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Research Article
Subfusion Frames

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Fusion frames are generalizations of frames in Hilbert spaces which were introduced by Casazza et al. (2008). In the present paper, we study the relations between fusion frames and subfusion frame operators. Specially, we introduce new construction of subfusion frames and derive new results.

1. Introduction

Frames were first introduced in 1924 by Duffin and Schaeffer [1]. Daubechies et al. in [2] found a fundamental new application. Nice properties of frames make them very useful in filter banks, sigma-delta quantization, signal and image processing. The theory of frames has been generalized rapidly, and various generalizations of frames have been proposed recently. Later general frame theory of subspaces was introduced by Casazza and Kutyniok [3] and Fornasier [4] as a natural generalization of the frame theory in Hilbert spaces. Since frames, particular frames of subspaces, are applied to signal processing, image processing, data compression, and sampling theory, we consider frames of subspaces on Hilbert spaces and extend some of the known results about bases and frames to frames of subspaces. Recently, the frames of subspaces have been renamed as fusion frames. This notion has been intensely studied earlier and several new applications have been discovered. The reader is referred to the works by Casazza and Kutyniok [3] and Găvruţa [5]. There exists a variety of applications which cannot be modeled naturally by one ordinary frame, for example, wireless sensor network [6], sensor geophones in geophysics measurement and studies [7], and the physiological structure of visual and hearing system [8].

Let $\nu = \{(V_i, \alpha_i)\}_{i\in I}$ be a fusion frame for H, W_i be a closed subspace of V_i and $\beta_i \le \alpha_i$ for all $i \in I$. If $\omega = \{(W_i, \beta_i)\}_{i\in I}$ is a fusion frame for H, then ω is called a subfusion frame of

ν. If ν and ω are Bessel fusion sequences for H, then ω is called a Bessel subfusion sequence of ν.

In [9], the authors introduced Bessel subfusion sequences and subfusion frames and they investigated the relationship between their operation. Also, the definition of the orthogonal complement of subfusion frames and the definition of the completion of Bessel fusion sequences were provided, and several results related with these notions were shown.

A notion related to subfusion frames has been brought in [10], which is called frame of subspaces refinement (shortly: FSR). A subfusion frame $\omega = \{(W_i, \beta_i)\}_{i\in I}$ of $\nu = \{(V_i, \alpha_i)\}_{i\in I}$ is a FSR if $\alpha_i = \beta_i$ for all $i \in I$. Therefore, an FSR is a special subfusion frame and the authors have studied the excess of FSR in [10].

In the present paper, we study the relations between fusion frames and subfusion frame operators. We also obtain some results about subfusion frame operators that these results are not true for fusion frame operators.

This paper is organized as follows. Section 2 briefly review the concept of frames, subfusion frames, and their properties. Section 3 includes some results of operator obtained of Bessel subfusion sequences. In [11], the authors tried to show that the frame operator for a pair of fusion frames is bounded below and invertible, but we show this is not true. We further prove that the frame operator for a pair of subfusion frames is bounded below and invertible. Also, we will study operators for a pair of Bessel subfusion sequences. In Section 4, we study some constructions of subfusion frames. Finally, Section 5 contains a discussion on dual subfusion frames. In [5], it has been shown that the dual fusion frame is a fusion frame. In this section, through an example, we show if $\omega = \{(W_i, \beta_i)\}_{i\in I}$ be a subfusion frame of $\nu = \{(V_i, \alpha_i)\}_{i\in I}$, then $\{(S_\omega^{-1} W_i, \beta_i)\}_{i\in I}$ is not necessarily a subfusion frame of $\{(S_\nu^{-1} V_i, \alpha_i)\}_{i\in I}$, and we show that under some conditions, $\{(S_\omega^{-1} W_i, \beta_i)\}_{i\in I}$ is a subfusion frame of $\{(S_\nu^{-1} V_i, \alpha_i)\}_{i\in I}$.

Through this paper, H is a separable Hilbert space, I is a countable index set, and $\{V_i\}_{i\in I}$ is a sequence of closed subspaces of H.

2. Review of Frames, Fusion Frames, and Subfusion Frames

In this section, we recall some definitions and basic properties of frames, fusion frames and subfusion frames. For more information, we refer the reader to [3, 9, 12, 13].

Definition 2.1. A sequence $\{f_i\}_{i\in I}$ of elements in H is a frame for H if there exist positive constants A and B (lower and upper frame bounds, resp.) such that

$$A\|f\|^2 \le \sum_{i\in I} \|\langle f, f_i\rangle\|^2 \le B\|f\|^2, \quad \forall f \in H. \tag{2.1}$$

Definition 2.2. Let $\{V_i\}_{i\in I}$ be a family of closed subspaces of a Hilbert space H and let $\{\alpha_i\}_{i\in I}$ be a family of weights, that is, $\alpha_i > 0$ for all $i \in I$. Then $\nu = \{(V_i, \alpha_i)\}_{i\in I}$ is a fusion frame, if there exist positive constants C and D (lower and upper fusion frame bounds, resp.) such that

$$C\|f\|^2 \le \sum_{i\in I} \alpha_i^2 \|\pi_{V_i}(f)\|^2 \le D\|f\|^2, \quad \forall f \in H, \tag{2.2}$$

where π_{V_i} is the orthogonal projection onto the subspace V_i. A fusion frame ν is called λ-tight fusion frame if $C = D = \lambda$, Parseval fusion frame if $C = D = 1$, and α-uniform fusion frame if $\alpha = \alpha_i$ for all $i \in I$. If the second part of the above inequality is satisfied, then ν is called a Bessel fusion sequence for H with bound D.

Similar to ordinary frames, the fusion frame operator S_ν is defined by

$$S_\nu(f) = \sum_{i\in I} \alpha_i^2 \pi_{V_i} f, \quad \forall f \in H. \tag{2.3}$$

S_ν is a linear, positive, self-adjoint, and invertible operator and we have

$$CI \leq S_\nu \leq DI. \tag{2.4}$$

A family of bounded operators $\{T_i\}_{i\in I}$ on H is called a resolution of identity on H if $f = \sum_{i\in I} T_i f$, for all $f \in H$.

Proposition 2.3. *Let $\{(V_i, \alpha_i)\}_{i\in I}$ be a fusion frame for H, W_i be a closed subspace of V_i and $\beta_i \leq \alpha_i$ for all $i \in I$. Then $\{(W_i, \beta_i)\}_{i\in I}$ is a Bessel fusion sequence for H.*

Proof. Since W_i is a closed subspace of V_i, $\pi_{W_i}\pi_{V_i} f = \pi_{V_i}\pi_{W_i} f = \pi_{W_i} f$ and $\|\pi_{W_i} f\|^2 \leq \|\pi_{V_i} f\|^2$ for all $f \in H$ and for all $i \in I$. Hence,

$$\sum_{i\in I} \beta_i^2 \|\pi_{W_i} f\|^2 \leq \sum_{i\in I} \alpha_i^2 \|\pi_{V_i} f\|^2 \tag{2.5}$$

implies that $\{(W_i, \beta_i)\}_{i\in I}$ is a Bessel fusion sequence. □

There are examples such that $\{(V_i, \alpha_i)\}_{i\in I}$ is a fusion frame, W_i is a closed subspace of V_i and $\beta_i \leq \alpha_i$ for all $i \in I$, while $\{(W_i, \alpha_i)\}_{i\in I}$ is not a fusion frame.

Example 2.4. Let $\{(V_i, \alpha_i)\}_{i\in \mathbb{N}}$ be a fusion frame for H and $V_1 \neq H$. Define

$$W_i = \begin{cases} V_1 & \text{if } i = 1, \\ 0 & \text{otherwise}. \end{cases} \tag{2.6}$$

Since $\overline{\operatorname{span}}_{i\in\mathbb{N}}\{W_i\} = V_1 \neq H$, $\{(W_i, \alpha_i)\}_{i\in I}$ is not a fusion frame for H [3, Lemma 3.4].

Definition 2.5. Let $\nu = \{(V_i, \alpha_i)\}_{i\in I}$ be a fusion frame for H, W_i be a closed subspace of V_i and $\beta_i \leq \alpha_i$ for all $i \in I$. If $\omega = \{(W_i, \beta_i)\}_{i\in I}$ is a fusion frame for H, then ω is called a subfusion frame of ν. If ν and ω are Bessel fusion sequences for H, then ω is called a Bessel subfusion sequence of ν [9].

3. Operators between a Pair of Bessel Subfusion Sequences

In this section, we will study operators for a pair of Bessel subfusion sequences. Alternate dual frames and Bessel fusion sequences are important in the literature of frame theory because of their important role in applications. The notions of operators for a pair of Bessel fusion sequences and alternative dual of a fusion frame in H are defined by Găvruţa in [5].

Let $\nu = \{(V_i, \alpha_i)\}_{i\in I}$ and $\omega = \{(W_i, \beta_i)\}_{i\in I}$ be two Bessel fusion sequences for H. Then the frame operator for them is defined by

$$S_{\omega\nu} f = \sum_{i\in I} \alpha_i \beta_i \pi_{W_i} \pi_{V_i} f, \quad \forall f \in H. \tag{3.1}$$

Moreover, if $\nu = \{(V_i, \alpha_i)\}_{i\in I}$ is a fusion frame for H, with fusion frame operatore S_ν, then ω is called an alternate dual of ν, if we have

$$f = \sum_{i\in I} \alpha_i \beta_i \pi_{W_i} S_\nu^{-1} \pi_{V_i} f, \quad \forall f \in H. \tag{3.2}$$

$S_{\omega\nu}$ is bounded and $S_{\omega\nu} = S^*_{\nu\omega}$. Recently, Khosravi and Musazadeh in [11, Proposition 2.9] tried to show that $S_{\omega\nu}$ is bounded below and invertible. "Let $\nu = \{(V_i, \alpha_i)\}_{i\in I}$ be a fusion frame with fusion frame bounds C and D and fusion frame operator S_ν for H. Let $\omega = \{(W_i, \beta_i)\}_{i\in I}$ be an alternate dual fusion frame for ν with required positivity. Then we have

$$CI \leq S_{\omega\nu} \leq DI, \tag{3.3}$$

and also $S_{\omega\nu}$ is invertible."

In the following example, we show that $S_{\omega\nu}$ is neither invertible nor bounded below.

Example 3.1. Set

$$V_1 = \langle (1,0,0) \rangle, \qquad V_2 = \langle (1,1,0) \rangle, \qquad V_3 = \langle (0,1,0) \rangle, \qquad V_4 = \langle (0,0,1) \rangle, \tag{3.4}$$

and $\alpha_1 = \alpha_3 = \alpha_4 = 1$, $\alpha_2 = \sqrt{2}$. Define $\pi_{V_i} : \mathbb{R}^3 \to V_i$ by

$$\begin{gathered} \pi_{V_1}(a,b,c) = (a,0,0), \qquad \pi_{V_2}(a,b,c) = \left(\frac{a+b}{2}, \frac{a+b}{2}, 0\right), \\ \pi_{V_3}(a,b,c) = (0,b,0), \qquad \pi_{V_4}(a,b,c) = (0,0,c). \end{gathered} \tag{3.5}$$

For any $f = (a,b,c)$ in $\mathbb{R}^3$.

Computation show that π_{V_i}, $i = \{1,2,3,4\}$ are projection (π_{V_i} is projection if $\pi_{V_i}^2 = \pi_{V_i}$ [14]). We have

$$S_\nu(f) = (a,0,0) + (a+b,\ a+b,\ 0) + (0,b,0) + (0,0,c) = (2a+b,\ 2b+a,\ c). \tag{3.6}$$

Then

$$S_\nu = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad S_\nu^{-1} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & 0 \\ -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{3.7}$$

Now, let

$$W_1 = \langle (0,1,0) \rangle, \qquad W_2 = \mathbb{R}^3, \qquad W_3 = \langle (1,0,0) \rangle, \qquad W_4 = \langle (0,0,1) \rangle, \tag{3.8}$$

and $\beta_1 = \beta_3 = 3$, $\beta_2 = 3\sqrt{2}$, $\beta_4 = 1$. Define $\pi_{W_i} : \mathbb{R}^3 \to W_i$ by

$$\begin{aligned} \pi_{W_1}(a,b,c) &= (0,b,0), \qquad \pi_{W_2}(a,b,c) = (a,b,c), \\ \pi_{W_3}(a,b,c) &= (a,0,0), \qquad \pi_{W_4}(a,b,c) = (0,0,c). \end{aligned} \tag{3.9}$$

Obviously, $\omega = \{(W_i, \beta_i)\}_{i\in I}$ is an alternate dual of $\nu = \{(V_i, \alpha_i)\}_{i\in I}$, where $I = \{1,2,3,4\}$. Then, we have

$$\begin{aligned} \sum_{i=1}^{4} \alpha_i \beta_i \pi_{W_i} S_\nu^{-1} \pi_{V_i}(f) &= 3\pi_{W_1} \begin{bmatrix} \frac{2a}{3} \\ -\frac{a}{3} \\ 0 \end{bmatrix} + 6\pi_{W_2} \begin{bmatrix} \frac{a+b}{6} \\ \frac{a+b}{6} \\ 0 \end{bmatrix} + 3\pi_{W_3} \begin{bmatrix} -\frac{b}{3} \\ \frac{2b}{3} \\ 0 \end{bmatrix} + \pi_{W_4} \begin{bmatrix} 0 \\ 0 \\ c \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ -a \\ 0 \end{bmatrix} + \begin{bmatrix} a+b \\ a+b \\ 0 \end{bmatrix} + \begin{bmatrix} -b \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ c \end{bmatrix} = (a,b,c) = f. \end{aligned} \tag{3.10}$$

Also

$$\begin{aligned} S_{\omega\nu} &= \sum_{i=1}^{4} \alpha_i \beta_i \pi_{W_i} \pi_{V_i}(f) = \alpha_2 \beta_2 \pi_{W_2} \pi_{V_2}(f) + \alpha_4 \beta_4 \pi_{W_4} \pi_{V_4}(f) = 6\pi_{W_2} \begin{bmatrix} \frac{a+b}{2} \\ \frac{a+b}{2} \\ 0 \end{bmatrix} + \pi_{W_4} \begin{bmatrix} 0 \\ 0 \\ c \end{bmatrix} \\ &= 3 \begin{bmatrix} a+b \\ a+b \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ c \end{bmatrix} = \begin{bmatrix} 3(a+b) \\ 3(a+b) \\ c \end{bmatrix}. \end{aligned} \tag{3.11}$$

Then $S_{\omega\nu} = \begin{bmatrix} 3 & 3 & 0 \\ 3 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Therefore, $S_{\omega\nu}$ is not invertible. If $f = (1, -1, 0)$, then there is not a positive number C such that

$$C\langle f, f\rangle \leq \langle S_{\omega\nu} f, f\rangle. \tag{3.12}$$

Next, we show that under some conditions, $S_{\omega\nu}$ is invertible and bounded below.

Proposition 3.2. *Let $\nu = \{(V_i, \alpha_i)\}_{i\in I}$ be a fusion frame with fusion frame bounds C and D and fusion frame operator S_ν for H. Let $\omega = \{(V_i, \beta_i)\}_{i\in I}$ be an alternate dual fusion frame for ν with required positivity. Then we have*

$$CI \leq S_{\omega\nu} \leq DI, \tag{3.13}$$

and $S_{\omega\nu}$ is invertible.

Proof. Let f be an arbitrary element of H. Then we have

$$\begin{aligned} \|f\|^2 = \langle f, f\rangle &= \left\langle \sum_{i\in I} \alpha_i\beta_i \pi_{V_i} S_\nu^{-1} \pi_{V_i}(f), f \right\rangle = \sum_{i\in I} \alpha_i\beta_i \left\langle S_\nu^{-1}\pi_{V_i} f, \pi_{V_i} f\right\rangle \\ &\leq \frac{1}{C}\sum_{i\in I}\alpha_i\beta_i\langle \pi_{V_i} f, \pi_{V_i} f\rangle = \frac{1}{C}\left\langle \sum_{i\in I}\alpha_i\beta_i\pi_{V_i} f, f\right\rangle \\ &= \frac{1}{C}\langle S_{\omega\nu} f, f\rangle. \end{aligned} \tag{3.14}$$

Similarly, we have $\langle S_{\omega\nu} g, f\rangle \leq D\|f\|^2$. Hence, $S_{\omega\nu}$ is injective and $S_{\omega\nu}H$ is closed in H

$$\operatorname{Range}(S_{\omega\nu}) = \overline{\operatorname{Range}(S_{\omega\nu})} = (N(S_{\omega\nu}^*))^\perp = (N(S_{\omega\nu}))^\perp = H. \tag{3.15}$$

Then $S_{\omega\nu}$ is invertible. □

Now, we are ready to describe the operator for a pair of subfusion frames. In this case, positivity, invertibility, and boundedness properties of these operators have been checked.

Proposition 3.3. *Let $\omega = \{(W_i, \beta_i)\}_{i\in I}$ be a Bessel subfusion frame of $\nu = \{(V_i, \alpha_i)\}_{i\in I}$. Then $S_{\omega\nu}$ is self-adjoint and positive.*

Proof. For any $f \in H$

$$S_{\nu\omega} f = \sum_{i\in I}\alpha_i\beta_i\pi_{V_i}\pi_{W_i} f = \sum_{i\in I}\alpha_i\beta_i\pi_{W_i} f = \sum_{i\in I}\alpha_i\beta_i\pi_{W_i}\pi_{V_i} f = S_{\omega\nu} f, \tag{3.16}$$

then $S_{\nu\omega} = S_{\omega\nu}$. Also,

$$\langle S_{\omega\nu} f, f\rangle = \sum_{i\in I}\alpha_i\beta_i\langle \pi_{W_i} f, f\rangle = \sum_{i\in I}\alpha_i\beta_i\|\pi_{W_i} f\|^2. \tag{3.17}$$

Then $S_{\omega\nu}$ is positive. □

Lemma 3.4. *Let $\omega = \{(W_i, \beta_i)\}_{i\in I}$ be Bessel subfusion sequence of $\nu = \{(V_i, \alpha_i)\}_{i\in I}$. If $S_{\omega\nu}$ is bounded below, then ω is a subfusion frame of ν.*

Proof. Suppose that there exist a number $\lambda > 0$ such that for all $f \in H$

$$\lambda\|f\| \leq \|S_{\omega\nu} f\|, \tag{3.18}$$

Then we have

$$\begin{aligned}
\lambda\|f\| \leq \|S_{\omega\nu} f\| &= \sup_{g\in H, \|g\|=1} \left\| \left\langle \sum_{i\in I} \alpha_i\beta_i\pi_{W_i}\pi_{V_i} f, g \right\rangle \right\| \\
&\leq \sup_{g\in H, \|g\|=1} \left(\sum_{i\in I} \beta_i^2 \|\pi_{W_i} f\|^2 \right)^{1/2} \left(\sum_{i\in I} \alpha_i^2 \|\pi_{V_i} g\|^2 \right)^{1/2} \\
&\leq \sqrt{D} \left(\sum_{i\in I} \beta_i^2 \|\pi_{W_i} f\|^2 \right)^{1/2},
\end{aligned} \tag{3.19}$$

where D is upper bound of ν. Hence,

$$\frac{\lambda^2}{D}\|f\|^2 \leq \sum_{i\in I} \beta_i^2 \|\pi_{W_i} f\|^2 \leq \sum_{i\in I} \alpha_i^2 \|\pi_{V_i} f\|^2. \tag{3.20}$$

□

Corollary 3.5. *Let $\omega = \{(W_i, \beta_i)\}_{i\in I}$ be a Bessel subfusion sequence of $\nu = \{(V_i, \alpha_i)\}_{i\in I}$ and $S_{\omega\nu}$ be invertible. Then ω is a subfusion frame of ν.*

Proof. Since $S_{\omega\nu}$ is invertible, $S_{\omega\nu}$ is a below bounded. Then ω is a subfusion frame of ν. □

Lemma 3.6. *Let $\omega = \{(W_i, \beta_i)\}_{i\in I}$ be Bessel subfusion frame of $\nu = \{(V_i, \alpha_i)\}_{i\in I}$. Then*

$$S_\omega \leq S_{\omega\nu} \leq S_\nu. \tag{3.21}$$

Proof.

$$\begin{aligned}
\langle (S_{\omega\nu} - S_\omega f), f \rangle &= \left\langle \sum_{i\in I} \alpha_i\beta_i\pi_{W_i} f - \sum_{i\in I} \beta_i^2\pi_{W_i} f, f \right\rangle = \sum_{i\in I} \beta_i(\alpha_i - \beta_i)\langle \pi_{W_i} f, f\rangle \\
&= \sum_{i\in I} \beta_i(\alpha_i - \beta_i)\|\pi_{W_i} f\|^2,
\end{aligned} \tag{3.22}$$

then $S_\omega \leq S_{\omega\nu}$. We have

$$\begin{aligned}\langle (S_\nu - S_{\omega\nu} f), f\rangle &= \left\langle \sum_{i\in I} \alpha_i^2 \pi_{V_i} f - \sum_{i\in I} \alpha_i\beta_i \pi_{W_i} f, f \right\rangle = \sum_{i\in I} \alpha_i \left(\langle \alpha_i \pi_{V_i} f, f\rangle - \langle \beta_i \pi_{W_i} f, f\rangle \right) \\ &= \sum_{i\in I} \alpha_i \left(\alpha_i \left\| \pi_{V_i} f \right\|^2 - \beta_i \left\| \pi_{V_i} f \right\|^2 \right),\end{aligned} \tag{3.23}$$

since $\beta_i \|\pi_{V_i} f\|^2 \leq \alpha_i \|\pi_{V_i} f\|^2$, $S_{\omega\nu} \leq S_\nu$. Hence, $S_\omega \leq S_{\omega\nu} \leq S_\nu$. □

Now we show that if ω be a subfusion frame of ν, then $S_{\omega\nu}$ is invertible.

Corollary 3.7. *Let $\omega = \{(W_i, \alpha_i)\}_{i\in I}$ be a subfusion frame of $\nu = \{(V_i, \alpha_i)\}_{i\in I}$. Then $S_{\omega\nu}$ is invertible.*

Proof. For all $f \in H$

$$S_{\omega\nu} f = \sum_{i\in I} \alpha_i^2 \pi_{W_i} \pi_{V_i} f = \sum_{i\in I} \alpha_i^2 \pi_{W_i} f = S_\omega f, \tag{3.24}$$

then $S_{\omega\nu} = S_\omega$. Since S_ω is invertible, hence, $S_{\omega\nu}$ is invertible. □

Lemma 3.8. *If $\omega = \{(W_i, \beta_i)\}_{i\in I}$ be a subfusion frame of $\nu = \{(V_i, \alpha_i)\}$ such that C is lower bound of ω and D is upper bound of ν, then*

$$CI \leq S_{\omega\nu} \leq DI, \tag{3.25}$$

and $S_{\omega\nu}$ is invertible.

Proof. By Lemma 3.6, we have

$$CI \leq S_{\omega\nu} \leq DI. \tag{3.26}$$

Hence, $S_{\omega\nu}$ is injective, $S_{\omega\nu} H$ is closed in H. We have

$$\operatorname{Range}(S_{\omega\nu}) = \overline{\operatorname{Range}(S_{\omega\nu})} = (N(S_{\omega\nu}^*))^\perp = (N(S_{\omega\nu}))^\perp = H. \tag{3.27}$$

Hence, $S_{\omega\nu}$ is onto and therefor, invertible on H. On the other hand, since $CI \leq S_{\omega\nu} \leq DI$, we have $1/D \leq \|S_{\omega\nu}^{-1}\| \leq 1/C$. □

Remark 3.9. By this lemma we have the following reconstruction formulas:

$$f = \sum_{i\in I} \alpha_i\beta_i \pi_{W_i} S_{\omega\nu}^{-1} f = \sum_{i\in I} \alpha_i\beta_i S_{\omega\nu}^{-1} \pi_{W_i} f, \tag{3.28}$$

for all $f \in H$. Therefore, two families of bounded operators $\{\alpha_i\beta_i\pi_{W_i} S_{\omega\nu}^{-1}\}_{i\in I}$ and $\{\alpha_i\beta_i S_{\omega\nu}^{-1}\pi_{W_i}\}_{i\in I}$ are resolutions of the identity.

Theorem 3.10. *Let $\nu = \{(V_i, \alpha_i)\}_{i\in I}$ be a fusion frame with upper bound D and $\omega = \{(W_i, \beta_i)\}_{i\in I}$ be a Bessel subfusion sequence of ν. Suppose that there exists a number $\lambda > D$ such that $\lambda\|f\| \leq \|(S_\nu - S_{\omega\nu})f\|$, for all $f \in H$. Then $S_{\omega\nu}$ is invertible and also ω is a subfusion frame of ν.*

Proof. Let f be an arbitrary element of H. Then we have

$$\begin{aligned} \|S_{\omega\nu} f\| &= \|(S_{\omega\nu} - S_\nu + S_\nu)f\| \geq \|(S_{\omega\nu} - S_\nu)f\| - \|S_\nu f\| \\ &\geq (\lambda - D)\|f\|. \end{aligned} \tag{3.29}$$

So $S_{\omega\nu}$ is bounded below, hence $S_{\omega\nu}$ is invertible and there exists a number $B > 0$ such that $B\|f\| \leq \|S_{\omega\nu} f\|$, for all $f \in H$. Then

$$\|S_{\omega\nu} f\| \leq \sqrt{D}\left(\sum_{i\in I}\beta_i^2\|\pi_{W_i} f\|^2\right)^{1/2}. \tag{3.30}$$

Hence,

$$\frac{B^2}{D}\|f\|^2 \leq \sum_{i\in I}\beta_i^2\|\pi_{W_i} f\|^2 \leq \sum_{i\in I}\alpha_i^2\|\pi_{V_i} f\|^2 \leq D\|f\|^2. \tag{3.31}$$

□

4. Construction of Subfusion Frames

In this section, we study some new constructions of subfusion frames. Dealing with Bessel subfusion frames is important, since there are easy ways to turn such a family into subfusion frames. One way is to just add the subspace $W_0 = V_0 = H$ to the families. We follow some ways that obtain new subfusion frames.

Lemma 4.1. *Let $\{(W_i, \alpha_i)\}_{i\in I}$ be a fusion frame for H with bound C and D, and Let $\{V_i\}_{i\in I}$ be a family of closed subspace H such that $W_i \subset V_i$ for all $i \in I$. If there exists $\lambda > 0$ such that*

$$\sum_{i\in I}\alpha_i^2\|\pi_{V_i} f - \pi_{W_i} f\|^2 \leq \lambda\|f\|^2, \tag{4.1}$$

for all $f \in H$, then $\{(W_i, \alpha_i)\}_{i\in I}$ is a subfusion frame of $\{(V_i, \alpha_i)\}_{i\in I}$.

Proof. By using the triangle inequality, for all $f \in H$, we have

$$\begin{aligned} \|\{\alpha_i\pi_{V_i} f\}_{i\in I}\|_{\ell^2} &\leq \|\{\alpha_i(\pi_{V_i} f - \pi_{W_i} f)\}_{i\in I}\|_{\ell^2} + \|\{\alpha_i\pi_{W_i} f\}_{i\in I}\|_{\ell^2} \\ &\leq \sqrt{\lambda}\|f\| + \sqrt{D}\|f\| = \sqrt{D}\left(1 + \frac{\sqrt{\lambda}}{D}\right)\|f\|; \end{aligned} \tag{4.2}$$

therefore,

$$C\|f\|^2 \le \sum_{i\in I}\alpha_i^2\|\pi_{W_i}f\|^2 \le \sum_{i\in I}\alpha_i^2\|\pi_{V_i}f\|^2 \le D\left(1+\frac{\sqrt{\lambda}}{D}\right)^2\|f\|^2. \tag{4.3}$$

□

Theorem 4.2. *Let $\omega = \{(W_i,\alpha_i)\}_{i\in I}$ be a fusion frame for H, and Let $\{V_i\}_{i\in I}$ be a family of closed subspace in H such that $W_i \subset V_i$ for $i \in I$. If $U : H \to H$ defined by*

$$U(f) = \sum_{i\in I}\alpha_i^2(\pi_{V_i}f - \pi_{W_i}f) \quad \forall f \in H, \tag{4.4}$$

is a bounded operator, then ω is a subfusion frame of $\{(V_i,\alpha_i)\}_{i\in I}$.

Proof. Let C and D be the frame bounds for ω. Then $\|S_\omega\| \le D$. A simple computation show that U is self-adjoint. So if $T : H \to H$ is defined by $T = S_\omega + U$, then T is a bounded, linear and self-adjoint operator. Therefore,

$$\begin{gathered}\|T\| = \sup_{\|f\|\le 1}\|\langle Tf,f\rangle\| = \sup_{\|f\|\le 1}\sum\alpha_i^2\|\pi_{V_i}f\|^2,\\ \sum_{i\in I}\alpha_i^2\|\pi_{V_i}f\|^2 \le \|T\|\|f\|^2 \le (\|S_\omega\|+\|U\|)\|f\|^2 \le (D+\|U\|)\|f\|^2,\end{gathered} \tag{4.5}$$

for all $f \in H$. Hence,

$$C\|f\|^2 \le \sum_{i\in I}\alpha_i^2\|\pi_{W_i}f\|^2 \le \sum_{i\in I}\alpha_i^2\|\pi_{V_i}f\|^2 \le (D+\|U\|)\|f\|^2. \tag{4.6}$$

□

Proposition 4.3. *Let $\omega = \{(W_i,\beta_i)\}_{i\in I}$ be a Bessel subfusion sequence of $\nu = \{(V_i,\alpha_i)\}_{i\in I}$ and $S_{\omega\nu}$ be below bounded. Then*

(i) $\{(S_{\omega\nu}W_i,\beta_i)\}_{i\in I}$ is a subfusion frame of $\{(S_{\omega\nu}V_i,\alpha_i)\}_{i\in I}$.
(ii) $\{(S_{\omega\nu}^{-1}W_i,\beta_i)\}_{i\in I}$ is a subfusion frame of $\{(S_{\omega\nu}^{-1}V_i,\alpha_i)\}_{i\in I}$.

Proof. The proof is straight forward. □

Lemma 4.4. *Let $\{W_i\}_{i\in I}$ and $\{V_i\}_{i\in I}$ be a closed subspaces of H and $J \subset I$. Suppose that*

$$W_i = \begin{cases} V_i & i \in J,\\ \langle 0\rangle & i \in I - J.\end{cases} \tag{4.7}$$

If $\omega = \{(W_i,\alpha_i)\}_{i\in I}$ is a fusion frame for H with bound C, D and $c = \sum_{i\in I-J}\alpha_i^2$, then ω is a subfusion frame of $\nu = \{(V_i,\alpha_i)\}_{i\in I}$ with bound $C, D+c$.

Proof. For all $f \in H$, we have

$$\begin{aligned}\sum_{i\in I}\alpha_i^2\|\pi_{V_i}f\|^2 &= \sum_{i\in I-J}\alpha_i^2\|\pi_{V_i}f\|^2+\sum_{i\in J}\alpha_i^2\|\pi_{V_i}f\|^2 = \sum_{i\in I-J}\alpha_i^2\|\pi_{V_i}f\|^2+\sum_{i\in J}\alpha_i^2\|\pi_{W_i}f\|^2 \\ &\le D\|f\|^2+\left(\sum_{i\in I-J}\alpha_i^2\right)\|f\|^2 \le (D+c)\|f\|^2.\end{aligned} \tag{4.8}$$

then ω is a subfusion frame of ν. □

5. Dual Subfusion Frames

Let $\nu = \{(V_i,\alpha_i)\}_{i\in I}$ be a fusion frame, then the family $\{(S_\nu^{-1}V_i,\alpha_i)\}_{i\in I}$ is called the dual fusion frame. In [5], it has been shown that the dual fusion frame is a fusion frame. In this section, through an example, we show that if $\omega = \{(W_i,\beta_i)\}_{i\in I}$ be a subfusion frame of $\nu = \{(V_i,\alpha_i)\}_{i\in I}$, then $\{(S_\omega^{-1}W_i,\beta_i)\}_{i\in I}$ is not necessarily a subfusion frame of $\{(S_\nu^{-1}V_i,\alpha_i)\}_{i\in I}$.

Example 5.1. Suppose $V_1 = \langle e_1\rangle$, $V_2 = V_3 = R^3$. We define $\pi_{V_i} : R^3 \to V_i$ by $\pi_{V_1}(a,b,c) = (a,0,0)$, $\pi_{V_2} = \pi_{V_3}(a,b,c) = (a,b,c)$. $\{(V_1,\alpha_1),(V_2,\alpha_2),(V_3,\alpha_3)\}$ is a fusion frame. We have

$$S_\nu = \begin{bmatrix}\alpha_1^2+\alpha_2^2+\alpha_3^2 & 0 & 0\\ 0 & \alpha_2^2+\alpha_3^2 & 0\\ 0 & 0 & \alpha_2^2+\alpha_3^2\end{bmatrix}. \tag{5.1}$$

Also we assume that $W_1 = \langle e_1\rangle$, $W_2 = \langle(1,1,0)\rangle$, $W_3 = R^3$.

We define $\pi_{W_i} : R^3 \to W_i$ by $\pi_{W_1}(a,b,c) = (a,0,0)$, $\pi_{W_2}(a,b,c) = ((a+b)/2,(a+b)/2,0)$, $\pi_{W_3}(a,b,c) = (a,b,c)$. The set $\{(W_1,\beta_1),(W_2,\beta_2),(W_3,\beta_3)\}$ is a fusion frame and

$$S_\omega = \begin{bmatrix}\beta_1^2+\dfrac{\beta_2^2}{2}+\beta_3^2 & \dfrac{\beta_2^2}{2} & 0\\ \dfrac{\beta_2^2}{2} & \dfrac{\beta_2^2}{2}+\beta_3^2 & 0\\ 0 & 0 & \beta_3^2\end{bmatrix}. \tag{5.2}$$

Set $\alpha_1^2 = \alpha_3^2 = 1$, $\alpha_2^2 = 3$, $\beta_1^2 = \beta_3^2 = 1$ and $\beta_2^2 = 2$. It is clear that $\{(S_\omega^{-1}W_i,\beta_i)\}_{i\in I}$ is not a subfusion frame of $\{(S_\nu^{-1}V_i,\alpha_i)\}_{i\in I}$ where $I = \{1,2,3\}$.

Now by another example we show that $\{(S_\nu^{-1}V_i,\alpha_i)\}_{i\in I}$ is not subfusion frame of $\{(S_\omega^{-1}W_i,\beta_i)\}_{i\in I}$.

Example 5.2. Suppose $V_1 = \langle e_1 \rangle$, $V_2 = V_3 = R^3$. We define $\pi_{V_i} : R^3 \to V_i$ by $\pi_{V_1}(a,b,c) = (a,0,0)$, $\pi_{V_2}(a,b,c) = \pi_{V_3}(a,b,c) = (a,b,c)$. $\{(V_i,\alpha_i)\}_{i\in I}$ where $I = \{1,2,3\}$ is a fusion frame, We assume that $W_1 = \langle e_1 \rangle$, $W_2 = \langle e_2 \rangle$, $W_3 = R^3$. We define $\pi_{W_i} : R^3 \to W_i$ by

$$\pi_{W_1}(a,b,c) = (a,0,0), \qquad \pi_{W_2}(a,b,c) = (0,b,0), \qquad \pi_{W_3}(a,b,c) = (a,b,c). \tag{5.3}$$

The set $\{(W_1,\beta_1),(W_2,\beta_2),(W_3,\beta_3)\}$ is a fusion frame and

$$S_\omega = \begin{bmatrix} \beta_1^2 + \beta_3^2 & 0 & 0 \\ 0 & \beta_2^2 + \beta_3^2 & 0 \\ 0 & 0 & \beta_3^2 \end{bmatrix}. \tag{5.4}$$

Set $\alpha_1^2 = \alpha_3^2 = 1$, $\alpha_2^2 = 3$, $\beta_1^2 = \beta_3^2 = 1$ and $\beta_2^2 = 3$. A simple computation shows that $\{(S_\nu^{-1}V_i,\alpha_i)\}_{i\in I}$ is not a subfusion frame of $\{(S_\omega^{-1}W_i,\beta_i)\}_{i\in I}$.

Proposition 5.3. *Let $\omega = \{(W_i,\beta_i)\}_i$ be a subfusion frame of $\nu = \{(V_i,\alpha_i)\}_{i\in I}$. If $V_i \perp V_j$ and $W_i \perp W_j$ for all $i \neq j$, then $\{(S_\omega^{-1}W_i,\beta_i)\}_{i\in I}$ is a subfusion frame of $\{(S_\nu^{-1}V_i,\alpha_i)\}_{i\in I}$.*

Proof. Set $f \in V_j$ and $f = \alpha_j(g)\ g \in V_j$, $f = \alpha_j(g) = \sum_{i\in I} \alpha_i^2 \pi_{V_i}(g) \in S_\nu V_j$ then $V_j \subset S_\nu V_j$ for all $j \in I$. If $f \in S_\nu V_j$, $f = \sum_{i\in} \alpha_i^2 \pi_{V_i}(g) = \alpha_j(g)\ g \in V_j$, then $S_\nu V_j \subset V_j$. Hence, $S_\nu^{-1} V_j = V_j$. □

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