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Research Article

General Iterative Methods for Equilibrium Problems and Infinitely Many Strict Pseudocontractions in Hilbert Spaces

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We propose an implicit iterative scheme and an explicit iterative scheme for finding a common element of the set of fixed point of infinitely many strict pseudocontractive mappings and the set of solutions of an equilibrium problem by the general iterative methods. In the setting of real Hilbert spaces, strong convergence theorems are proved. Our results improve and extend the corresponding results reported by many others.

1. Introduction

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let F be a bifunction from $C \times C$ to $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers.

The equilibrium problem for $F : C \times C \to \mathbb{R}$ is to find $x \in C$ such that

$$F(x, y) \geq 0 \tag{1.1}$$

for all $y \in C$. The set of such solutions is denoted by $\mathrm{EP}(F)$.

A mapping S of C is said to be a κ-strict pseudocontraction if there exists a constant $\kappa \in [0, 1)$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + \kappa\|(I - S)x - (I - S)y\|^2 \tag{1.2}$$

for all $x, y \in C$; see [1]. We denote the set of fixed points of S by $F(S)$ (i.e., $F(S) = \{x \in C : Sx = x\}$).

Note that the class of strict pseudocontractions strictly includes the class of nonexpansive mappings which are mapping S on C such that

$$\|Sx - Sy\| \leq \|x - y\| \tag{1.3}$$

for all $x, y \in C$. That is, S is nonexpansive if and only if S is a 0-strict pseudocontraction.

Numerous problems in physics, optimization, and economics reduce to finding a solution of the equilibrium problem. Some methods have been proposed to solve the equilibrium problem (1.1); see, for instance, [2–4]. In particular, Combettes and Hirstoaga [5] proposed several methods for solving the equilibrium problem. On the other hand, Mann [6], Shimoji and Takahashi [7] considered iterative schemes for finding a fixed point of a nonexpansive mapping. Further, Acedo and Xu [8] projected new iterative methods for finding a fixed point of strict pseudocontractions.

In 2006, Marino and Xu [3] introduced the general iterative method and proved that the algorithm converged strongly. Recently, Liu [2] considered a general iterative method for equilibrium problems and strict pseudocontractions. Tian [9] proposed a new general iterative algorithm combining an L-Lipschitzian and η-strong monotone operator. Very recently, Wang [10] considered a general composite iterative method for infinite family strict pseudocontractions.

In this paper, motivated by the above facts, we introduce two iterative schemes and obtain strong convergence theorems for finding a common element of the set of fixed points of a infinite family of strict pseudocontractions and the set of solutions of the equilibrium problem (1.1).

2. Preliminaries

Throughout this paper, we always write $\rightharpoonup$ for weak convergence and $\rightarrow$ for strong convergence. We need some facts and tools in a real Hilbert space H which are listed as below.

Lemma 2.1. *Let H be a real Hilbert space. There hold the following identities:*

(i) $\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle,\ \forall x, y \in H$;

(ii) $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x - y\|^2,\ \forall t \in [0,1],\ \forall x, y \in H$.

Lemma 2.2 (see [11]). *Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + \delta_n, \tag{2.1}$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(ii) $\lim_{n\to\infty} \sup(\delta_n/\gamma_n) \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then, $\lim_{n\to\infty}\alpha_n = 0$.

Recall that given a nonempty closed convex subset C of a real Hilbert space H, for any $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\| \tag{2.2}$$

for all $y \in C$. Such a P_C is called the metric (or the nearest point) projection of H onto C. As known, $y = P_C x$ if and only if there holds the relation:

$$\langle x - y, y - z\rangle \geq 0 \quad \forall z \in C. \tag{2.3}$$

Lemma 2.3 (see [10]). *Let $A : H \to H$ be a L-Lipschitzian and η-strongly monotone operator on a Hilbert space H with $L > 0$, $\eta > 0$, $0 < \mu < 2\eta/L^2$, and $0 < t < 1$. Then, $S = (I - t\mu A) : H \to H$ is a contraction with contractive coefficient $1 - t\tau$ and $\tau = (1/2)\mu(2\eta - \mu L^2)$.*

Lemma 2.4 (see [1]). *Let $S : C \to C$ be a κ-strict pseudocontraction. Define $T : C \to C$ by $Tx = \lambda x + (1 - \lambda)Sx$ for each $x \in C$. Then, as $\lambda \in [\kappa, 1)$, T is a nonexpansive mapping such that $F(T) = F(S)$.*

Lemma 2.5 (see [9]). *Let H be a Hilbert space and $f : H \to H$ be a contraction with coefficient $0 < \alpha < 1$, and $A : H \to H$ an L-Lipschitzian continuous operator and η-strongly monotone with $L > 0$, $\eta > 0$. Then for $0 < \gamma < \mu\eta/\alpha$:*

$$\langle x - y, (\mu A - \gamma f)x - (\mu A - \gamma f)y\rangle \geq (\mu\eta - \gamma\alpha)\|x - y\|^2, \quad x, y \in H. \tag{2.4}$$

That is, $\mu A - \gamma f$ is strongly monotone with coefficient $\mu\eta - \gamma\alpha$.

Let $\{S_n\}$ be a sequence of κ_n-strict pseudo-contractions. Define $S'_n = \theta_n I + (1 - \theta_n)S_n, \theta_n \in [\kappa_n, 1)$. Then, by Lemma 2.4, S'_n is nonexpansive. In this paper, consider the mapping W_n defined by

$$\begin{aligned}
U_{n,n+1} &= I,\\
U_{n,n} &= t_n S'_n U_{n,n+1} + (1 - t_n)I,\\
U_{n,n-1} &= t_{n-1} S'_{n-1} U_{n,n} + (1 - t_{n-1})I,\\
&\ldots,\\
U_{n,i} &= t_i S'_i U_{n,i+1} + (1 - t_i)I,\\
&\ldots,\\
U_{n,2} &= t_2 S'_2 U_{n,3} + (1 - t_2)I,\\
W_n = U_{n,1} &= t_1 S'_1 U_{n,2} + (1 - t_1)I,
\end{aligned} \tag{2.5}$$

where $t_1, t_2, \ldots$ are real numbers such that $0 \leq t_n < 1$. Such a mapping W_n is called a W-mapping generated by $S'_1, S'_2, \ldots$ and $t_1, t_2, \ldots$. It is easy to see W_n is nonexpansive.

Lemma 2.6 (see [7]). *Let C be a nonempty closed convex subset of a strictly convex Banach space E, let $S'_1, S'_2, \ldots$ be nonexpansive mappings of C into itself such that $\cap_{i=1}^{\infty} F(S'_i) \neq \emptyset$ and let $t_1, t_2, \ldots$ be* *real numbers such that* $0 < t_i \leq b < 1$, *for every* $i = 1, 2, \ldots$. *Then, for any* $x \in C$ *and* $k \in N$, *the limit* $\lim_{n\to\infty} U_{n,k}x$ *exists.*

Using Lemma 2.6, one can define the mapping W *of* C *into itself as follows:*

$$Wx := \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x, \quad x \in C. \tag{2.6}$$

Lemma 2.7 (see [7]). *Let* C *be a nonempty closed convex subset of a strictly convex Banach space* E. *Let* $S_1', S_2', \ldots$ *be nonexpansive mappings of* C *into itself such that* $\cap_{i=1}^{\infty} F(S_i') \neq \emptyset$ *and let* $t_1, t_2, \ldots$ *be real numbers such that* $0 < t_i \leq b < 1$, *for all* $i \geq 1$. *If* K *is any bounded subset of* C, *then*

$$\lim_{n\to\infty} \sup_{x\in K} \|Wx - W_n x\| = 0. \tag{2.7}$$

Lemma 2.8 (see [12]). *Let* C *be a nonempty closed convex subset of a Hilbert space* H, *let* $\{S_i' : C \to C\}$ *be a family of infinite nonexpansive mappings with* $\cap_{i=1}^{\infty} F(S_i') \neq \emptyset$, *let* $t_1, t_2, \ldots$ *be real numbers such that* $0 < t_i \leq b < 1$, *for every* $i = 1, 2, \ldots$. *Then* $F(W) = \cap_{i=1}^{\infty} F(S_i')$.

For solving the equilibrium problem, assume that the bifunction F *satisfies the following conditions:*

(A1) $F(x, x) = 0$ *for all* $x \in C$;

(A2) F *is monotone, that is,* $F(x, y) + F(y, x) \leq 0$ *for any* $x, y \in C$;

(A3) *for each* $x, y, z \in C$, $\limsup_{t\to 0} F(tz + (1-t)x, y) \leq F(x, y)$;

(A4) $F(x, \cdot)$ *is convex and lower semicontinuous for each* $x \in C$.

Recall some lemmas which will be needed in the rest of this paper.

Lemma 2.9 (see [13]). *Let* C *be a nonempty closed convex subset of* H, *let* F *be bifunction from* $C \times C$ *to* $\mathbb{R}$ *satisfying (A1)–(A4), and let* $r > 0$ *and* $x \in H$. *Then, there exists* $z \in C$ *such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C. \tag{2.8}$$

Lemma 2.10 (see [5]). *For* $r > 0$, $x \in H$, *define a mapping* $T_r : H \to C$ *as follows:*

$$T_r(x) = \left\{ z \in C \mid F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \forall y \in C \right\} \tag{2.9}$$

for all $x \in H$. *Then, the following statements hold:*

(i) T_r *is single-valued;*

(ii) T_r *is firmly nonexpansive, that is, for any* $x, y \in H$,

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle; \tag{2.10}$$

(iii) $F(T_r) = EP(F)$;

(iv) $EP(F)$ *is closed and convex.*

Lemma 2.11 (see [14]). *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space and let $\{\beta_n\}$ be a sequence of real numbers such that $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$ for all $n = 0, 1, 2, \ldots$. Suppose that $x_{n+1} = (1-\beta_n)z_n + \beta_n x_n$ for all $n = 0, 1, 2, \ldots$ and $\limsup_{n\to\infty}\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \leq 0$. Then $\lim_{n\to\infty}\|z_n - x_n\| = 0$.*

Lemma 2.12 (see [4]). *Let C, H, F, and $T_r x$ be as in Lemma 2.10. Then, the following holds:*

$$\|T_s x - T_t x\|^2 \leq \frac{s-t}{s}\langle T_s x - T_t x, T_s x - x\rangle \tag{2.11}$$

for all $s, t > 0$ and $x \in H$.

Lemma 2.13 (see [10]). *Let H be a Hilbert space and let C be a nonempty closed convex subset of H, and $T : C \to C$ a nonexpansive mapping with $F(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C weakly converging to x and if $\{(I - T)x_n\}$ converges strongly to y, then $(I - T)x = y$.*

3. Main Result

Throughout the rest of this paper, we always assume that f is a contraction of H into itself with coefficient $\alpha \in (0, 1)$, and A is a L-Lipschitzian continuous operator and η-strongly monotone on H with $L > 0$, $\eta > 0$. Assume that $0 < \mu < 2\eta/L^2$ and $0 < \gamma < \mu(\eta - (\mu L^2/2))/\alpha = \tau/\alpha$.

Define a mapping $V_n = \beta_n I + (1 - \beta_n)W_n T_{r_n}$. Since both W_n and T_{r_n} are nonexpansive, it is easy to get V_n is also nonexpansive. Consider the following mapping G_n on H defined by

$$G_n x = \alpha_n \gamma f(x) + (I - \alpha_n \mu A)V_n x, \quad \forall x \in H,\ n \in N, \tag{3.1}$$

where $\alpha_n \in (0, 1)$. By Lemmas 2.3 and 2.10, we have

$$\begin{aligned}\|G_n x - G_n y\| &\leq \alpha_n \gamma \|f(x) - f(y)\| + (1 - \alpha_n \tau)\|V_n x - V_n y\| \\ &\leq \alpha_n \gamma \alpha \|x - y\| + (1 - \alpha_n \tau)\|x - y\| \\ &= (1 - \alpha_n(\tau - \gamma\alpha))\|x - y\|.\end{aligned} \tag{3.2}$$

Since $0 < 1 - \alpha_n(\tau - \gamma\alpha) < 1$, it follows that G_n is a contraction. Therefore, by the Banach contraction principle, G_n has a unique fixed pointed $x_n^f \in H$ such that

$$x_n^f = \alpha_n \gamma f\left(x_n^f\right) + (I - \alpha_n \mu A)V_n x_n^f. \tag{3.3}$$

For simplicity, we will write x_n for x_n^f provided no confusion occurs. Next we prove the sequences $\{x_n\}$ converges strongly to a $x^* \in \Omega = \cap_{i=1}^{\infty} F(S_i) \cap EP(F)$ which solves the variational inequality:

$$\langle (\gamma f - \mu A)x^*, p - x^*\rangle \leq 0, \quad \forall p \in \Omega. \tag{3.4}$$

Equivalently, $x^* = P_\Omega(I - \mu A + \gamma f)x^*$.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H and F a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4). Let $S_i : C \to C$ be a family κ_i-strict pseudocontractions for some $0 \le \kappa_i < 1$. Assume the set $\Omega = \cap_{i=1}^{\infty} F(S_i) \cap EP(F) \neq \emptyset$. Let f be a contraction of H into itself with $\alpha \in (0,1)$ and let A be a L-Lipschitzian continuous operator and η-strongly monotone with $L > 0$, $\eta > 0$, $0 < \mu < 2\eta/L^2$ and $0 < \gamma < \mu(\eta - (\mu L^2/2))/\alpha = \tau/\alpha$. For every $n \in \mathbb{N}$, let W_n be the mapping generated by S_i' and t_i as in* (2.5). *Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by the following algorithm:*

$$\begin{aligned} u_n &= T_{r_n} x_n, \\ y_n &= \beta_n x_n + (1-\beta_n) W_n u_n, \\ x_n &= \alpha_n \gamma f(x_n) + (I - \mu \alpha_n A) y_n. \end{aligned} \tag{3.5}$$

If $\{\alpha_n\}$, $\{\beta_n\}$, and $\{r_n\}$ satisfy the following conditions:

(i) $\{\alpha_n\} \subset (0,1)$, $\lim_{n\to\infty} \alpha_n = 0$;

(ii) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$;

(iii) $\{r_n\} \subset (0,\infty)$, $\liminf_{n\to\infty} r_n > 0$.

Then, $\{x_n\}$ converges strongly to a point $x^* \in \Omega$, which solves the variational inequality (3.4).

Proof. The proof is divided into several steps.

Step 1. Show first that $\{x_n\}$ is bounded.

Take any $p \in \Omega$, by (3.5) and Lemma 2.3, we derive that

$$\begin{aligned} \|x_n - p\| &= \|\alpha_n(\gamma f(x_n) - \mu A p) + (I - \mu\alpha_n A) y_n - (I - \mu\alpha_n A) p\| \\ &\le \alpha_n \alpha \gamma \|x_n - p\| + \alpha_n \|\gamma f(p) - \mu A p\| + (1 - \alpha_n \tau)\|y_n - p\| \\ &\le (1 - \alpha_n(\tau - \gamma\alpha))\|x_n - p\| + \alpha_n \|\gamma f(p) - \mu A p\|. \end{aligned} \tag{3.6}$$

It follows that $\|x_n - p\| \le (\|\gamma f(p) - \mu A p\|)/(\tau - \gamma\alpha)$.

Hence, $\{x_n\}$ is bounded, so are $\{u_n\}$ and $\{y_n\}$. It follows from the Lipschitz continuity of A that $\{A x_n\}$ and $\{A u_n\}$ are also bounded. From the nonexpansivity of f and W_n, it follows that $\{f(x_n)\}$ and $\{W_n x_n\}$ are also bounded.

Step 2. Show that

$$\lim_{n\to\infty} \|u_n - x_n\| = 0, \qquad \lim_{n\to\infty} \|u_n - y_n\| = 0. \tag{3.7}$$

Notice that

$$\|u_n - y_n\| \le \|u_n - x_n\| + \|x_n - y_n\| = \|u_n - x_n\| + \alpha_n \|\gamma f(x_n) - \mu A y_n\|. \tag{3.8}$$

By Lemma 2.10, we have

$$\|u_n - p\|^2 = \|T_{r_n}x_n - T_{r_n}p\|^2 \leq \langle x_n - p, u_n - p\rangle = \frac{1}{2}\left(\|u_n - p\|^2 + \|x_n - p\|^2 - \|x_n - u_n\|^2\right). \tag{3.9}$$

It follows that

$$\|u_n - p\|^2 \leq \|x_n - p\|^2 - \|x_n - u_n\|^2. \tag{3.10}$$

Thus, from Lemma 2.1 and (3.10), we get

$$\begin{aligned}
&\|x_n - p\|^2 \\
&\quad = \|\alpha_n(\gamma f(x_n) - \mu A p) + (I - \mu\alpha_n A)y_n - (I - \mu\alpha_n A)p\|^2 \\
&\quad \leq (1 - \alpha_n\tau)^2\|y_n - p\|^2 + 2\alpha_n\langle \gamma f(x_n) - \gamma f(p) + \gamma f(p) - \mu A p, x_n - p\rangle \\
&\quad \leq (1 - \alpha_n\tau)^2\|u_n - p\|^2 + 2\alpha_n\langle \gamma f(x_n) - \gamma f(p) + \gamma f(p) - \mu A p, x_n - p\rangle \\
&\quad \leq (1 - \alpha_n\tau)^2\left(\|x_n - p\|^2 - \|x_n - u_n\|^2\right) + 2\alpha_n\gamma\alpha\|x_n - p\|^2 + 2\alpha_n\|\gamma f(p) - \mu A p\|\|x_n - p\| \\
&\quad = \left(1 - 2\alpha_n(\tau - \gamma\alpha) + (\alpha_n\tau)^2\right)\|x_n - p\|^2 - (1 - \alpha_n\tau)^2\|x_n - u_n\|^2 + 2\alpha_n\|\gamma f(p) - \mu A p\|\|x_n - p\| \\
&\quad \leq \|x_n - p\|^2 + (\alpha_n\tau)^2\|x_n - p\|^2 - (1 - \alpha_n\tau)^2\|x_n - u_n\|^2 + 2\alpha_n\|\gamma f(p) - \mu A p\|\|x_n - p\|.
\end{aligned} \tag{3.11}$$

It follows that

$$(1 - \alpha_n\tau)^2\|x_n - u_n\|^2 \leq (\alpha_n\tau)^2\|x_n - p\|^2 + 2\alpha_n\|\gamma f(p) - \mu A p\|\|x_n - p\|. \tag{3.12}$$

Since $\alpha_n \to 0$, we have

$$\lim_{n\to\infty}\|u_n - x_n\| = 0. \tag{3.13}$$

From (3.8), it is easy to get

$$\lim_{n\to\infty}\|u_n - y_n\| = 0. \tag{3.14}$$

Step 3. Show that

$$\lim_{n\to\infty}\|u_n - W u_n\| = 0, \tag{3.15}$$

$$\|u_n - W_n u_n\| \leq \|u_n - y_n\| + \|y_n - W_n u_n\| = \|u_n - y_n\| + \beta_n(\|x_n - u_n\| + \|u_n - W_n u_n\|). \tag{3.16}$$

This implies that

$$(1-\beta_n)\|u_n - W_n u_n\| \leq \|u_n - y_n\| + \beta_n \|x_n - u_n\|. \tag{3.17}$$

From condition (ii), (3.13), and (3.14), we have

$$\|u_n - W_n u_n\| \longrightarrow 0. \tag{3.18}$$

Notice that

$$\|u_n - W u_n\| \leq \|u_n - W_n u_n\| + \|W_n u_n - W u_n\|. \tag{3.19}$$

By Lemma 2.7 and (3.18), we get (3.15).

Since $\{u_n\}$ is bounded, so there exists a subsequence $\{u_{n_j}\}$ which converges weakly to x^*.

Step 4. Show that $x^* \in \Omega$.

Since C is closed and convex, C is weakly closed. So, we have $x^* \in C$.

From (3.15), we obtain $W u_{n_j} \rightharpoonup x^*$. From Lemmas 2.8, 2.4, and 2.13, we have $x^* \in F(W) = \cap_{i=1}^{\infty} F(S_i') = \cap_{i=1}^{\infty} F(S_i)$.

By $u_n = T_{r_n} x_n$, for all $n \geq 1$, we have

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C. \tag{3.20}$$

It follows from (A2) that

$$\frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq F(y, u_n), \quad \forall y \in C. \tag{3.21}$$

Hence, we get

$$\frac{1}{r_{n_j}}\left\langle y - u_{n_j}, u_{n_j} - x_{n_j} \right\rangle \geq F\left(y, u_{n_j}\right), \quad \forall y \in C. \tag{3.22}$$

It follows from condition (iii), (3.13), and (A4) that

$$0 \geq F(y, x^*), \quad \forall \mathrm{y} \in C. \tag{3.23}$$

For s with $0 < s \leq 1$ and $y \in C$, let $y_s = sy + (1-s)x^*$. Since $y \in C$ and $x^* \in C$, we obtain $y_s \in C$ and hence $F(y_s, x^*) \leq 0$. So, we have

$$0 = f(y_s, y_s) \leq sF(y_s, y) + (1-s)F(y_s, x^*) \leq sF(y_s, y). \tag{3.24}$$

Dividing by s, we get

$$F(y_s, y) \geq 0, \quad \forall y \in C. \tag{3.25}$$

Letting $s \to 0$ and from (A3), we get

$$F(x^*, y) \geq 0 \tag{3.26}$$

for all $y \in C$ and $x^* \in \mathrm{EP}(F)$. Hence $x^* \in \Omega$.

Step 5. Show that $x_n \to x^*$, where $x^* = P_\Omega(I - \mu A + \gamma f)x^*$:

$$x_n - x^* = \alpha_n(\gamma f(x_n) - \mu A x^*) + (I - \mu\alpha_n A)y_n - (I - \mu\alpha_n A)x^*. \tag{3.27}$$

Hence, we obtain

$$\begin{aligned}\|x_n - x^*\|^2 &= \alpha_n\langle \gamma f(x_n) - \mu A x^*, x_n - x^*\rangle + \langle (I - \mu\alpha_n A)y_n - (I - \mu\alpha_n A)x^*, x_n - x^*\rangle; \\ &\leq \alpha_n\langle \gamma f(x_n) - \mu A x^*, x_n - x^*\rangle + (1 - \alpha_n\tau)\|x_n - x^*\|^2.\end{aligned} \tag{3.28}$$

It follows that

$$\begin{aligned}\|x_n - x^*\|^2 &\leq \frac{1}{\tau}\langle \gamma f(x_n) - \mu A x^*, x_n - x^*\rangle \\ &= \frac{1}{\tau}(\gamma\langle f(x_n) - f(x^*), x_n - x^*\rangle + \langle \gamma f(x^*) - \mu A x^*, x_n - x^*\rangle) \\ &\leq \frac{1}{\tau}\Big(\gamma\alpha\|x_n - x^*\|^2 + \langle \gamma f(x^*) - \mu A x^*, x_n - x^*\rangle\Big).\end{aligned} \tag{3.29}$$

This implies that

$$\|x_n - x^*\|^2 \leq \frac{\langle \gamma f(x^*) - \mu A x^*, x_n - x^*\rangle}{\tau - \gamma\alpha}. \tag{3.30}$$

In particular,

$$\left\|x_{n_j} - x^*\right\|^2 \leq \frac{\left\langle \gamma f(x^*) - \mu A x^*, x_{n_j} - x^*\right\rangle}{\tau - \gamma\alpha}. \tag{3.31}$$

Since $x_{n_j} \rightharpoonup x^*$, it follows from (3.31) that $x_{n_j} \to x^*$ as $j \to \infty$. Next, we show that x^* solves the variational inequality (3.4).

By the iterative algorithm (3.5), we have

$$x_n = \alpha_n\gamma f(x_n) + (I - \mu\alpha_n A)y_n = \alpha_n\gamma f(x_n) + (I - \mu\alpha_n A)V_n x_n. \tag{3.32}$$

Therefore, we have

$$\mu\alpha_n A x_n - \alpha_n \gamma f(x_n) = \mu\alpha_n A x_n - x_n + (I - \mu\alpha_n A) V_n x_n, \tag{3.33}$$

that is,

$$(\mu A - \gamma f) x_n = -\frac{1}{\alpha_n}\big((I - V_n) x_n - \mu\alpha_n (A x_n - A V_n) x_n\big). \tag{3.34}$$

Hence, for $p \in \Omega$,

$$\begin{aligned}\langle(\mu A - \gamma f) x_n, x_n - p\rangle &= -\frac{1}{\alpha_n}\langle(I - V_n) x_n - \mu\alpha_n (A x_n - A V_n x_n), x_n - p\rangle \\ &= -\frac{1}{\alpha_n}\langle(I - V_n) x_n - (I - V_n) p, x_n - p\rangle + \mu\langle A x_n - A V_n x_n, x_n - p\rangle \\ &\leq \mu\langle A x_n - A V_n x_n, x_n - p\rangle.\end{aligned} \tag{3.35}$$

Since $I - V_n$ is monotone (i.e., $\langle x - y, (I - V_n) x - (I - V_n) y\rangle \geq 0$, for all $x, y \in H$). This is due to the nonexpansivity of V_n.

Now replacing n in (3.35) with n_j and letting $j \to \infty$, we obtain

$$\langle(\mu A - \gamma f) x^*, x^* - p\rangle = \lim_{j\to\infty}\left\langle(\mu A - \gamma f) x_{n_j}, x_{n_j} - p\right\rangle \leq \lim_{j\to\infty}\mu\left\langle A x_{n_j} - A V_n x_{n_j}, x_{n_j} - p\right\rangle = 0. \tag{3.36}$$

That is, $x^* \in \Omega$ is a solution of (3.4). To show that the sequence $\{x_n\}$ converges strongly to x^*, we assume that $x_{n_k} \to \widehat{x}$. By the same processing as the proof above, we derive $\widehat{x} \in \Omega$. Moreover, it follows from the inequality (3.36) that

$$\langle(\mu A - \gamma f) x^*, x^* - \widehat{x}\rangle \leq 0. \tag{3.37}$$

Interchanging x^* and $\widehat{x}$, we get

$$\langle(\mu A - \gamma f)\widehat{x}, \widehat{x} - x^*\rangle \leq 0. \tag{3.38}$$

By Lemma 2.5, adding up (3.37) and (3.38) yields

$$(\mu\eta - \gamma\alpha)\|x^* - \widehat{x}\|^2 \leq \langle(\mu A - \gamma f) x^* - (\mu A - \gamma f)\widehat{x}, x^* - \widehat{x}\rangle \leq 0. \tag{3.39}$$

Hence $x^* = \widehat{x}$ and, therefore, $x_n \to x^*$ as $n \to \infty$,

$$\langle(I - \mu A + \gamma f) x^* - x^*, x^* - p\rangle \geq 0, \quad \forall p \in \Omega. \tag{3.40}$$

This is equivalent to the fixed point equation:

$$P_{\Omega}(I - \mu A + \gamma f)x^* = x^*. \tag{3.41}$$

□

Theorem 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H and F a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4). Let $S_i : C \to C$ be a family κ_i-strict pseudocontractions for some $0 \le \kappa_i < 1$. Assume the set $\Omega = \cap_{i=1}^{\infty} F(S_i) \cap EP(F) \neq \emptyset$. Let f be a contraction of H into itself with $\alpha \in (0,1)$ and let A be a L-Lipschitzian continuous operator and η-strongly monotone with $L > 0$, $\eta > 0, 0 < \mu < 2\eta/L^2$, and $0 < \gamma < \mu(\eta - (\mu L^2/2))/\alpha = \tau/\alpha$. For every $n \in \mathbb{N}$, let W_n be the mapping generated by S_i' and $0 < t_i \le b < 1$. Given $x_1 \in H$, let $\{x_n\}$ and $\{u_n\}$ be sequences generated by the following algorithm:*

$$\begin{aligned} u_n &= T_{r_n} x_n, \\ y_n &= \beta_n x_n + (1 - \beta_n) W_n u_n, \\ x_{n+1} &= \alpha_n \gamma f(x_n) + (I - \mu \alpha_n A) y_n. \end{aligned} \tag{3.42}$$

If $\{\alpha_n\}$, $\{\beta_n\}$ and $\{r_n\}$ satisfy the following conditions:

(i) $\{\alpha_n\} \subset (0,1)$, $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$;

(iii) $\{r_n\} \subset (0,\infty)$, $\liminf_{n\to\infty} r_n > 0$ *and* $\lim_{n\to\infty}|r_{n+1} - r_n| = 0$.

Then, $\{x_n\}$ converges strongly to $x^* \in \Omega$, which solves the variational inequality (3.4).

Proof. The proof is divided into several steps.

Step 1. Show first that $\{x_n\}$ is bounded.
Taking any $p \in \Omega$, we have

$$\begin{aligned} \|x_{n+1} - p\| &= \|\alpha_n(\gamma f(x_n) - \mu A p) + (I - \mu\alpha_n A) y_n - (I - \mu\alpha_n A) p\| \\ &\le \alpha_n(\|\gamma f(x_n) - \gamma f(p)\| + \|\gamma f(p) - \mu A p\|) + (1 - \alpha_n \tau)\|y_n - p\| \\ &\le \alpha_n \alpha\gamma \|x_n - p\| + \alpha_n \|\gamma f(p) - \mu A p\| + (1 - \alpha_n\tau)\|y_n - p\| \\ &= (1 - \alpha_n(\tau - \alpha\gamma))\|x_n - p\| + \alpha_n(\tau - \alpha\gamma)\frac{\|\gamma f(p) - \mu A p\|}{\tau - \alpha\gamma} \\ &\le \max\left\{\|x_n - p\|, \frac{\|\gamma f(p) - \mu A p\|}{\tau - \alpha\gamma}\right\}. \end{aligned} \tag{3.43}$$

By induction, we obtain $\|x_n - p\| \le \max\{\|x_1 - p\|, \|\gamma f(p) - \mu A p)\|/(\tau - \alpha\gamma)\}$, $n \ge 1$. Hence, $\{x_n\}$ is bounded, so are $\{u_n\}$ and $\{y_n\}$. It follows from the Lipschitz continuity of A that $\{Ax_n\}$ and $\{Au_n\}$ are also bounded. From the nonexpansivity of f and W_n, it follows that $\{f(x_n)\}$ and $\{W_n x_n\}$ are also bounded.

Step 2. Show that

$$\|x_{n+1} - x_n\| \longrightarrow 0. \tag{3.44}$$

Observe that

$$\begin{aligned}\|u_{n+1} - u_n\| &= \|T_{r_{n+1}} x_{n+1} - T_{r_n} x_n\| \\ &\leq \|T_{r_{n+1}} x_{n+1} - T_{r_{n+1}} x_n\| + \|T_{r_{n+1}} x_n - T_{r_n} x_n\| \\ &\leq \|x_{n+1} - x_n\| + \|T_{r_{n+1}} x_n - T_{r_n} x_n\|,\end{aligned} \tag{3.45}$$

and from (2.5), we have

$$\begin{aligned}\|W_{n+1}u_n - W_n u_n\| &= \left\|t_1 S_1' U_{n+1,2} u_n - t_1 S_1' U_{n,2} u_n\right\| \\ &\leq t_1 \|U_{n+1,2} u_n - U_{n,2} u_n\| \\ &= t_1 \left\|t_2 S_2' U_{n+1,3} u_n - t_2 S_2' U_{n,3} u_n\right\| \\ &\leq t_1 t_2 \|U_{n+1,3} u_n - U_{n,3} u_n\| \\ &\leq \cdots \\ &\leq \prod_{i=1}^{n} t_i \|U_{n+1,n+1} u_n - U_{n,n+1} u_n\| \\ &\leq M_1 \prod_{i=1}^{n} t_i,\end{aligned} \tag{3.46}$$

where $M_1 = \sup_n \{\|U_{n+1,n+1} u_n - U_{n,n+1} u_n\|\}$.

Suppose $x_{n+1} = \beta_n x_n + (1 - \beta_n) z_n$, then $z_n = (x_{n+1} - \beta_n x_n)/(1 - \beta_n) = (\alpha_n \gamma f(x_n) + (I - \mu\alpha_n A) y_n - \beta_n x_n)/(1 - \beta_n)$.

Hence, we have

$$\begin{aligned}z_{n+1} - z_n &= \frac{\alpha_{n+1}\gamma f(x_{n+1}) + (I - \mu\alpha_{n+1}A) y_{n+1} - \beta_{n+1} x_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n \gamma f(x_n) + (I - \mu\alpha_n A) y_n - \beta_n x_n}{1 - \beta_n} \\ &= \frac{\alpha_{n+1}(\gamma f(x_{n+1}) - \mu A y_{n+1})}{1 - \beta_{n+1}} + \frac{y_{n+1} - \beta_{n+1} x_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n(\gamma f(x_n) - \mu A y_n)}{1 - \beta_n} - \frac{y_n - \beta_n x_n}{1 - \beta_n} \\ &= \frac{\alpha_{n+1}(\gamma f(x_{n+1}) - \mu A y_{n+1})}{1 - \beta_{n+1}} + \frac{\beta_{n+1} x_{n+1} + (1 - \beta_{n+1}) W_{n+1} u_{n+1} - \beta_{n+1} x_{n+1}}{1 - \beta_{n+1}} \\ &\quad - \frac{\alpha_n(\gamma f(x_n) - \mu A y_n)}{1 - \beta_n} - \frac{\beta_n x_n + (1 - \beta_n) W_n u_n - \beta_n x_n}{1 - \beta_n} \\ &\leq \frac{\alpha_{n+1}(\gamma f(x_{n+1}) - \mu A y_{n+1})}{1 - \beta_{n+1}} - \frac{\alpha_n(\gamma f(x_n) - \mu A y_n)}{1 - \beta_n} + W_{n+1} u_{n+1} - W_n u_n.\end{aligned} \tag{3.47}$$

It follows from (3.45), (3.46), and the above result that

$$
\begin{aligned}
&\|z_{n+1} - z_n\| \\
&\quad \le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\left(\|\gamma f(x_{n+1})\| + \|\mu A y_{n+1}\|\right) + \frac{\alpha_n}{1-\beta_n}\left(\|\gamma f(x_n)\| + \|\mu A y_n\|\right) + \|W_{n+1}u_{n+1} - W_n u_n\| \\
&\quad \le \left(\frac{\alpha_{n+1}}{1-\beta_{n+1}} + \frac{\alpha_n}{1-\beta_n}\right)M_2 + \|W_{n+1}u_{n+1} - W_{n+1}u_n\| + \|W_{n+1}u_n - W_n u_n\| \\
&\quad \le \left(\frac{\alpha_{n+1}}{1-\beta_{n+1}} + \frac{\alpha_n}{1-\beta_n}\right)M_2 + \|u_{n+1} - u_n\| + \|W_{n+1}u_n - W_n u_n\| \\
&\quad \le \|x_{n+1} - x_n\| + \|T_{r_{n+1}}x_n - T_{r_n}x_n\| + \left(\frac{\alpha_{n+1}}{1-\beta_{n+1}} + \frac{\alpha_n}{1-\beta_n}\right)M_2 + M_1\prod_{i=1}^{n} t_i,
\end{aligned}
\tag{3.48}
$$

where $M_2 = \sup_n\{\|\gamma f(x_n)\| + \|\mu A y_n\|\}$. Hence, we get

$$
\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \le \|T_{r_{n+1}}x_n - T_{r_n}x_n\| + \left(\frac{\alpha_{n+1}}{1-\beta_{n+1}} + \frac{\alpha_n}{1-\beta_n}\right)M_2 + M_1\prod_{i=1}^{n} t_i. \tag{3.49}
$$

From condition (i), (iii), $0 < t_n \le b < 1$, and Lemma 2.12, we obtain

$$
\limsup_{n\to\infty}\left(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|\right) \le 0. \tag{3.50}
$$

By Lemma 2.11,we have $\lim_{n\to\infty}\|z_n - x_n\| = 0$. Thus,

$$
\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}(1-\beta_n)\|z_n - x_n\| = 0. \tag{3.51}
$$

By Lemma 2.12, (3.45) and (3.44), we obtain

$$
\|u_{n+1} - u_n\| \longrightarrow 0. \tag{3.52}
$$

Step 3. Show that

$$
\|x_n - W x_n\| \longrightarrow 0. \tag{3.53}
$$

Observe that

$$
\begin{aligned}
&\|x_n - W_n x_n\| \le \|x_n - W_n u_n\| + \|W_n u_n - W_n x_n\| \le \|x_n - W_n u_n\| + \|u_n - x_n\|, \\
&\|x_n - W_n u_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| + \|y_n - W_n u_n\| = \|x_n - x_{n+1}\| \\
&\qquad + \|x_{n+1} - y_n\| + \beta_n\left(\|u_n - x_n\| + \|x_n - W_n u_n\|\right).
\end{aligned}
\tag{3.54}
$$

From condition (i) and (3.5),we can obtain

$$(1-\beta_n)\|x_n - W_n u_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| + \beta_n\|u_n - x_n\| \le \|x_n - x_{n+1}\| + \alpha_n\|\gamma f(x_n) - \mu A y_n\| + \beta_n\|u_n - x_n\|. \tag{3.55}$$

By Lemma 2.10, we get

$$\|u_n - p\|^2 = \|T_{r_n}x_n - T_{r_n}p\|^2 \le \langle T_{r_n}x_n - T_{r_n}p, x_n - p\rangle = \frac{1}{2}\left(\|u_n - p\|^2 + \|x_n - p\|^2 + \|x_n - u_n\|^2\right). \tag{3.56}$$

This implies that

$$\|u_n - p\|^2 \le \|x_n - p\|^2 - \|x_n - u_n\|^2. \tag{3.57}$$

By nonexpansivity of W_n, we have

$$\|y_n - p\|^2 \le \beta_n\|x_n - p\|^2 + (1-\beta_n)\|u_n - p\|^2 \le \|x_n - p\|^2 - (1-\beta_n)\|x_n - u_n\|^2. \tag{3.58}$$

It follows from (3.42) that

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|\alpha_n(\gamma f(x_n) - p) + (I - \mu\alpha_n A)y_n - (I - \mu\alpha_n A)p + \alpha_n(p - \mu A p)\|^2 \\ &\le \alpha_n\|\gamma f(x_n) - p\|^2 + (1-\alpha_n\tau)\|y_n - p\|^2 + \alpha_n\|p - \mu A p\|^2 \\ &\le \alpha_n\|\gamma f(x_n) - p\|^2 + (1-\alpha_n\tau)\left(\|x_n - p\|^2 - (1-\beta_n)\|x_n - u_n\|^2\right) + \alpha_n\|p - \mu A p\|^2 \\ &\le \alpha_n\|\gamma f(x_n) - p\|^2 + \|x_n - p\|^2 - (1-\beta_n)\|x_n - u_n\|^2 + \alpha_n\|p - \mu A p\|^2.\end{aligned} \tag{3.59}$$

This implies that

$$\begin{aligned}(1-\beta_n)\|x_n - u_n\|^2 &\le \alpha_n\left(\|\gamma f(x_n) - p\|^2 + \|p - \mu A p\|^2\right) + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\ &\le \alpha_n\left(\|\gamma f(x_n) - p\|^2 + \|p - \mu A p\|^2\right) + (\|x_n - p\| + \|x_{n+1} - p\|)\|x_{n+1} - x_n\|.\end{aligned} \tag{3.60}$$

From condition (i), (ii), and (3.44), we have

$$\|x_n - u_n\| \longrightarrow 0. \tag{3.61}$$

Further we have $\|x_n - W_n u_n\| \to 0$. Thus we get

$$\|x_n - W_n x_n\| \longrightarrow 0. \tag{3.62}$$

On the other hand, we have

$$\|x_n - Wx_n\| \leq \|x_n - W_n x_n\| + \|W_n x_n - Wx_n\| \leq \|x_n - W_n x_n\| + \sup_{x_n \in C} \|W_n x_n - Wx_n\|. \tag{3.63}$$

Combining (3.62), the last inequality, and Lemma 2.7, we obtain (3.53).

Step 4. Show that

$$\limsup_{n \to \infty} \langle (\gamma f - \mu A)x^*, x_n - x^* \rangle \leq 0, \tag{3.64}$$

where $x^* = P_\Omega(I - \mu A + \gamma f)x^*$ is a unique solution of the variational inequality (3.4). Indeed, take a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n \to \infty} \langle (\gamma f - \mu A)x^*, x_n - x^* \rangle = \lim_{j \to \infty} \left\langle (\gamma f - \mu A)x^*, x_{n_j} - x^* \right\rangle. \tag{3.65}$$

Since $\{x_{n_j}\}$ is bounded, there exists a subsequence $\{x_{n_{j_k}}\}$ of $\{x_{n_j}\}$ which converges weakly to q. Without loss of generality, we can assume $x_{n_j} \rightharpoonup q$. From (3.53), we obtain $Wx_{n_j} \rightharpoonup q$.

By the same argument as in the proof of Theorem 3.1, we have $q \in \Omega$. Since $x^* = P_\Omega(I - \mu A + \gamma f)x^*$, it follows that

$$\limsup_{n \to \infty} \langle (\gamma f - \mu A)x^*, x_n - x^* \rangle = \lim_{j \to \infty} \left\langle (\gamma f - \mu A)x^*, x_{n_j} - x^* \right\rangle = \langle (\gamma f - \mu A)x^*, q - x^* \rangle \leq 0. \tag{3.66}$$

Step 5. Show that

$$x_n \longrightarrow x^*. \tag{3.67}$$

Since

$$\begin{aligned} \langle (\gamma f - \mu A)x^*, x_{n+1} - x^* \rangle &= \langle (\gamma f - \mu A)x^*, x_{n+1} - x_n \rangle + \langle (\gamma f - \mu A)x^*, x_n - x^* \rangle \\ &\leq \left\| (\gamma f - \mu A)x^* \right\| \|x_{n+1} - x_n\| + \langle (\gamma \mathrm{f} - \mu A)x^*, x_n - x^* \rangle. \end{aligned} \tag{3.68}$$

It follows from (3.44) and (3.66) that

$$\limsup_{n \to \infty} \langle (\gamma f - \mu A)x^*, x_{n+1} - x^* \rangle \leq 0.$$

$$\begin{aligned} &\|x_{n+1} - x^*\|^2 \\ &= \left\| \alpha_n \gamma f(x_n) + (I - \mu \alpha_n A)y_n - x^* \right\|^2 \\ &= \left\| (I - \mu \alpha_n A)y_n - (I - \mu \alpha_n A)x^* + \alpha_n(\gamma f(x_n) - \mu A x^*) \right\|^2 \end{aligned}$$

$$\leq \left\| (I-\mu\alpha_n A)y_n - (I-\mu\alpha_n A)x^* \right\|^2 + 2\alpha_n\langle \gamma f(x_n) - \mu A x^*, x_{n+1}-x^*\rangle$$
$$\leq (1-\alpha_n\tau)^2\left\| y_n - x^*\right\|^2 + 2\alpha_n\langle \gamma f(x_n) - \gamma f(x^*), x_{n+1}-x^*\rangle + 2\alpha_n\langle (\gamma f - \mu A)x^*, x_{n+1}-x^*\rangle$$
$$\leq (1-\alpha_n\tau)^2\| x_n - x^*\|^2 + \alpha_n\alpha\gamma\left(\| x_n - x^*\|^2 + \| x_{n+1} - x^*\|^2\right) + 2\alpha_n\langle (\gamma f - \mu A)x^*, x_{n+1}-x^*\rangle. \tag{3.69}$$

This implies that

$$\| x_{n+1} - x^*\|^2$$
$$\leq \frac{(1-\alpha_n\tau)^2 + \alpha_n\alpha\gamma}{1-\alpha_n\alpha\gamma}\| x_n - x^*\|^2 + \frac{2\alpha_n}{1-\alpha_n\alpha\gamma}\langle (\gamma f - \mu A)x^*, x_{n+1}-x^*\rangle$$
$$\leq \left(1 - \frac{2\alpha_n(\tau-\alpha\gamma)}{1-\alpha_n\alpha\gamma}\right)\| x_n - x^*\|^2 + \frac{2\alpha_n}{1-\alpha_n\alpha\gamma}\langle (\gamma f - \mu A)x^*, x_{n+1}-x^*\rangle + \frac{(\alpha_n\tau)^2}{1-\alpha_n\alpha\gamma}M_3, \tag{3.70}$$

where $M_3 = \sup_n\| x_n - x^*\|^2,\ n \geq 1$. It is easily to see that $\gamma_n = 2\alpha_n(\tau-\alpha\gamma)/(1-\alpha_n\alpha\gamma)$. Hence, by Lemma 2.2, the sequence $\{x_n\}$ converges strongly to x^*. □

Remark 3.3. If $F \equiv 0$, then Theorem 3.2 reduces to Theorem 3.1 of Wang [10].

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References

[1] F. E. Browder and W. V. Petryshyn, "Construction of fixed points of nonlinear mappings in Hilbert space," *Journal of Mathematical Analysis and Applications*, vol. 20, pp. 197–228, 1967.
[2] Y. Liu, "A general iterative method for equilibrium problems and strict pseudo-contractions in Hilbert spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 10, pp. 4852–4861, 2009.
[3] G. Marino and H.-K. Xu, "Weak and strong convergence theorems for strict pseudo-contractions in Hilbert spaces," *Journal of Mathematical Analysis and Applications*, vol. 329, no. 1, pp. 336–346, 2007.
[4] S. Takahashi and W. Takahashi, "Strong convergence theorem for a generalized equilibrium problem and a nonexpansive mapping in a Hilbert space," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 69, no. 3, pp. 1025–1033, 2008.
[5] P. L. Combettes and S. A. Hirstoaga, "Equilibrium programming in Hilbert spaces," *Journal of Nonlinear and Convex Analysis*, vol. 6, no. 1, pp. 117–136, 2005.
[6] W. R. Mann, "Mean value methods in iteration," *Proceedings of the American Mathematical Society*, vol. 4, pp. 506–510, 1953.
[7] K. Shimoji and W. Takahashi, "Strong convergence to common fixed points of infinite nonexpansive mappings and applications," *Taiwanese Journal of Mathematics*, vol. 5, no. 2, pp. 387–404, 2001.
[8] G. L. Acedo and H.-K. Xu, "Iterative methods for strict pseudo-contractions in Hilbert spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 67, no. 7, pp. 2258–2271, 2007.

[9] M. Tian, "A general iterative algorithm for nonexpansive mappings in Hilbert spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 73, no. 3, pp. 689–694, 2010.

[10] S. Wang, "A general iterative method for obtaining an infinite family of strictly pseudo-contractive mappings in Hilbert spaces," *Applied Mathematics Letters*, vol. 24, no. 6, pp. 901–907, 2011.

[11] H. K. Xu, "An iterative approach to quadratic optimization," *Journal of Optimization Theory and Applications*, vol. 116, no. 3, pp. 659–678, 2003.

[12] S.-S. Chang, H. W. Joseph Lee, and C. K. Chan, "A new method for solving equilibrium problem fixed point problem and variational inequality problem with application to optimization," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 70, no. 9, pp. 3307–3319, 2009.

[13] E. Blum and W. Oettli, "From optimization and variational inequalities to equilibrium problems," *The Mathematics Student*, vol. 63, no. 1–4, pp. 123–145, 1994.

[14] T. Suzuki, "Strong convergence of Krasnoselskii and Mann's type sequences for one-parameter nonexpansive semigroups without Bochner integrals," *Journal of Mathematical Analysis and Applications*, vol. 305, no. 1, pp. 227–239, 2005.