \frac{0.6/0.5 + 1.0/0.6}{h_5} + \frac{0.8/0.9}{h_6}.
\end{aligned}$$

(3.12)

That is,

$$\begin{aligned}
(\mathcal{F}, A) \vee (\mathcal{G}, B) &= \left\{ \begin{array}{l} \text{expensive and beautiful houses} \\ \\ \text{expensive and convenient traffic houses} \\ \\ \text{wooden and beautiful houses} \end{array} \right. \\
&= \frac{0.3/0.1 + 0.3/0.2 + 1.0/0.5}{h_1} + \frac{1.0/0.6}{h_2} + \frac{0.6/0.8 + 1/0.9}{h_3} \\
&\quad + \frac{0.7/0.5 + 0.2/0.6}{h_4} + \frac{0.5/0.9}{h_5} + \frac{1/0.6 + 0.5/0.8}{h_6}, \\
&= \frac{0.5/0.6 + 1.0/0.8}{h_1} + \frac{1.0/0.5 + 0.3/0.6 + 0.9/0.7}{h_2} + \frac{1.0/0.8}{h_3} \\
&\quad + \frac{0.7/0.5 + 0.2/0.6}{h_4} + \frac{0.5/0.9}{h_5} + \frac{0.8/0.6}{h_6}, \\
&= \frac{0.3/0.6 + 0.9/0.8}{h_1} + \frac{0.6/0.6 + 0.9/0.7}{h_2} + \frac{0.6/0.5 + 1.0/0.9}{h_3} \\
&\quad + \frac{0.9/0.6 + 0.4/0.8}{h_4} + \frac{0.6/0.9}{h_5} + \frac{1/0.9}{h_6},
\end{aligned}$$

wooden and convenient traffic houses

$$\begin{aligned}
 &= \frac{0.3/0.6 + 0.9/0.8}{h_1} + \frac{0.6/0.4 + 0.9/0.7}{h_2} + \frac{1.0/0.5}{h_3} \\
 &\quad \left. + \frac{0.9/0.6 + 0.4/0.8}{h_4} + \frac{0.6/0.5 + 1.0/0.6}{h_5} + \frac{0.8/0.9}{h_6} \right\}.
 \end{aligned}
 \tag{3.13}$$

*Definition 3.12.* The union of two type-2 fuzzy soft sets  $(\mathcal{F}, A)$  and  $(\mathcal{G}, B)$  over a common universe  $U$  is the type-2 fuzzy soft set  $(\mathcal{H}, C)$ , where  $C = A \cup B$ , and for all  $\varepsilon \in C$ ,

$$\mathcal{H}(\varepsilon) = \begin{cases} \mathcal{F}(\varepsilon), & \text{if } \varepsilon \in A - B, \\ \mathcal{G}(\varepsilon), & \text{if } \varepsilon \in B - A, \\ \mathcal{F}(\varepsilon) \cup \mathcal{G}(\varepsilon), & \text{if } \varepsilon \in A \cap B. \end{cases}
 \tag{3.14}$$

One denotes it by  $(\mathcal{F}, A) \tilde{\cup} (\mathcal{G}, B) = (\mathcal{H}, C)$ .

*Definition 3.13.* The intersection of two type-2 fuzzy soft sets  $(\mathcal{F}, A)$  and  $(\mathcal{G}, B)$  over a common universe  $U$  is the type-2 fuzzy soft set  $(\mathcal{K}, C)$ , where  $C = A \cup B$ , and for all  $\varepsilon \in C$ ,

$$\mathcal{K}(\varepsilon) = \begin{cases} \mathcal{F}(\varepsilon), & \text{if } \varepsilon \in A - B, \\ \mathcal{G}(\varepsilon), & \text{if } \varepsilon \in B - A, \\ \mathcal{F}(\varepsilon) \cap \mathcal{G}(\varepsilon), & \text{if } \varepsilon \in A \cap B. \end{cases}
 \tag{3.15}$$

One denotes it by  $(\mathcal{F}, A) \tilde{\cap} (\mathcal{G}, B) = (\mathcal{K}, C)$ .

*Example 3.14.* Let  $(\mathcal{F}, A)$  and  $(\mathcal{G}, B)$  be two type-2 fuzzy soft sets showed in Tables 2 and 3, where  $B = \{e_1, e_2, e_3\} = \{\text{beautiful; convenient traffic; modern style}\}$ .

The union and intersection of  $(\mathcal{F}, A)$  and  $(\mathcal{G}, B)$  are, respectively, given below.

Let  $(\mathcal{F}, A) \tilde{\cup} (\mathcal{G}, B) = (\mathcal{H}, A \cup B)$ . Then by Definition 3.12, we have that

$$\begin{aligned}
 \mathcal{H}(\text{expensive}) = \mathcal{F}(\varepsilon_1) &= \frac{0.3/0.1 + 1/0.2 + 0.7/0.3}{h_1} \\
 &+ \frac{0.4/0.3 + 1/0.5 + 0.3/0.6}{h_2} \\
 &+ \frac{0.6/0.5 + 1/0.9}{h_3} + \frac{0.3/0.4 + 0.8/0.5 + 0.2/0.6}{h_4} \\
 &+ \frac{0.5/0.7}{h_5} + \frac{0.3/0.1 + 1/0.6 + 0.5/0.7 + 0.2/0.9}{h_6},
 \end{aligned}$$

**Table 3:** A type-2 fuzzy soft set  $(G, B)$ .

$U$	$e_1$	$e_2$	$e_3$
$h_1$	$\frac{0.3}{0.4} + \frac{0.7}{0.6} + \frac{0.2}{0.9}$	$\frac{0.3}{0.4} + \frac{0.8}{0.5} + \frac{0.6}{0.9}$	$\frac{0.3}{0.2} + \frac{0.9}{0.4} + \frac{1}{0.6}$
$h_2$	$\frac{0.2}{0.3} + \frac{0.1}{0.6} + \frac{0.9}{0.7}$	$\frac{0.6}{0.6} + \frac{0.9}{0.7}$	$\frac{1.0}{0.9}$
$h_3$	$\frac{0.7}{0.4}$	$\frac{0.5}{0.1} + \frac{1}{0.4} + \frac{0.4}{0.5}$	$\frac{0.7}{0.5} + \frac{1.0}{0.8}$
$h_4$	$\frac{0.8}{1.0}$	$\frac{0.9}{0.4}$	$\frac{0.2}{0.4} + \frac{0.7}{0.5} + \frac{0.5}{0.6}$
$h_5$	$\frac{0.3}{0.2} + \frac{0.4}{0.5} + \frac{1}{0.7} + \frac{0.6}{0.9}$	$\frac{0.7}{0.3} + \frac{1}{0.6}$	$\frac{0.7}{0.2} + \frac{0.8}{0.3} + \frac{1.0}{0.5} + \frac{0.5}{0.8}$
$h_6$	$\frac{0.6}{0.4} + \frac{0.8}{0.7}$	$\frac{0.1}{0.1} + \frac{0.9}{0.5} + \frac{0.6}{0.6}$	$\frac{0.4}{0.2} + \frac{1.0}{0.5} + \frac{0.6}{0.7}$

$$\mathcal{L}(\text{beautiful}) = \mathcal{F}(e_2) \cup \mathcal{G}(e_1)$$

$$= \frac{0.3/0.4 + 0.6/0.6 + 0.5/0.7 + 0.5/0.8 + 0.2/0.9}{h_1} + \frac{0.2/0.3 + 0.2/0.6 + 0.9/0.7}{h_2} + \frac{0.7/0.8}{h_3} + \frac{0.8/1.0}{h_4} + \frac{0.4/0.5 + 0.4/0.6 + 0.6/0.7 + 0.6/0.9}{h_5} + \frac{0.6/0.4 + 0.7/0.7}{h_6},$$

$$\mathcal{L}(\text{wooden}) = \mathcal{F}(e_3) = \frac{0.2/0.6 + 0.8/0.8 + 0.6/0.9}{h_1} + \frac{0.6/0.4 + 0.8/0.7}{h_2} + \frac{0.5/0.3 + 0.9/0.4 + 0.4/0.6}{h_3} + \frac{0.3/0.5 + 0.9/0.6 + 0.4/0.8}{h_4} + \frac{0.6/0.5 + 1/0.6}{h_5} + \frac{1/0.9}{h_6},$$

$$\mathcal{L}(\text{in the green surroundings}) = \mathcal{F}(e_4) = \frac{0.5/0.7}{h_1} + \frac{0.3/0.1 + 0.8/0.5 + 0.5/0.8}{h_2} + \frac{0.7/0.4 + 0.7/0.8}{h_3} + \frac{0.2/0.4 + 0.7/0.5 + 0.5/0.6}{h_4}$$

$$\begin{aligned}
& + \frac{0.7/0.1 + 0.6/0.2 + 0.3/0.5 + 0.5/0.6}{h_5} \\
& + \frac{0.4/0.5 + 0.3/0.7}{h_6},
\end{aligned} \tag{3.16}$$

$$\begin{aligned}
\mathcal{H}(\text{convenient traffic}) = \mathcal{F}(e_5) \cup \mathcal{G}(e_2) &= \frac{0.8/0.5 + 0.8/0.8 + 0.6/0.9}{h_1} \\
& + \frac{0.6/0.6 + 0.9/0.7}{h_2} \\
& + \frac{0.3/0.2 + 0.7/0.4 + 0.8/0.5 + 0.9/0.7}{h_3}
\end{aligned} \tag{3.17}$$

$$\begin{aligned}
& + \frac{0.7/0.5 + 0.5/0.6 + 0.7/0.8}{h_4} \\
& + \frac{0.9/0.5 + 0.5/0.8}{h_5} + \frac{0.2/0.6 + 0.5/0.7 + 0.4/0.9}{h_6}, \\
\mathcal{H}(\text{modern style}) = \mathcal{G}(e_3) &= \frac{0.3/0.2 + 0.9/0.4 + 1/0.6}{h_1} + \frac{1.0/0.9}{h_2} \\
& + \frac{0.7/0.5 + 1.0/0.8}{h_3} + \frac{0.2/0.4 + 0.7/0.5 + 0.5/0.6}{h_4} \\
& + \frac{0.7/0.2 + 0.8/0.3 + 1.0/0.5 + 0.5/0.8}{h_5} \\
& + \frac{0.4/0.2 + 1.0/0.5 + 0.6/0.7}{h_6}.
\end{aligned} \tag{3.18}$$

That is,

$$\begin{aligned}
(\mathcal{F}, A) \tilde{\cup} (\mathcal{G}, B) &= \left\{ \begin{array}{l} \text{expensive houses} \\ \\ \\ \\ \end{array} \right. \\
&= \frac{0.3/0.1 + 1/0.2 + 0.7/0.3}{h_1} + \frac{0.4/0.3 + 1/0.5 + 0.3/0.6}{h_2} \\
&+ \frac{0.6/0.5 + 1/0.9}{h_3} + \frac{0.3/0.4 + 0.8/0.5 + 0.2/0.6}{h_4} \\
&+ \frac{0.5/0.7}{h_5} + \frac{0.3/0.1 + 1/0.6 + 0.5/0.7 + 0.2/0.9}{h_6},
\end{aligned}$$

beautiful houses

$$\begin{aligned}
&= \frac{0.3/0.4 + 0.6/0.6 + 0.5/0.7 + 0.5/0.8 + 0.2/0.9}{h_1} \\
&+ \frac{0.2/0.3 + 0.2/0.6 + 0.9/0.7}{h_2} + \frac{0.7/0.8}{h_3} + \frac{0.8/1.0}{h_4}
\end{aligned}$$

$$+ \frac{0.4/0.5 + 0.4/0.6 + 0.6/0.7 + 0.6/0.9}{h_5} + \frac{0.6/0.4 + 0.7/0.7}{h_6},$$

wooden houses

$$\begin{aligned} &= \frac{0.2/0.6 + 0.8/0.8 + 0.6/0.9}{h_1} + \frac{0.6/0.4 + 0.8/0.7}{h_2} \\ &+ \frac{0.5/0.3 + 0.9/0.4 + 0.4/0.6}{h_3} + \frac{0.3/0.5 + 0.9/0.6 + 0.4/0.8}{h_4} \\ &+ \frac{0.6/0.5 + 1/0.6}{h_5} + \frac{1/0.9}{h_6}, \end{aligned}$$

in the green surroundings houses

$$\begin{aligned} &= \frac{0.5/0.7}{h_1} + \frac{0.3/0.1 + 0.8/0.5 + 0.5/0.8}{h_2} \\ &+ \frac{0.7/0.4 + 0.7/0.8}{h_3} + \frac{0.2/0.4 + 0.7/0.5 + 0.5/0.6}{h_4} \\ &+ \frac{0.7/0.1 + 0.6/0.2 + 0.3/0.5 + 0.5/0.6}{h_5} + \frac{0.4/0.5 + 0.3/0.7}{h_6}, \end{aligned}$$

convenient traffic houses

$$\begin{aligned} &= \frac{0.8/0.5 + 0.8/0.8 + 0.6/0.9}{h_1} + \frac{0.6/0.6 + 0.9/0.7}{h_2} \\ &+ \frac{0.3/0.2 + 0.7/0.4 + 0.8/0.5 + 0.9/0.7}{h_3} \\ &+ \frac{0.7/0.5 + 0.5/0.6 + 0.7/0.8}{h_4} \\ &+ \frac{0.9/0.5 + 0.5/0.8}{h_5} + \frac{0.2/0.6 + 0.5/0.7 + 0.4/0.9}{h_6}, \end{aligned}$$

modern style houses

$$\begin{aligned} &= \frac{0.3/0.2 + 0.9/0.4 + 1/0.6}{h_1} + \frac{1.0/0.9}{h_2} \\ &+ \frac{0.7/0.5 + 1.0/0.8}{h_3} + \frac{0.2/0.4 + 0.7/0.5 + 0.5/0.6}{h_4} \\ &+ \frac{0.7/0.2 + 0.8/0.3 + 1.0/0.5 + 0.5/0.8}{h_5} \\ &+ \frac{0.4/0.2 + 1.0/0.5 + 0.6/0.7}{h_6} \}. \end{aligned}$$

(3.19)

Let  $(\mathcal{F}, A)\tilde{\cap}(\mathcal{G}, B) = (\mathcal{K}, A \cup B)$ . Then by Definition 3.13, we have that

$$\begin{aligned}\mathcal{K}(\text{expensive}) = \mathcal{F}(\varepsilon_1) &= \frac{0.3/0.1 + 1/0.2 + 0.7/0.3}{h_1} \\ &+ \frac{0.4/0.3 + 1/0.5 + 0.3/0.6}{h_2} \\ &+ \frac{0.6/0.5 + 1/0.9}{h_3} + \frac{0.3/0.4 + 0.8/0.5 + 0.2/0.6}{h_4} \\ &+ \frac{0.5/0.7}{h_5} + \frac{0.3/0.1 + 1/0.6 + 0.5/0.7 + 0.2/0.9}{h_6},\end{aligned}$$

$$\begin{aligned}\mathcal{K}(\text{beautiful}) &= \mathcal{F}(\varepsilon_2) \cap \mathcal{G}(\varepsilon_1) \\ &= \frac{0.3/0.1 + 0.6/0.4 + 0.5/0.6 + 0.2/0.7 + 0.2/0.8}{h_1} \\ &+ \frac{0.1/0.1 + 0.6/0.2 + 0.2/0.3 + 0.9/0.6}{h_2} + \frac{0.7/0.4}{h_3} \\ &+ \frac{0.8/0.4}{h_4} + \frac{0.3/0.2 + 0.6/0.5 + 1/0.6}{h_5} \\ &+ \frac{0.6/0.1 + 0.7/0.4}{h_6},\end{aligned}$$

$$\begin{aligned}\mathcal{K}(\text{wooden}) = \mathcal{F}(\varepsilon_3) &= \frac{0.2/0.6 + 0.8/0.8 + 0.6/0.9}{h_1} + \frac{0.6/0.4 + 0.8/0.7}{h_2} \\ &+ \frac{0.5/0.3 + 0.9/0.4 + 0.4/0.6}{h_3} \\ &+ \frac{0.3/0.5 + 0.9/0.6 + 0.4/0.8}{h_4} + \frac{0.6/0.5 + 1/0.6}{h_5} + \frac{1/0.9}{h_6},\end{aligned}$$

$$\begin{aligned}\mathcal{K}(\text{in the green surroundings}) = \mathcal{F}(\varepsilon_4) &= \frac{0.5/0.7}{h_1} + \frac{0.3/0.1 + 0.8/0.5 + 0.5/0.8}{h_2} \\ &+ \frac{0.7/0.4 + 0.7/0.8}{h_3} + \frac{0.2/0.4 + 0.7/0.5 + 0.5/0.6}{h_4} \\ &+ \frac{0.7/0.1 + 0.6/0.2 + 0.3/0.5 + 0.5/0.6}{h_5} + \frac{0.4/0.5 + 0.3/0.7}{h_6},\end{aligned}$$

$$\begin{aligned}
\mathcal{K}(\text{convenient traffic}) &= \mathcal{F}(\varepsilon_5) \cap \mathcal{G}(e_2) = \frac{0.3/0.4 + 0.8/0.5 + 0.6/0.8}{h_1} \\
&+ \frac{0.1/0.1 + 0.9/0.5 + 0.6/0.6}{h_2} \\
&+ \frac{0.5/0.1 + 0.3/0.2 + 0.9/0.4 + 0.4/0.5}{h_3} + \frac{0.9/0.4}{h_4} \\
&+ \frac{0.7/0.3 + 0.9/0.5}{h_5} + \frac{0.1/0.1 + 0.5/0.5 + 0.5/0.6}{h_6}, \\
\mathcal{K}(\text{modern style}) &= \mathcal{G}(e_3) = \frac{0.3/0.2 + 0.9/0.4 + 1/0.6}{h_1} + \frac{1.0/0.9}{h_2} \\
&+ \frac{0.7/0.5 + 1.0/0.8}{h_3} + \frac{0.2/0.4 + 0.7/0.5 + 0.5/0.6}{h_4} \\
&+ \frac{0.7/0.2 + 0.8/0.3 + 1.0/0.5 + 0.5/0.8}{h_5} \\
&+ \frac{0.4/0.2 + 1.0/0.5 + 0.6/0.7}{h_6}.
\end{aligned} \tag{3.20}$$

That is,

$$\begin{aligned}
(\mathcal{F}, A) \tilde{\cap} (\mathcal{G}, B) &= \left\{ \begin{array}{l} \text{expensive houses} \\ \\ \text{beautiful houses} \\ \\ \text{wooden houses} \end{array} \right. \\
&= \frac{0.3/0.1 + 1/0.2 + 0.7/0.3}{h_1} + \frac{0.4/0.3 + 1/0.5 + 0.3/0.6}{h_2} \\
&+ \frac{0.6/0.5 + 1/0.9}{h_3} + \frac{0.3/0.4 + 0.8/0.5 + 0.2/0.6}{h_4} + \frac{0.5/0.7}{h_5} \\
&+ \frac{0.3/0.1 + 1/0.6 + 0.5/0.7 + 0.2/0.9}{h_6}, \\
&= \frac{0.3/0.1 + 0.6/0.4 + 0.5/0.6 + 0.2/0.7 + 0.2/0.8}{h_1} \\
&+ \frac{0.1/0.1 + 0.6/0.2 + 0.2/0.3 + 0.9/0.6}{h_2} + \frac{0.7/0.4}{h_3} + \frac{0.8/0.4}{h_4} \\
&+ \frac{0.3/0.2 + 0.6/0.5 + 1/0.6}{h_5} + \frac{0.6/0.1 + 0.7/0.4}{h_6}, \\
&= \frac{0.2/0.6 + 0.8/0.8 + 0.6/0.9}{h_1} + \frac{0.6/0.4 + 0.8/0.7}{h_2}
\end{aligned}$$



$$+ \frac{0.5/0.3 + 0.9/0.4 + 0.4/0.6}{h_3} + \frac{0.3/0.5 + 0.9/0.6 + 0.4/0.8}{h_4}$$

$$+ \frac{0.6/0.5 + 1/0.6}{h_5} + \frac{1/0.9}{h_6},$$

in the green surroundings houses

$$= \frac{0.5/0.7}{h_1} + \frac{0.3/0.1 + 0.8/0.5 + 0.5/0.8}{h_2} + \frac{0.7/0.4 + 0.7/0.8}{h_3}$$

$$+ \frac{0.2/0.4 + 0.7/0.5 + 0.5/0.6}{h_4} + \frac{0.7/0.1 + 0.6/0.2 + 0.3/0.5 + 0.5/0.6}{h_5}$$

$$+ \frac{0.4/0.5 + 0.3/0.7}{h_6},$$

convenient traffic houses

$$= \frac{0.3/0.4 + 0.8/0.5 + 0.6/0.8}{h_1} + \frac{0.1/0.1 + 0.9/0.5 + 0.6/0.6}{h_2}$$

$$+ \frac{0.5/0.1 + 0.3/0.2 + 0.9/0.4 + 0.4/0.5}{h_3} + \frac{0.9/0.4}{h_4}$$

$$+ \frac{0.7/0.3 + 0.9/0.5}{h_5} + \frac{0.1/0.1 + 0.5/0.5 + 0.5/0.6}{h_6},$$

modern style houses

$$= \frac{0.3/0.2 + 0.9/0.4 + 1/0.6}{h_1} + \frac{1.0/0.9}{h_2} + \frac{0.7/0.5 + 1.0/0.8}{h_3}$$

$$+ \frac{0.2/0.4 + 0.7/0.5 + 0.5/0.6}{h_4} + \frac{0.7/0.2 + 0.8/0.3 + 1.0/0.5 + 0.5/0.8}{h_5}$$

$$+ \left. \frac{0.4/0.2 + 1.0/0.5 + 0.6/0.7}{h_6} \right\}.$$

(3.21)

*Definition 3.15.* A type-2 fuzzy soft set  $(\mathcal{F}, A)$  over  $U$  is said to be a null type-2 fuzzy soft set denoted by  $\emptyset_A$ , if for all  $\varepsilon \in A$ , for all  $x \in U$ ,

$$\mu_{\tilde{A}}(x, u) = \begin{cases} 1, & u = 0, \\ 0, & u \neq 0. \end{cases} \quad (3.22)$$

*Definition 3.16.* A type-2 fuzzy soft set  $(\mathcal{F}, A)$  over  $U$  is said to be an absolute type-2 fuzzy soft set denoted by  $\mathcal{U}_A$ , if for all  $\varepsilon \in A$ , for all  $x \in U$ ,

$$\mu_{\tilde{A}}(x, u) = \begin{cases} 1, & u = 1, \\ 0, & u \neq 1. \end{cases} \quad (3.23)$$

#### 4. Type-2 Fuzzy-Soft-Set-Based Decision Making

Since its appearance, soft set theory has a wide application in many practical problems, especially the use of soft sets in decision making. Maji et al. [5] first introduced the soft set into the decision making problems. Furthermore, Roy and Maji [44] presented an algorithm to solve the recognition problem by means of fuzzy soft sets. Later, Kong et al. [45] gave a counterexample to illustrate that the optimal choice could not be obtained in general by using Roy and Maji's algorithm [44] and presented a modified version. Feng et al. [46] gave more deeper insights into decision making based on fuzzy soft sets. They presented an adjustable approach to fuzzy-soft-set-based decision making by means of level soft sets. By generalizing the adjustable approach to fuzzy-soft-set-based decision making [46], Jiang et al. [47] presented an adjustable approach to intuitionistic fuzzy-soft-set-based decision making and gave some illustrative examples.

In this section, we will present an adjustable approach to type-2 fuzzy-soft-set-based decision making problems. This approach is based on the following concept called level soft sets.

*Definition 4.1.* Let  $\mathfrak{S} = (\mathcal{F}, A)$  be a type-2 fuzzy soft set over  $U$ , where  $A \subseteq E$  and  $E$  is the parameter set.

For  $\alpha \in [0, 1]$ ,  $\lambda \in [0, 1]$ , the  $(\alpha, \lambda)$ -level soft set of  $\mathfrak{S}$  is a crisp soft set  $L(\mathfrak{S}; \alpha, \lambda) = (\mathcal{F}_{(\alpha, \lambda)}, A)$  defined as follows: for any  $\varepsilon \in A$ ,

$$\mathcal{F}_{(\alpha, \lambda)}(\varepsilon) = \mathcal{F}(\varepsilon)_{\lambda}^{\alpha} = \{x \in U \mid u(x) \geq \lambda, \forall u(x) \in J_x^{\alpha}\}, \quad (4.1)$$

where  $J_x^{\alpha} = \{u \mid \mu_{\mathcal{F}(\varepsilon)}(x, u) \geq \alpha, u \in J_x\}$ .

Here,  $\alpha \in [0, 1]$  can be viewed as a given least threshold on degrees of secondary membership, and  $\lambda \in [0, 1]$  can be viewed as a given least threshold on degrees of primary membership. In practical applications of type-2 fuzzy-soft-set-based decision making, usually the thresholds  $\alpha$  and  $\lambda$  are in advance given by decision makers, and they represent decision makers' requirements on "secondary membership levels" and "primary membership levels," respectively.

To illustrate this idea, let us consider the type-2 fuzzy soft set  $\mathfrak{S} = (\mathcal{F}, A)$  in Example 3.2.

*Example 4.2.* Suppose that we take  $\alpha = 0.6$  and  $\lambda = 0.5$ ; then we have the following results:

$$\begin{aligned} \mathcal{F}(\varepsilon_1)_{0.5}^{0.6} &= \{h_3, h_4\}, & \mathcal{F}(\varepsilon_2)_{0.5}^{0.6} &= \{h_3, h_5\}, \\ \mathcal{F}(\varepsilon_3)_{0.5}^{0.6} &= \{h_1, h_4, h_5, h_6\}, & \mathcal{F}(\varepsilon_4)_{0.5}^{0.6} &= \{h_2, h_4\}, & \mathcal{F}(\varepsilon_5)_{0.5}^{0.6} &= \{h_1, h_2, h_4, h_5\}. \end{aligned} \quad (4.2)$$

Therefore, the  $(0.6, 0.5)$ -level soft set of  $\mathfrak{S} = (\mathcal{F}, A)$  is a soft set  $L(\mathfrak{S}; 0.6, 0.5) = (\mathcal{F}_{(0.6, 0.5)}, A)$ , where the set-valued mapping  $\mathcal{F}_{(0.6, 0.5)} : A \rightarrow P(U)$  is defined by  $\mathcal{F}_{(0.6, 0.5)}(\varepsilon_i) = \mathcal{F}(\varepsilon_i)_{0.5}^{0.6}$  for  $i = 1, 2, 3, 4, 5$ . Table 4 gives the tabular representation of the  $(0.6, 0.5)$ -level soft set  $L(\mathfrak{S}; 0.6, 0.5)$ .

In Definition 4.1 the level pair (or threshold pair) assigned to each parameter is always the constant value pair  $(\alpha, \lambda)$ . However, in some decision making problems, decision makers would like to impose different threshold pairs on different parameters. To cope with such

**Table 4:** Tabular representation of the level soft set  $L(\mathfrak{S}; 0.6, 0.5)$ .

$U$	$\varepsilon_1$	$\varepsilon_2$	$\varepsilon_3$	$\varepsilon_4$	$\varepsilon_5$
$h_1$	0	0	1	0	1
$h_2$	0	0	0	1	1
$h_3$	1	1	0	0	0
$h_4$	1	0	1	1	1
$h_5$	0	1	1	0	1
$h_6$	0	0	1	0	0

**Table 5:** Tabular representation of the level soft set  $L(\mathfrak{S}; \alpha, \lambda)$ .

$U$	$\varepsilon_1$	$\varepsilon_2$	$\varepsilon_3$	$\varepsilon_4$	$\varepsilon_5$
$h_1$	0	0	1	0	0
$h_2$	0	0	1	1	0
$h_3$	1	1	0	0	1
$h_4$	1	0	0	1	1
$h_5$	0	0	0	0	0
$h_6$	1	0	1	0	0

problems, we need to use two functions instead of a constant value pair as the thresholds on secondary membership values and primary membership values, respectively.

*Definition 4.3.* Let  $\mathfrak{S} = (\mathcal{F}, A)$  be a type-2 fuzzy soft set over  $U$ , where  $A \subseteq E$  and  $E$  is the parameter set.

Let  $\alpha : A \rightarrow [0, 1]$  and  $\lambda : A \rightarrow [0, 1]$  be, respectively, two fuzzy sets in  $A$  which are called two threshold fuzzy sets. The level soft set of  $\mathfrak{S} = (\mathcal{F}, A)$  with respect to  $\alpha$  and  $\lambda$  is a crisp soft set  $L(\mathfrak{S}; \alpha, \lambda) = (\mathcal{F}_{(\alpha, \lambda)}, A)$  defined as follows: for any  $\varepsilon \in A$ ,

$$\mathcal{F}_{(\alpha, \lambda)}(\varepsilon) = \mathcal{F}(\varepsilon)_{\lambda}^{\alpha} = \{x \in U \mid u(x) \geq \lambda(\varepsilon), \forall u(x) \in J_x^{\alpha}\}, \tag{4.3}$$

where  $J_x^{\alpha} = \{u \mid \mu_{\mathcal{F}(\varepsilon)}(x, u) \geq \alpha(\varepsilon), u \in J_x\}$ .

To illustrate this idea, let us consider the following examples.

*Example 4.4.* Let  $\mathfrak{S} = (\mathcal{F}, A)$  be a type-2 fuzzy soft set over  $U$ , where  $A \subseteq E$  and  $E$  is a set of parameters. We define two threshold fuzzy sets  $\alpha : A \rightarrow [0, 1]$  and  $\lambda : A \rightarrow [0, 1]$  by

$$\alpha = \frac{0.6}{\varepsilon_1} + \frac{0.6}{\varepsilon_2} + \frac{0.8}{\varepsilon_3} + \frac{0.7}{\varepsilon_4} + \frac{0.9}{\varepsilon_5}, \quad \lambda = \frac{0.5}{\varepsilon_1} + \frac{0.6}{\varepsilon_2} + \frac{0.7}{\varepsilon_3} + \frac{0.5}{\varepsilon_4} + \frac{0.7}{\varepsilon_5}. \tag{4.4}$$

Then the level soft set of  $\mathfrak{S} = (\mathcal{F}, A)$  with respect to  $\alpha$  and  $\lambda$  is a soft set  $L(\mathfrak{S}; \alpha, \lambda)$  with its tabular representation given by Table 5.

We redefine two threshold fuzzy sets  $\alpha' : A \rightarrow [0, 1]$  and  $\lambda' : A \rightarrow [0, 1]$  by

$$\alpha' = \frac{0.7}{\varepsilon_1} + \frac{0.7}{\varepsilon_2} + \frac{0.8}{\varepsilon_3} + \frac{0.7}{\varepsilon_4} + \frac{0.9}{\varepsilon_5}, \quad \lambda' = \frac{0.6}{\varepsilon_1} + \frac{0.6}{\varepsilon_2} + \frac{0.8}{\varepsilon_3} + \frac{0.6}{\varepsilon_4} + \frac{0.7}{\varepsilon_5}. \tag{4.5}$$

**Table 6:** Tabular representation of the level soft set  $L(\mathfrak{S}; \alpha', \lambda')$ .

$U$	$\varepsilon_1$	$\varepsilon_2$	$\varepsilon_3$	$\varepsilon_4$	$\varepsilon_5$
$h_1$	0	0	1	0	0
$h_2$	0	1	0	0	0
$h_3$	1	1	0	0	1
$h_4$	0	0	0	0	1
$h_5$	0	1	0	0	0
$h_6$	1	0	1	0	0

Then the level soft set of  $\mathfrak{S} = (\mathcal{F}, A)$  with respect to  $\alpha'$  and  $\lambda'$  is a soft set  $L(\mathfrak{S}; \alpha', \lambda')$  with its tabular representation given by Table 6.

In the following we will show an adjustable approach to type-2 fuzzy-soft-set-based decision making by using the concepts of level soft sets.

*Algorithm 4.5.* Consider the following.

Step 1: Input the (resultant) type-2 fuzzy soft set  $\mathfrak{S} = (\mathcal{F}, A)$ .

Step 2: Input two threshold fuzzy sets  $\alpha : A \rightarrow [0, 1]$  and  $\lambda : A \rightarrow [0, 1]$  for decision making.

Step 3: Compute the level soft set  $L(\mathfrak{S}; \alpha, \lambda)$  of  $\mathfrak{S}$  with respect to the two threshold fuzzy sets  $\alpha$  and  $\lambda$ .

Step 4: Present the level soft set  $L(\mathfrak{S}; \alpha, \lambda)$  in tabular form. For any  $h_i \in U$ , compute the choice value  $c_i$  of  $h_i$ .

Step 5: The optimal decision is to select  $h_k$  if  $c_k = \max_{h_i \in U} \{c_i\}$ .

Step 6: If  $k$  has more than one value then any one of  $h_k$  may be chosen.

*Remark 4.6.* In the last step of Algorithm 4.5 given above, one may go back to the second step and change the thresholds that he/she once used so as to adjust the final optimal decision, especially when there are too many “optimal choices” to be chosen.

To illustrate the basic idea of Algorithm 4.5, let us consider the following example.

*Example 4.7.* Suppose that  $\mathfrak{S} = (\mathcal{F}, A)$  is a type-2 fuzzy soft set with its tabular representation given by Table 1.

Following Example 4.4, we deal with the decision making problem involving  $\mathfrak{S} = (\mathcal{F}, A)$  by using the threshold fuzzy sets  $\alpha$  and  $\lambda$  and obtain the level soft set  $L(\mathfrak{S}; \alpha, \lambda)$ . Table 7 gives the tabular representation of  $L(\mathfrak{S}; \alpha, \lambda)$  with choice values.

From Table 7, since the optimal choice value  $\max_{1 \leq i \leq 6} \{c_i\} = \{c_3, c_4\}$ ,  $h_3$  and  $h_4$  are the optimal choice objects.

If we deal with this problem by using the threshold fuzzy sets  $\alpha'$  and  $\lambda'$ , then we obtain the level soft set  $L(\mathfrak{S}; \alpha', \lambda')$ . Table 8 gives the tabular representation of  $L(\mathfrak{S}; \alpha', \lambda')$  with choice values.

From Table 8, since the optimal choice value  $\max_{1 \leq i \leq 6} \{c_i\} = \{c_3\}$ ,  $h_3$  is the optimal choice object.

**Table 7:** Tabular representation of  $L(\mathfrak{S}; \alpha, \lambda)$  with choice values.

$U$	$\varepsilon_1$	$\varepsilon_2$	$\varepsilon_3$	$\varepsilon_4$	$\varepsilon_5$	Choice value ( $c_i$ )
$h_1$	0	0	1	0	0	$c_1 = 1$
$h_2$	0	0	1	1	0	$c_2 = 2$
$h_3$	1	1	0	0	1	$c_3 = 3$
$h_4$	1	0	0	1	1	$c_4 = 3$
$h_5$	0	0	0	0	0	$c_5 = 0$
$h_6$	1	0	1	0	0	$c_6 = 2$

**Table 8:** Tabular representation of  $L(\mathfrak{S}; \alpha', \lambda')$  with choice values.

$U$	$\varepsilon_1$	$\varepsilon_2$	$\varepsilon_3$	$\varepsilon_4$	$\varepsilon_5$	Choice value ( $c_i$ )
$h_1$	0	0	1	0	0	$c_1 = 1$
$h_2$	0	1	0	0	0	$c_2 = 1$
$h_3$	1	1	0	0	1	$c_3 = 3$
$h_4$	0	0	0	0	1	$c_4 = 1$
$h_5$	0	1	0	0	0	$c_5 = 1$
$h_6$	1	0	1	0	0	$c_6 = 2$

## 5. Weighted Type-2 Fuzzy-Soft-Set-Based Decision Making

Maji et al. [5] defined the weighted table of a soft set, which is presented by having  $d_{ij} = w_j \times h_{ij}$  instead of 0 and 1 only, where  $h_{ij}$  are the entries in the table of the soft set and  $w_j$  are the weights of the attribute  $e_j$ . Feng et al. [46] introduced the notion of weighted fuzzy soft sets and discussed its applications to decision making problems. Recently, Jiang et al. [47] proposed an adjustable approach to weighted intuitionistic fuzzy soft-sets-based decision making problems by extending the approach to weighted fuzzy soft-sets-based decision making [46].

In Section 4, we have examined the application of type-2 fuzzy soft sets in decision making problems. A further representational capability can be added by associating each parameter  $e_j$  with a value  $w_j \in [0, 1]$  called its weight. In the case of multicriteria decision making, these weights can be used to represent the different importance of the concerned criteria. In this section, we will investigate the application of weighted type-2 fuzzy soft set in decision making problems. To begin with, we give the concept of weighted type-2 fuzzy soft sets.

*Definition 5.1.* Let  $E$  be a set of parameters and  $A \subseteq E$ . A weighted type-2 fuzzy soft set is a triple  $(\mathcal{F}, A, w)$  where  $(\mathcal{F}, A)$  is a type-2 fuzzy soft set over  $U$ , and  $w : A \rightarrow [0, 1]$  is a weight function specifying the weight  $w_j = w(\varepsilon_j)$  for each attribute  $\varepsilon_j \in A$ .

According to the above definition, every type-2 fuzzy soft set can be considered as a weighted type-2 fuzzy soft set. The notion of weighted type-2 fuzzy soft sets provides a mathematical framework for modeling and analyzing the decision making problems in which all the choice parameters may not be of equal importance [46]. These differences between the importance of parameters are characterized by the weight function in a weighted type-2 fuzzy soft set [46].

A revised version of Algorithm 4.5 can be proposed to cope with the decision making problems based on weighted type-2 fuzzy soft sets (see Algorithm 5.2). In the revised

algorithm, we consider the weights of parameters and compute the weighted choice values  $\bar{c}_i$  instead of choice values  $c_i$ .

*Algorithm 5.2.* Consider the following.

Step 1: Input a weighted type-2 fuzzy soft set  $\mathfrak{T} = (\mathcal{F}, A, w)$ .

Step 2: Input two threshold fuzzy sets  $\alpha : A \rightarrow [0, 1]$  and  $\lambda : A \rightarrow [0, 1]$  for decision making.

Step 3: Compute the level soft set  $L((\mathcal{F}, A); \alpha, \lambda)$  of  $\mathfrak{T}$  with respect to the two threshold fuzzy sets  $\alpha$  and  $\lambda$ .

Step 4: Present the level soft set  $L((\mathcal{F}, A); \alpha, \lambda)$  in tabular form. For each  $h_i \in U$ , compute the weighted choice value  $\bar{c}_i$  of  $h_i$ , where  $\bar{c}_i = \sum_{\varepsilon \in A} \mathcal{F}(\varepsilon)(h_i) \cdot w(\varepsilon)$ .

Step 5: The optimal decision is to select  $h_k$  if  $\bar{c}_k = \max_{h_i \in U} \{\bar{c}_i\}$ .

Step 6: If  $k$  has more than one value then any one of  $h_k$  may be chosen.

*Remark 5.3.* Similar to Algorithm 4.5, in the last step of Algorithm 5.2 given above, one may go back to the second step and change the threshold (or decision rule) that he/she once used so as to adjust the final optimal decision, especially when there are too many "optimal choices" to be chosen.

To illustrate the basic idea of Algorithm 5.2, let us consider the following example.

*Example 5.4.* Let  $\mathfrak{S} = (\mathcal{F}, A)$  be a type-2 fuzzy soft set shown in Table 1. Now assume that we have imposed the following weights for the parameters in  $A$ : for the parameter "expensive",  $w_1 = 0.9$ ; for the parameter "beautiful",  $w_2 = 0.6$ ; for the parameter "wooden",  $w_3 = 0.2$ ; for the parameter "in the green surroundings",  $w_4 = 0.5$ ; for the parameter "convenient traffic",  $w_5 = 0.8$ . Thus we have a weight function  $w : A \rightarrow [0, 1]$  and the type-2 fuzzy soft set  $\mathfrak{S} = (\mathcal{F}, A)$  in Example 3.2 is changed into a weighted type-2 fuzzy soft set  $\mathfrak{T} = (\mathcal{F}, A, w)$ . Table 9 gives the tabular representation of  $\mathfrak{T} = (\mathcal{F}, A, w)$ .

In the following, we utilize Algorithm 5.2 to find the best object, which involves the following steps.

Step 1: Input the type-2 fuzzy soft set  $\mathfrak{T} = (\mathcal{F}, A, w)$ .

Step 2: Input two threshold fuzzy sets  $\alpha$  and  $\lambda$  given in Example 4.4.

Step 3: Compute the level soft set  $L((\mathcal{F}, A); \alpha, \lambda)$  of  $\mathfrak{T}$  w.r.t.  $\alpha$  and  $\lambda$ .

Step 4: For each  $h_i \in U$ , compute the weighted choice value  $\bar{c}_i$  of  $h_i$ . The tabular representation of  $L((\mathcal{F}, A); \alpha, \lambda)$  with weighted choice values is shown in Table 10.

Step 5: From Table 10, since the weighted optimal choice value  $\max_{1 \leq i \leq 6} \{\bar{c}_i\} = \bar{c}_3$ ,  $h_3$  is the optimal choice object.

If we deal with this problem by using the threshold fuzzy sets  $\alpha'$  and  $\lambda'$  given in Example 4.4, we shall obtain level soft set  $L((\mathcal{F}, A); \alpha', \lambda')$ . Table 11 gives the tabular representation of  $L((\mathcal{F}, A); \alpha', \lambda')$  with weighted choice values.

From Table 11, since the weighted optimal choice value  $\max_{1 \leq i \leq 6} \{\bar{c}_i\} = \bar{c}_3$ ,  $h_3$  is the optimal choice object.

**Table 9:** Tabular representation of  $\mathcal{F} = (\mathcal{F}, A, w)$ .

$U$	$\varepsilon_1, w_1 = 0.9$	$\varepsilon_2, w_2 = 0.6$	$\varepsilon_3, w_3 = 0.2$	$\varepsilon_4, w_4 = 0.5$	$\varepsilon_5, w_5 = 0.8$
$h_1$	$\frac{0.3}{0.1} + \frac{1}{0.2} + \frac{0.7}{0.3}$	$\frac{0.3}{0.1} + \frac{0.6}{0.4} + \frac{0.5}{0.7} + \frac{0.5}{0.8}$	$\frac{0.2}{0.6} + \frac{0.8}{0.8} + \frac{0.6}{0.9}$	$\frac{0.5}{0.7}$	$\frac{0.9}{0.5} + \frac{1}{0.8}$
$h_2$	$\frac{0.4}{0.3} + \frac{1}{0.5} + \frac{0.3}{0.6}$	$\frac{0.1}{0.1} + \frac{0.6}{0.2} + \frac{0.9}{0.6}$	$\frac{0.6}{0.4} + \frac{0.8}{0.7}$	$\frac{0.3}{0.1} + \frac{0.8}{0.5} + \frac{0.5}{0.8}$	$\frac{0.1}{0.1} + \frac{0.9}{0.5} + \frac{0.6}{0.6}$
$h_3$	$\frac{0.6}{0.5} + \frac{1}{0.9}$	$\frac{0.7}{0.8}$	$\frac{0.5}{0.3} + \frac{0.9}{0.4} + \frac{0.4}{0.6}$	$\frac{0.7}{0.4} + \frac{0.7}{0.8}$	$\frac{0.3}{0.2} + \frac{0.7}{0.4} + \frac{0.8}{0.5} + \frac{0.9}{0.7}$
$h_4$	$\frac{0.3}{0.4} + \frac{0.8}{0.5} + \frac{0.2}{0.6}$	$\frac{0.9}{0.4}$	$\frac{0.3}{0.5} + \frac{0.9}{0.6} + \frac{0.4}{0.8}$	$\frac{0.2}{0.4} + \frac{0.7}{0.5} + \frac{0.5}{0.6}$	$\frac{0.7}{0.5} + \frac{1}{0.8}$
$h_5$	$\frac{0.5}{0.7}$	$\frac{0.6}{0.5} + \frac{1}{0.6}$	$\frac{0.6}{0.5} + \frac{1}{0.6}$	$\frac{0.7}{0.1} + \frac{0.6}{0.2} + \frac{0.3}{0.5} + \frac{0.5}{0.6}$	$\frac{0.9}{0.5}$
$h_6$	$\frac{0.3}{0.1} + \frac{1}{0.6} + \frac{0.5}{0.7} + \frac{0.2}{0.9}$	$\frac{0.6}{0.1} + \frac{0.7}{0.4}$	$\frac{1}{0.9}$	$\frac{0.4}{0.5} + \frac{0.3}{0.7}$	$\frac{0.2}{0.6} + \frac{0.5}{0.7} + \frac{0.4}{0.9}$

**Table 10:** Tabular representation of  $L((\mathcal{F}, A); \alpha, \lambda)$  with weighted choice values.

$U$	$\varepsilon_1, w_1 = 0.9$	$\varepsilon_2, w_2 = 0.6$	$\varepsilon_3, w_3 = 0.2$	$\varepsilon_4, w_4 = 0.5$	$\varepsilon_5, w_5 = 0.8$	Weighted choice value ( $\bar{c}_i$ )
$h_1$	0	0	1	0	0	$\bar{c}_1 = 0.2$
$h_2$	0	0	1	1	0	$\bar{c}_2 = 0.7$
$h_3$	1	1	0	0	1	$\bar{c}_3 = 2.3$
$h_4$	1	0	0	1	1	$\bar{c}_4 = 2.2$
$h_5$	0	0	0	0	0	$\bar{c}_5 = 0$
$h_6$	1	0	1	0	0	$\bar{c}_6 = 1.1$

**Table 11:** Tabular representation of  $L((\mathcal{F}, A); \alpha', \lambda')$  with weighted choice values.

$U$	$\varepsilon_1, w_1 = 0.9$	$\varepsilon_2, w_2 = 0.6$	$\varepsilon_3, w_3 = 0.2$	$\varepsilon_4, w_4 = 0.5$	$\varepsilon_5, w_5 = 0.8$	Weighted choice value ( $\bar{c}_i$ )
$h_1$	0	0	1	0	0	$\bar{c}_1 = 0.2$
$h_2$	0	1	0	0	0	$\bar{c}_2 = 0.6$
$h_3$	1	1	0	0	1	$\bar{c}_3 = 2.3$
$h_4$	0	0	0	0	1	$\bar{c}_4 = 0.8$
$h_5$	0	1	0	0	0	$\bar{c}_5 = 0.6$
$h_6$	1	0	1	0	0	$\bar{c}_6 = 1.1$

*Remark 5.5.* The advantages of Algorithms 4.5 and 5.2 are mainly twofold. First, level soft sets build bridges between type-2 fuzzy soft sets and crisp soft sets. By using Algorithms 4.5 and 5.2, in decision making process we actually do not work directly on type-2 fuzzy soft sets, but only need to deal with the crisp level soft sets derived from the initial type-2 fuzzy soft sets after choosing certain thresholds. This makes our algorithms simpler in computational complexity and thus easier for application in real life problems. Second, Algorithms 4.5 and 5.2 can be seen as an adjustable approach to (weighted) type-2 fuzzy-soft-set-based decision making because the final optimal decision is with respect to the (weighted) thresholds. By choosing different types of thresholds, we can derive different level soft sets from the original type-2 fuzzy soft set. In general, the final optimal decisions based on different level soft sets could be different. Thus the newly proposed approach is actually an adjustable method which captures an important feature for decision making in an imprecise environment: some of these problems are essentially humanistic and thus subjective in nature; there actually does not exist a unique or uniform criterion for evaluating the alternatives [46]. This adjustable feature makes Algorithms 4.5 and 5.2 not only efficient but more appropriate for many real world applications.

Based on the above analysis, the advantage of Algorithms 4.5 and 5.2 is that they have great flexibility, less computations, and wide applications. These advantages ensure that we may use (weighted) type-2 fuzzy soft sets to efficiently address multiple attribute decision making problems in which the attribute values take the form of fuzzy sets. Because the attribute values are given in the form of fuzzy sets rather than exact numerical values, interval numbers, intuitionistic fuzzy numbers, and interval-valued intuitionistic fuzzy number, these decision making problems cannot be solved by some approaches based on fuzzy sets, hesitant fuzzy sets, hesitant fuzzy linguistic term sets, intuitionistic fuzzy sets, and interval-valued intuitionistic fuzzy sets. Though these decision making problems can be solved by some existing approaches based on types-2 fuzzy sets, these existing approaches may have some inherent limitations. First, these approaches are not in essence some adjustable



methods which cannot capture an important feature for decision making in an imprecise environment: some of these problems are essentially humanistic and thus subjective in nature (e.g., human understanding and vision systems); there actually does not exist a unique or uniform criterion for evaluating the alternatives. In addition, these approaches generally use some aggregation operators to aggregate the characteristics of each alternative under all the attributes. This means that the computational complexity of these approaches may be high, especially when the decision problem is associated with a large parameter set or a great number of objects. The newly proposed approaches in this paper can overcome all the above difficulties. Therefore, our new proposals are not only more suitable but more feasible for dealing with multiple attribute decision making problems where the attribute values take the form of fuzzy sets.

Based on the above discussion, in the following, we point out the differences between type-2 fuzzy soft set and type-2 fuzzy set in detail.

- (1) Through Definition 3.1, the type-2 fuzzy soft set is different from the type-2 fuzzy set. A type-2 fuzzy soft set is not a type-2 fuzzy set but a parameterized family of type-2 fuzzy subsets of  $U$ .
- (2) In Section 3.2, we introduce some computations and operations that can be performed on type-2 fuzzy soft sets, such as complement, AND, OR, union, and intersection. These operations are different from the operations on type-2 fuzzy sets.
- (3) Through Examples 4.2 and 5.4, our approaches are different from the existing approaches to multiple attribute decision making (MADM) with type-2 fuzzy information. Generally speaking, the existing approaches to MADM with type-2 fuzzy information first utilize some aggregation operators to aggregate all the individual type-2 fuzzy decision matrices into the collective type-2 fuzzy decision matrix, then utilize these operators to aggregate all the preference values in the each line of the collective type-2 fuzzy decision matrix, and derive the collective overall preference value of each alternative. However, our approaches first utilize the thresholds to derive level soft set from the original type-2 fuzzy soft set and then utilize level soft set to calculate the choice values of each alternative with respect to all the parameters.

## 6. Conclusion

Considering that soft set and its existing extension models cannot deal with the situations in which the evaluations of parameters are fuzzy concepts, we in the current paper introduce the notion of the type-2 fuzzy soft set as an extension to the soft set model. We also define some operations on the type-2 fuzzy soft sets. Furthermore, some illustrative examples are provided to show the validity of type-2 fuzzy soft set and weighted type-2 fuzzy soft set in decision making problems. Our new proposals can be used to solve multiple attribute decision making problems in which the attribute values take the form of fuzzy sets, which cannot be addressed by some approaches based on fuzzy sets, hesitant fuzzy sets, hesitant fuzzy linguistic term sets, intuitionistic fuzzy sets, and interval-valued intuitionistic fuzzy sets. Compared with some existing approaches to multiple attribute decision making problems in which the attribute values take the form of fuzzy sets, our approaches have great flexibility, less computations, and wide applications.

For future research, it is desirable to further explore the parameterization reduction of the type-2 fuzzy soft set.

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## References

- [1] D. Molodtsov, "Soft set theory-first results," *Computers & Mathematics with Applications*, vol. 37, no. 4-5, pp. 19–31, 1999.
- [2] L. A. Zadeh, "Fuzzy sets," *Information and Computation*, vol. 8, pp. 338–353, 1965.
- [3] Z. Pawlak, "Rough sets," *International Journal of Computer and Information Sciences*, vol. 11, no. 5, pp. 341–356, 1982.
- [4] P. K. Maji, R. Biswas, and A. R. Roy, "Soft set theory," *Computers & Mathematics with Applications*, vol. 45, no. 4-5, pp. 555–562, 2003.
- [5] P. K. Maji, A. R. Roy, and R. Biswas, "An application of soft sets in a decision making problem," *Computers & Mathematics with Applications*, vol. 44, no. 8-9, pp. 1077–1083, 2002.
- [6] D. Chen, E. C. C. Tsang, D. S. Yeung, and X. Wang, "The parameterization reduction of soft sets and its applications," *Computers & Mathematics with Applications*, vol. 49, no. 5-6, pp. 757–763, 2005.
- [7] Z. Kong, L. Gao, L. Wang, and S. Li, "The normal parameter reduction of soft sets and its algorithm," *Computers & Mathematics with Applications*, vol. 56, no. 12, pp. 3029–3037, 2008.
- [8] M. I. Ali, F. Feng, X. Liu, W. K. Min, and M. Shabir, "On some new operations in soft set theory," *Computers & Mathematics with Applications*, vol. 57, no. 9, pp. 1547–1553, 2009.
- [9] Y. Zou and Z. Xiao, "Data analysis approaches of soft sets under incomplete information," *Knowledge-Based Systems*, vol. 21, no. 8, pp. 941–945, 2008.
- [10] N. Çağman and S. Enginoğlu, "Soft matrix theory and its decision making," *Computers & Mathematics with Applications*, vol. 59, no. 10, pp. 3308–3314, 2010.
- [11] N. Çağman and S. Enginoğlu, "Soft set theory and uni-int decision making," *European Journal of Operational Research*, vol. 207, no. 2, pp. 848–855, 2010.
- [12] T. Herawan and M. M. Deris, "A soft set approach for association rules mining," *Knowledge-Based Systems*, vol. 24, no. 1, pp. 186–195, 2011.
- [13] K. Gong, Z. Xiao, and X. Zhang, "The bijective soft set with its operations," *Computers & Mathematics with Applications*, vol. 60, no. 8, pp. 2270–2278, 2010.
- [14] H. Aktaş and N. Çağman, "Soft sets and soft groups," *Information Sciences*, vol. 177, no. 13, pp. 2726–2735, 2007.
- [15] Y. B. Jun, "Soft BCK/BCI-algebras," *Computers & Mathematics with Applications*, vol. 56, no. 5, pp. 1408–1413, 2008.
- [16] Y. B. Jun and C. H. Park, "Applications of soft sets in ideal theory of BCK/BCI-algebras," *Information Sciences*, vol. 178, no. 11, pp. 2466–2475, 2008.
- [17] F. Feng, Y. B. Jun, and X. Zhao, "Soft semirings," *Computers & Mathematics with Applications*, vol. 56, no. 10, pp. 2621–2628, 2008.
- [18] U. Acar, F. Koyuncu, and B. Tanay, "Soft sets and soft rings," *Computers & Mathematics with Applications*, vol. 59, no. 11, pp. 3458–3463, 2010.
- [19] S. Yamak, O. Kazancı, and B. Davvaz, "Soft hyperstructure," *Computers & Mathematics with Applications*, vol. 62, no. 2, pp. 797–803, 2011.
- [20] S. M. Anvariye, S. Mirvakili, O. Kazancı, and B. Davvaz, "Algebraic hyperstructures of soft sets associated to semihypergroups," *Southeast Asian Bulletin of Mathematics*, vol. 35, no. 6, pp. 911–925, 2011.
- [21] P. K. Maji, R. Biswas, and A. R. Roy, "Fuzzy soft sets," *Journal of Fuzzy Mathematics*, vol. 9, no. 3, pp. 589–602, 2001.

- [22] P. Majumdar and S. K. Samanta, "Generalised fuzzy soft sets," *Computers & Mathematics with Applications*, vol. 59, no. 4, pp. 1425–1432, 2010.
- [23] W. Xu, J. Ma, S. Wang, and G. Hao, "Vague soft sets and their properties," *Computers & Mathematics with Applications*, vol. 59, no. 2, pp. 787–794, 2010.
- [24] X. Yang, T. Y. Lin, J. Yang, Y. Li, and D. Yu, "Combination of interval-valued fuzzy set and soft set," *Computers & Mathematics with Applications*, vol. 58, no. 3, pp. 521–527, 2009.
- [25] F. Feng, C. Li, B. Davvaz, and M. I. Ali, "Soft sets combined with fuzzy sets and rough sets: a tentative approach," *Soft Computing*, vol. 14, no. 9, pp. 899–911, 2010.
- [26] S. Bhattacharya and B. Davvaz, "Some more results on IF soft rough approximation space," *International Journal of Combinatorics*, Article ID 893061, 10 pages, 2011.
- [27] P. K. Maji, R. Biswas, and A. R. Roy, "Intuitionistic fuzzy soft sets," *Journal of Fuzzy Mathematics*, vol. 9, no. 3, pp. 677–692, 2001.
- [28] P. K. Maji, A. R. Roy, and R. Biswas, "On intuitionistic fuzzy soft sets," *Journal of Fuzzy Mathematics*, vol. 12, no. 3, pp. 669–683, 2004.
- [29] K. T. Atanassov, "Intuitionistic fuzzy sets," *Fuzzy Sets and Systems*, vol. 20, no. 1, pp. 87–96, 1986.
- [30] Y. Jiang, Y. Tang, Q. Chen, H. Liu, and J. Tang, "Interval-valued intuitionistic fuzzy soft sets and their properties," *Computers & Mathematics with Applications*, vol. 60, no. 3, pp. 906–918, 2010.
- [31] A. Aygünöglu and H. Aygün, "Introduction to fuzzy soft groups," *Computers & Mathematics with Applications*, vol. 58, no. 6, pp. 1279–1286, 2009.
- [32] J. M. Mendel, *Uncertain Rule-Based Fuzzy Logic Systems: Introduction and New Directions*, Prentice-Hall, Upper-Saddle River, NJ, USA, 2001.
- [33] L. A. Zadeh, "The concept of a linguistic variable and its application to approximate reasoning. I," *Information Sciences*, vol. 8, pp. 199–249, 1975.
- [34] N. N. Karnik and J. M. Mendel, "Applications of type-2 fuzzy logic systems to forecasting of time-series," *Information Sciences*, vol. 120, no. 1, pp. 89–111, 1999.
- [35] N. N. Karnik and J. M. Mendel, "Operations on type-2 fuzzy sets," *Fuzzy Sets and Systems*, vol. 122, no. 2, pp. 327–348, 2001.
- [36] J. M. Mendel, R. I. John, and F. Liu, "Interval type-2 fuzzy logic systems made simple," *IEEE Transactions on Fuzzy Systems*, vol. 122, pp. 327–348, 2001.
- [37] H. B. Mitchell, "Pattern recognition using type-II fuzzy sets," *Information Sciences*, vol. 170, no. 2–4, pp. 409–418, 2005.
- [38] M. Mizumoto and K. Tanaka, "Fuzzy sets of type-2 under algebraic product and algebraic sum," *Fuzzy Sets and Systems*, vol. 5, no. 3, pp. 277–290, 1981.
- [39] H. Wu, Y. Wu, and J. Luo, "An interval type-2 fuzzy rough set model for attribute reduction," *IEEE Transactions on Fuzzy Systems*, vol. 17, no. 2, pp. 301–315, 2009.
- [40] J. Zeng and Z. Q. Liu, "Type-2 fuzzy hidden Markov models and their application to speech recognition," *IEEE Transactions on Fuzzy Systems*, vol. 14, no. 3, pp. 454–467, 2006.
- [41] F. Liu, "An efficient centroid type-reduction strategy for general type-2 fuzzy logic system," *Information Sciences*, vol. 178, no. 9, pp. 2224–2236, 2008.
- [42] J. M. Mendel, F. Liu, and D. Zhai, " $\alpha$ -Plane representation for type-2 fuzzy sets: theory and applications," *IEEE Transactions on Fuzzy Systems*, vol. 17, no. 5, pp. 1189–1207, 2009.
- [43] J. M. Mendel and D. R. Wu, *Perceptual Computing: Aiding People in Making Subjective Judgments*, Wiley and IEEE Press, 2010.
- [44] A. R. Roy and P. K. Maji, "A fuzzy soft set theoretic approach to decision making problems," *Journal of Computational and Applied Mathematics*, vol. 203, no. 2, pp. 412–418, 2007.
- [45] Z. Kong, L. Gao, and L. Wang, "Comment on A fuzzy soft set theoretic approach to decision making problems," *Journal of Computational and Applied Mathematics*, vol. 223, no. 2, pp. 540–542, 2009.
- [46] F. Feng, Y. B. Jun, X. Liu, and L. Li, "An adjustable approach to fuzzy soft set based decision making," *Journal of Computational and Applied Mathematics*, vol. 234, no. 1, pp. 10–20, 2010.
- [47] Y. Jiang, Y. Tang, and Q. Chen, "An adjustable approach to intuitionistic fuzzy soft sets based decision making," *Applied Mathematical Modelling*, vol. 35, no. 2, pp. 824–836, 2011.