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Research Article

Mathematical Model and Cluster Synchronization for a Complex Dynamical Network with Two Types of Chaotic Oscillators

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We propose a mathematical model of a complex dynamical network consisting of two types of chaotic oscillators and investigate the schemes and corresponding criteria for cluster synchronization. The global asymptotically stable criteria for the linearly or adaptively coupled network are derived to ensure that each group of oscillators is synchronized to the same behavior. The cluster synchronization can be guaranteed by increasing the inner coupling strength in each cluster or enhancing the external excitation. Theoretical analysis and numerical simulation results show that the external excitation is more conducive to the cluster synchronization. All of the results are proved rigorously. Finally, a network with a scale-free subnetwork and a small-world subnetwork is illustrated, and the corresponding numerical simulations verify the theoretical analysis.

1. Introduction

Since the pioneering works by Watts and Strogatz on the small-world network [1] and by Barabási and Albert on the scale-free network [2], complex networks have been studied extensively in various disciplines, such as social, biological, mathematical, and engineering sciences [3]. Besides the properties of "small-world" and "scale-free," another common property in real-world complex networks is cluster (or community, module) structure [4]. Many real networks are composed of several clusters within which the connection of nodes is more than that of nodes between different clusters, or the nodes have some common properties in a cluster. This feature can be seen in many networks such as social networks [5], biological networks [6], and citation networks [7].

Synchronization of coupled chaotic oscillators is one of commonly collective coherent behaviors attracting a growing interest in physics, biology, communication, and other fields of science and technology. Synchronization of complex networks has attracted tremendous attention in recent years. Different synchronization phenomena in complex networks have been studied, such as global synchronization [8, 9], partial synchronization and cluster synchronization [10–15]. In particular, the cluster synchronization, which implies that nodes in the same group achieve the same synchronization state while nodes in different groups achieve different synchronization state, is considered to be significant in biological science [6], laser technology, and communication engineering [11].

More recently, some new progress in cluster synchronization of complex dynamical networks have been reported [16–18]. In [16], Belykh et al. have studied the cluster synchronization for conditional clusters and unconditional clusters in an oscillator network with given configuration based on a graph theoretical approach, and the corresponding existence and stability conditions are proposed. In [17], authors presented a linear feedback control strategy to achieve the cluster synchronization for a network with identical oscillators. In [18] authors have investigated the cluster synchronization in a dynamical network consisting of two groups of nonidentical oscillators, and upper bounds of input strength for the synchrony of each cluster are derived under the "same-input" condition.

In this paper, based on [18], we further investigate the cluster synchronization of a complex network containing two groups of different oscillators. But different from [18], our research mainly focuses on the inner coupling strength and the external excitation intensity to ensure the cluster synchronization. Intuitively, the inner coupling of clusters is beneficial to the cluster synchronization. Yet our study shows that the external excitation intensity is more conducive to the cluster synchronization under the "same-input" condition. Theoretical analysis and numerical simulation results show that, even without any connection within a cluster, the cluster synchronization can be still achieved by enhancing the external excitation.

The rest of the paper is organized as follows. In Section 2, the mathematical model of our research network is proposed and preliminaries are introduced. The linear coupling criteria for the cluster synchronization are derived in Section 3. The adaptive inner coupling and external excitation schemes and corresponding conditions for the cluster synchronization are presented in Section 4. The numerical simulations are provided to verify the effectiveness of the theoretical analysis in Section 5. Finally, a brief summary of the obtained results is given in Section 6.

2. Mathematical Model and Preliminaries

Let's consider a dynamical network with two clusters, each cluster contains a number of identical dynamical systems, however, the subsystems composing the two clusters can be different, that is, the individual dynamical system in one cluster can differ from that in the other cluster. Suppose that two clusters are composed of N_1 and N_2 nodes which are n_1 and n_2 dimensional dynamical, oscillator as $\dot{x} = f(x)$, $\dot{y} = g(y)$, respectively. We call each cluster as x-cluster and y-cluster respectively. The two-cluster-network is described by

$$
\begin{aligned}
\dot{x}_i &= f(x_i) + c_1 \sum_{k=1, k\neq i}^{N_1} a_{ik}(x_k - x_i) + d_1 \sum_{k=1}^{N_2} b_{ik}(\Gamma_1 y_k - x_i), \quad i = 1,2,\ldots,N_1, \\
\dot{y}_j &= g(y_j) + c_2 \sum_{k=1, k\neq j}^{N_2} \overline{a}_{jk}(y_k - y_j) + d_2 \sum_{k=1}^{N_1} \overline{b}_{jk}(\Gamma_2 x_k - y_j), \quad j = 1,2,\ldots,N_2.
\end{aligned}
\tag{2.1}
$$

Equivalently, (2.1) can be rewritten as

$$\dot{x}_i = f(x_i) + c_1 \sum_{k=1}^{N_1} a_{ik} x_k + d_1 \sum_{k=1}^{N_2} b_{ik} \Gamma_1 y_k - d_1 \sum_{k=1}^{N_2} b_{ik} x_i, \quad i = 1, 2, \dots, N_1,$$
$$\dot{y}_j = g(y_j) + c_2 \sum_{k=1}^{N_2} \overline{a}_{jk} y_k + d_2 \sum_{k=1}^{N_1} \overline{b}_{jk} \Gamma_2 x_k - d_2 \sum_{k=1}^{N_1} \overline{b}_{jk} y_j, \quad j = 1, 2, \dots, N_2, \tag{2.1$'$}$$

where $x_i = (x_{i1}, x_{i2}, \dots, x_{in_1})^T \in R^{n_1}$ and $y_j = (y_{j1}, y_{j2}, \dots, y_{jn_2})^T \in R^{n_1}$ are state vectors of the nodes. $f : \Omega_1 \to R^{n_1}$ ($\Omega_1 \subseteq R^{n_1}$) and $g : \Omega_2 \to R^{n_2}$ ($\Omega_2 \subseteq R^{n_2}$) are smooth nonlinear function vectors. $c_1, c_2 > 0$ are the inner coupling strength of each cluster, $d_1, d_2 > 0$ are the external excitation intensity acting on each cluster. $\Gamma_1 \in R^{n_1 \times n_2}$, $\Gamma_2 \in R^{n_2 \times n_1}$ are internal coupling matrices when the corresponding oscillators used to output, and have form $[I, 0]$ or $[I, 0]^T$ (where I denotes identity matrix, 0 is a proper dimension zero matrix). Obviously, there has $\Gamma_1 = {\Gamma_2}^T$. The matrix $A_1 = (a_{ij}) \in R^{N_1 \times N_1}$ is a diffusive symmetric irreducible matrix which represents the internal connection in x-cluster. If node i and node j are connected in x-cluster, then $a_{ij} = a_{ji} = 1$, otherwise $a_{ij} = a_{ji} = 0$. The diagonal elements of A_1 are $a_{ii} = -\sum_{j=1}^{N_1} a_{ij}$ ($i = 1, 2, \dots, N_1$). With these assumptions, the eigenvalues of matrix A_1 can be given by $0 = \lambda_1 > \lambda_2 \geq \cdots \geq \lambda_{N_1}$. The matrix $A_2 = (\overline{a}_{ij}) \in R^{N_2 \times N_2}$ which represents the internal connection in y-cluster is the same as A_1, and its eigenvalues can be given by $0 = \mu_1 > \mu_2 \geq \cdots \geq \mu_{N_2}$. The matrices $B_1 = (b_{ij}) \in R^{N_1 \times N_2}$ and $B_2 = (\overline{b}_{ij}) \in R^{N_2 \times N_1}$ represent external connection between two clusters which satisfy "same-input" condition (see Definition 2.1), and all the elements b_{ij} and $\overline{b}_{ij}$ take 0 or 1.

Definition 2.1. A matrix $C = (c_{ij})_{m \times n}$ is said to satisfy condition SI, if its elements satisfy

$$c_{ik} = c_{jk}, \quad i, j = 1, \dots, m, \; k = 1, \dots, n. \tag{2.2}$$

Moreover, if the external input matrices B_1 and B_2 satisfy the condition SI, then the network (2.1) is said to satisfy the "same-input" condition.

Note that, we suppose that all network models throughout this paper satisfy "same-input" condition. From this condition, we have $b_{ik} = b_{1k}$, $\sum_{k=1}^{N_2} b_{ik} = b$ ($i = 1, 2, \dots, N_1$), and $\overline{b}_{ik} = \overline{b}_{1k}$, $\sum_{k=1}^{N_2} \overline{b}_{ik} = \overline{b}$ ($i = 1, 2, \dots, N_2$). The positive constants b and $\overline{b}$ are regarded as the external excitation acting on each cluster. It implies that the input of nodes in the same cluster is equal.

Denote $s_1 = (1/N_1) \sum_{i=1}^{N_1} x_i$, $s_2 = (1/N_2) \sum_{j=1}^{N_2} y_j$ and $\overline{f}(s_1) = (1/N_1) \sum_{i=1}^{N_1} f(x_i)$, $\overline{g}(s_2) = (1/N_2) \sum_{j=1}^{N_2} g(y_j)$.

Then, we have

$$\dot{s}_1 = \overline{f}(s_1) + d_1 \sum_{k=1}^{N_2} b_{1k} \Gamma_1 y_k - d_1 b s_1, \qquad \dot{s}_2 = \overline{g}(s_2) + d_2 \sum_{k=1}^{N_1} \overline{b}_{1k} \Gamma_2 x_k - d_2 \overline{b} s_2. \tag{2.3}$$

Definition 2.2. The set

$$S = \left\{ \left(x_1^T, \dots, x_{N_1}^T, y_1^T, \dots, y_{N_2}^T \right)^T : x_i = s_1,\ y_j = s_2,\ i = 1, \dots, N_1,\ j = 1, \dots, N_2 \right\} \tag{2.4}$$

is called cluster synchronous manifold of the network (2.1).

Therefore, the synchronous errors are denoted as $\tilde{x}_i = x_i - s_1$ $(i = 1, \dots, N_1)$ for the x-cluster and $\tilde{y}_j = y_j - s_2$ $(j = 1, \dots, N_2)$ for the y-cluster. Denote $\tilde{X} = (\tilde{x}_1^T, \dots, \tilde{x}_{N_1}^T)^T$, $\tilde{Y} = (\tilde{y}_1^T, \dots, \tilde{y}_{N_2}^T)^T$. Then the error equations are given by

$$\begin{aligned} \dot{\tilde{x}}_i &= f(x_i) - \overline{f}(s_1) + c_1 \sum_{k=1}^{N_1} a_{ik} \tilde{x}_k - d_1 b \tilde{x}_i, \quad i = 1, 2, \dots, N_1, \\ \dot{\tilde{y}}_i &= g(y_j) - \overline{g}(s_2) + c_2 \sum_{k=1}^{N_2} \overline{a}_{jk} \tilde{y}_k - d_2 \overline{b} \tilde{y}_j, \quad j = 1, 2, \dots, N_2. \end{aligned} \tag{2.5}$$

Obviously, the stability problem of cluster synchronous manifold S in the network (2.1) is equivalent to the stability of system (2.5) at zero. Our objective is to find the criteria for the coupling strength such that the network (2.1) achieves cluster synchronization, that is, $\lim_{t \to +\infty} \|\tilde{x}_i\| = 0$ $(i = 1, 2, \dots, N_1)$ and $\lim_{t \to +\infty} \|\tilde{y}_j\| = 0$ $(j = 1, 2, \dots, N_2)$, which implies that the x-cluster and y-cluster achieve synchronization, respectively.

In order to achieve cluster synchronization, a useful assumption and a lemma are introduced as follows.

Assumption 2.3. $(A1)$ Suppose that there exists positive constants δ_f and δ_g such that

$$\| f(\mathbf{z}_1) - f(\mathbf{z}_2) \| \le \delta_f \| \mathbf{z}_1 - \mathbf{z}_2 \|, \qquad \| g(\mathbf{z}_1) - g(\mathbf{z}_2) \| \le \delta_g \| \mathbf{z}_1 - \mathbf{z}_2 \|, \tag{2.6}$$

where $\mathbf{z}_1, \mathbf{z}_2$ are time-varying vectors.

Note that we assume that $(A1)$ is satisfied by all models in this paper.

Lemma 2.4. *For the above matrices A_1 and A_2, one has*

$$\tilde{X}^T (A_1 \otimes I_{n_1}) \tilde{X} \le \lambda_2 \sum_{i=1}^{N_1} \tilde{x}_i^T \tilde{x}_i, \qquad \tilde{Y}^T (A_2 \otimes I_{n_2}) \tilde{Y} \le \mu_2 \sum_{j=1}^{N_2} \tilde{y}_j^T \tilde{y}_j. \tag{2.7}$$

Proof. Since A_1 is a real symmetric matrix, there exists an orthogonal matrix Q such that $A_1 = Q \Lambda_1 Q^T$, where $QQ^T = I$, $\Lambda_1 = \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_{N_1})$.

Take transformation $\zeta = (Q^T \otimes I_{n_1}) \tilde{X}$, that is, $\tilde{X} = (Q \otimes I_{n_1}) \zeta$, where $\zeta = (\zeta_1^T, \zeta_2^T, \dots, \zeta_{N_1}^T)$, $\zeta_i^T \in R^{n_1}$. Denote $Q = (q_1, q_2, \dots, q_{N_1})$, where $q_1 = (1/\sqrt{N_1})(1, 1, \dots, 1)^T$ is the eigenvector corresponding to $\lambda_1 = 0$, and we have

$$\zeta_1 = \frac{1}{\sqrt{N_1}} \sum_{i=1}^{N_1} \tilde{x}_i = \frac{1}{\sqrt{N_1}} \sum_{i=1}^{N_1} (x_i - s_1) = 0. \tag{2.8}$$

Moreover,

$$\zeta^T \zeta = \widetilde{X}^T (Q \otimes I_{n_1}) \left(Q^T \otimes I_{n_1} \right) \widetilde{X} = \widetilde{X}^T \widetilde{X}, \tag{2.9}$$

that is, $\sum_{i=1}^{N_1} \zeta_i^T \zeta_i = \sum_{i=1}^{N_1} \widetilde{x}_i^T \widetilde{x}_i$. So one has

$$\begin{aligned} \widetilde{X}^T (A_1 \otimes I_{n_1}) \widetilde{X} &= \zeta^T \left(\left(Q^T A_1 Q \right) \otimes I_{n_1} \right) \zeta = \zeta^T (\Lambda_1 \otimes I_{n_1}) \zeta \\ &= \sum_{i=1}^{N_1} \zeta_i^T \lambda_i \zeta_i \leq \lambda_2 \sum_{i=1}^{N_1} \zeta_i^T \zeta_i = \lambda_2 \sum_{i=1}^{N_1} \widetilde{x}_i^T \widetilde{x}_i. \end{aligned} \tag{2.10}$$

Similarly, one can obtain $\widetilde{Y}^T (A_2 \otimes I_{n_2}) \widetilde{Y} \leq \mu_2 \sum_{j=1}^{N_2} \widetilde{y}_j^T \widetilde{y}_j$. This completes the proof. □

3. Linear Coupling Scheme and Criteria for Cluster Synchronization

In this section, we propose linear coupling schemes to achieve the cluster synchronization, and derive the corresponding criteria for the coupling strength in the network (2.1).

For the linearly coupled network (2.1), the following criteria can be derived.

Theorem 3.1. *For the linearly coupled network* (2.1), *the cluster synchronous manifold S is globally exponentially stable under the following condition:*

$$\delta_f + c_1 \lambda_2 - d_1 b < 0, \qquad \delta_g + c_2 \mu_2 - d_2 \overline{b} < 0. \tag{3.1}$$

Proof. Consider the function $V(t) = (1/2) \sum_{i=1}^{N_1} \widetilde{x}_i^T \widetilde{x}_i + (1/2) \sum_{j=1}^{N_2} \widetilde{y}_j^T \widetilde{y}_j$.

Its time derivative along the trajectory of (2.5) is

$$\begin{aligned} \dot{V}(t) = &\sum_{i=1}^{N_1} \widetilde{x}_i^T (f(x_i) - f(s_1)) + \sum_{i=1}^{N_1} \widetilde{x}_i^T \left(f(s_1) - \overline{f}(s_1) \right) \\ &+ c_1 \sum_{i=1}^{N_1} \sum_{k=1}^{N_1} \widetilde{x}_i^T a_{ik} \widetilde{x}_k - d_1 b \sum_{i=1}^{N_1} \widetilde{x}_i^T \widetilde{x}_i \\ &+ \sum_{j=1}^{N_2} \widetilde{y}_j^T (g(y_j) - g(s_2)) + \sum_{j=1}^{N_2} \widetilde{y}_j^T (g(s_2) - \overline{g}(s_2)) \\ &+ c_2 \sum_{j=1}^{N_2} \sum_{k=1}^{N_2} \widetilde{y}_j^T \overline{a}_{jk} \widetilde{y}_k - d_2 \widetilde{b} \sum_{j=1}^{N_2} \widetilde{y}_j^T \widetilde{y}_j, \end{aligned} \tag{3.2}$$

since

$$\sum_{i=1}^{N_1} \widetilde{x}_i^T \left(f(s_1) - \overline{f}(s_1) \right) = \left(f(s_1) - \overline{f}(s_1) \right) \sum_{i=1}^{N_1} (x_i - s_1)^T = 0. \tag{3.3}$$

Similarly, one has $\sum_{j=1}^{N_2} \widetilde{y}_j^T (g(s_2) - \overline{g}(s_2)) = 0$.

One has

$$\begin{aligned}
\dot{V}(t) &\le \delta_f \sum_{i=1}^{N_1} \widetilde{x}_i^T \widetilde{x}_i + c_1 \sum_{i=1}^{N_1} \sum_{k=1}^{N_1} \widetilde{x}_i^T a_{ik} \widetilde{x}_k - d_1 b \sum_{i=1}^{N_1} \widetilde{x}_i^T \widetilde{x}_i \\
&\quad + \delta_g \sum_{j=1}^{N_2} \widetilde{y}_j^T \widetilde{y}_j + c_2 \sum_{j=1}^{N_2} \sum_{k=1}^{N_2} \widetilde{y}_j^T \overline{a}_{jk} \widetilde{y}_k - d_2 \widetilde{b} \sum_{j=1}^{N_2} \widetilde{y}_j^T \widetilde{y}_j \\
&= \left(\delta_f - d_1 b\right) \sum_{i=1}^{N_1} \widetilde{x}_i^T \widetilde{x}_i + c_1 \widetilde{X}^T (A_1 \otimes I_{n_1}) \widetilde{X} \\
&\quad + \left(\delta_g - d_2 \overline{b}\right) \sum_{j=1}^{N_2} \widetilde{y}_j^T \widetilde{y}_j + c_2 \widetilde{Y}^T (A_2 \otimes I_{n_2}) \widetilde{Y}.
\end{aligned} \tag{3.4}$$

By Lemma 2.4, one has

$$\dot{V}(t) \le \left(\delta_f + c_1 \lambda_2 - d_1 b\right) \sum_{i=1}^{N_1} \widetilde{x}_i^T \widetilde{x}_i + \left(\delta_g + c_2 \mu_2 - d_2 \overline{b}\right) \sum_{j=1}^{N_2} \widetilde{y}_j^T \widetilde{y}_j \le -2\vartheta V(t), \tag{3.5}$$

where $\vartheta = \min\{|\delta_f + c_1 \lambda_2 - d_1 b|, |\delta_g + c_2 \mu_2 - d_2 \overline{b}|\} > 0$. Then one has $V(t) \le V(0) e^{-2\vartheta t}$.

Notice that $V(t) \ge (1/2)\|\widetilde{x}_i\|^2$, so $\|\widetilde{x}_i\| \le \sqrt{2V(0)}\, e^{-\vartheta t} \to 0$, that is, $\lim_{t \to +\infty} \|\widetilde{x}_i\| = 0$ ($i = 1, 2, \ldots, N_1$). Similarly, $\lim_{t \to +\infty} \|\widetilde{y}_j\| \le \lim_{t \to +\infty} \sqrt{2V(0)}\, e^{-\vartheta t} = 0$ ($j = 1, 2, \ldots, N_2$), which implies that the cluster synchronization manifold S of dynamical network (2.1) is globally exponentially stable. Now the proof is completed. □

According to Theorem 3.1, for the cluster synchronization, c_1, c_2 may take any positive number, even zero, if the external excitation intensity is sufficiently large such that $d_1 b > \delta_f$ and $d_2 \overline{b} > \delta_g$. Thus, the following corollary is derived.

Corollary 3.2. *For the following network:*

$$\begin{aligned}
\dot{x}_i &= f(x_i) + d_1 \sum_{k=1}^{N_2} b_{ik} (\Gamma_1 y_k - x_i), \quad i = 1, 2, \ldots, N_1, \\
\dot{y}_j &= g(y_j) + d_2 \sum_{k=1}^{N_1} \overline{b}_{jk} (\Gamma_2 x_k - y_j), \quad j = 1, 2, \ldots, N_2,
\end{aligned} \tag{3.6}$$

which is a special case of network (2.1)*, the cluster synchronous manifold S is globally exponentially stable under the conditions $d_1 b > \delta_f$ and $d_2 \overline{b} > \delta_g$.*

Remark 3.3. In model (3.6), there is no connection inside clusters. Corollary 3.2 shows that the cluster synchronization can be achieved even if without any connection within a cluster. It implies that the external excitation intensity is more conducive to the cluster synchronization than the cluster's interconnection under the "same-input" condition.

4. Adaptive Coupling Scheme and Criteria for Cluster Synchronization

Note that, the Lipschitz constants δ_f and δ_g are required to be known in Theorem 3.1. However, it is often difficult to obtain the precise values of δ_f and δ_g in some practical systems, hence the constants δ_f and δ_g are often selected to be larger, which leads to the coupling strengths c_1 and c_2 being larger than their necessary values. To overcome this drawback, we design c_1 and c_2 as adaptive variables, and present a local adaptive coupling scheme to realize cluster synchronization as follows.

Theorem 4.1. *For the network* (2.1), *take the inner coupling strengths c_1 and c_2 as adaptive variables, then the cluster synchronous manifold S is globally asymptotically stable under the following adaptive laws:*

$$\begin{aligned} \dot{c}_1(t) &= \sum_{i=1}^{N_1} \tilde{x}_i^T \tilde{x}_i, \qquad c_1(0) > 0, \\ \dot{c}_2(t) &= \sum_{j=1}^{N_2} \tilde{y}_j^T \tilde{y}_j, \qquad c_2(0) > 0. \end{aligned} \tag{4.1}$$

Proof. Consider the Lyapunov function

$$\begin{aligned} V(t) = {} & \frac{1}{2}\sum_{k=1}^{N_1} \tilde{x}_k^T \tilde{x}_k + \frac{1}{2}\sum_{j=1}^{N_2} \tilde{y}_j^T \tilde{y}_j \\ & + \frac{1}{2\beta_1}\left(\beta_1 c_1(t) + d_1 b - \delta_f - 1\right)^2 + \frac{1}{2\beta_2}\left(\beta_2 c_2(t) + d_2 \overline{b} - \delta_g - 1\right)^2, \end{aligned} \tag{4.2}$$

where $\beta_1 = -\lambda_2 > 0$ and $\beta_2 = -\mu_2 > 0$.

Its time derivative along the trajectory of (2.5) is

$$\begin{aligned} \dot{V}(t) = {} & \sum_{i=1}^{N_1} \tilde{x}_i^T \dot{\tilde{x}}_i + \sum_{j=1}^{N_2} \tilde{y}_j^T \dot{\tilde{y}}_j + \left(\beta_1 c_1(t) + d_1 b - \delta_f - 1\right)\dot{c}_1(t) \\ & + \left(\beta_2 c_2(t) + d_2 \overline{b} - \delta_g - 1\right)\dot{c}_2(t) \\ = {} & \sum_{i=1}^{N_1} \tilde{x}_i^T \left(f(x_i) - \overline{f}(s_1)\right) + c_1(t) \sum_{i=1}^{N_1}\sum_{k=1}^{N_1} \tilde{x}_i^T a_{ik} \tilde{x}_k - d_1 b \sum_{i=1}^{N_1} \tilde{x}_i^T \tilde{x}_i \\ & + \left(\beta_1 c_1(t) + d_1 b - \delta_f - 1\right) \sum_{i=1}^{N_1} \tilde{x}_i^T \tilde{x}_i \end{aligned}$$

$$+\sum_{j=1}^{N_2}\tilde{y}_j^T\left(g(y_j)-\overline{g}(s_2)\right)+c_2(t)\sum_{j=1}^{N_2}\sum_{k=1}^{N_2}\tilde{y}_j^T\overline{a}_{jk}\tilde{y}_k$$
$$-d_2\overline{b}\sum_{j=1}^{N_2}\tilde{y}_j^T\tilde{y}_j+\left(\beta_2c_2(t)+d_2\overline{b}-\delta_g-1\right)\sum_{j=1}^{N_2}\tilde{y}_j^T\tilde{y}_j. \tag{4.3}$$

Similar to the proof of Theorem 3.1, one has

$$\begin{aligned}\dot{V}(t)&\le\delta_f\sum_{i=1}^{N_1}\tilde{x}_i^T\tilde{x}_i+c_1(t)\lambda_2\sum_{i=1}^{N_1}\tilde{x}_i^T\tilde{x}_i+\left(\beta_1c_1(t)-\delta_f-1\right)\sum_{i=1}^{N_1}\tilde{x}_i^T\tilde{x}_i\\&\quad+\delta_g\sum_{j=1}^{N_2}\tilde{y}_j^T\tilde{y}_j+c_2(t)\mu_2\sum_{j=1}^{N_2}\tilde{y}_j^T\tilde{y}_j+\left(\beta_2c_2(t)-\delta_g-1\right)\sum_{j=1}^{N_2}\tilde{y}_j^T\tilde{y}_j\\&=-\sum_{i=1}^{N_1}\tilde{x}_i^T\tilde{x}_i-\sum_{j=1}^{N_2}\tilde{y}_j^T\tilde{y}_j\le 0.\end{aligned} \tag{4.4}$$

By the LaSalle-Yoshizawa theorem [19, 20], we have $\lim_{t\to+\infty}\|\tilde{x}_i\|=0$ $(i=1,2,\ldots,N_1)$ and $\lim_{t\to+\infty}\|\tilde{y}_j\|=0$ $(j=1,2,\ldots,N_2)$, which means that the cluster synchronous manifold S is globally asymptotically stable. Now the proof is completed. □

Similarly, we can further obtain the following theorem.

Theorem 4.2. *For the network* (2.1), *take the external excitation intensities d_1, d_2 as adaptive variables, then the cluster synchronous manifold S is globally asymptotically stable under the following adaptive laws:*

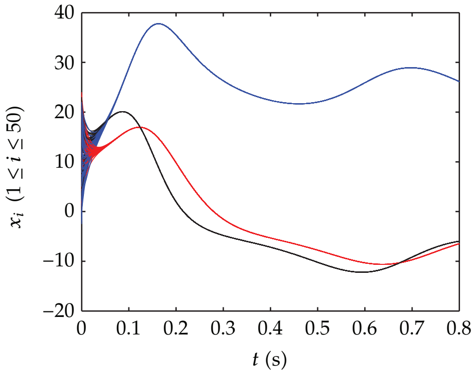
$$\begin{aligned}\dot{d}_1(t)&=\sum_{i=1}^{N_1}\tilde{x}_i^T\tilde{x}_i,\quad d_1(0)>0,\\\dot{d}_2(t)&=\sum_{j=1}^{N_2}\tilde{y}_j^T\tilde{y}_j,\quad d_2(0)>0.\end{aligned} \tag{4.5}$$

Proof. Consider the Lyapunov function $V(t)=(1/2)\sum_{k=1}^{N_1}\tilde{x}_k^T\tilde{x}_k+(1/2)\sum_{j=1}^{N_2}\tilde{y}_j^T\tilde{y}_j+(1/2b)(bd_1(t)-c_1\lambda_2-\delta_f-1)^2+(1/2\overline{b}\ )(\overline{b}d_2(t)-c_2\mu_2-\delta_g-1)^2$, and similar to the proof of Theorem 4.1, one can obtain the conclusion. □

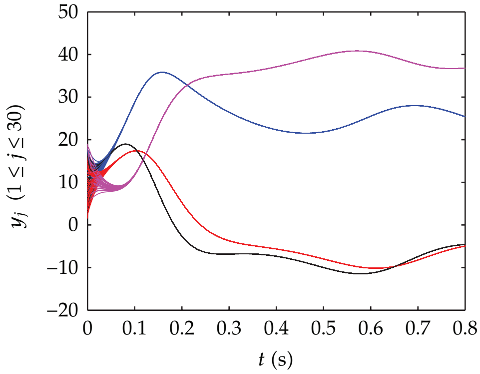
Corollary 4.3. *For the network* (3.6), *the cluster synchronous manifold S is globally asymptotically stable under the adaptive law* (4.5).

5. Numerical Simulations

In this section, illustrative examples are provided to verify the above theoretical analysis.



(a) The synchrony state x_i $(1 \le i \le 50)$



(b) The synchrony states y_j $(1 \le j \le 30)$

Figure 1: Cluster synchronization in the linearly coupled network (2.1) with 50 lorenz oscillators in the x-cluster and 30 hyperchaotic Lü oscillators in the y-cluster.

We consider a network which consists of two clusters with a scale-free sub-network with 50 Lorenz chaotic oscillators [21] as the x-cluster and a small-world sub-network with 30 hyperchaotic Lü oscillators [22] as the y-cluster. That is, the node's dynamics is $f(x_i) = (10(x_{i2} - x_{i1}),\ 28x_{i1} - x_{i2} - x_{i1}x_{i3}, x_{i1}x_{i2} - 8/3x_{i3})^T$ $(i = 1, \ldots, 50)$, or $g(y_j) = (36(y_{j2} - y_{j1}) + y_{j4}, 20y_{i2} - y_{j1}y_{j3}, y_{j1}y_{j2} - 3y_{j3}, y_{j1}y_{j3} + y_{j4})^T$ $(j = 1, \ldots, 30)$. The coupling matrix A_1 of the x-cluster is taken as an adjacency matrix of scale-free network with 50 nodes, The coupling matrix A_2 of the y-cluster is taken as an adjacency matrix of small-world network with 30 nodes, and the matrices B_1, B_2, Γ_1, and Γ_2 satisfy the conditions of models (2.1) and (3.6).

In the simulation, we take initial values $x_i(0) = (-1 + 0.5i, -2 + 0.5i, -3 + 0.5i)^T$, $y_j(0) = (1 + 0.5j, 2 + 0.5j, 3 + 0.5j, 4 + 0.5j)^T$, where $1 \le i \le 50$, $1 \le j \le 30$, and $c_1(0) = c_2(0) = d_1(0) = d_2(0) = 1$.

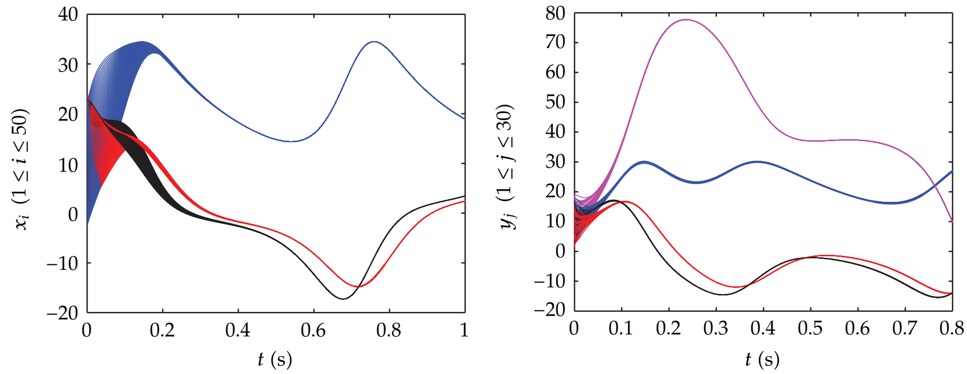
Figures 1 and 3 display the numerical simulation results of network (2.1) linear coupling and adaptive coupling, respectively. It shows that a set of nodes belonging to each cluster synchronize to the same behavior, that is, the cluster synchronization has been achieved quite soon.

Figures 2 and 4 show the numerical simulation results of network (3.6) with linear coupling and the adaptive coupling, respectively, where there are no connections inside the cluster. Obviously, synchronization has been reached quite soon.

Remark 5.1. Figures 1 and 2 show that the cluster synchronization (x-cluster and y-cluster) can be reached while the complete synchronization cannot be achieved in the network. Furthermore, compared with the coupling strength in Figures 3 and 4, one can also see that the external excitation d_1 and d_2 are more conducive to the cluster synchronization.

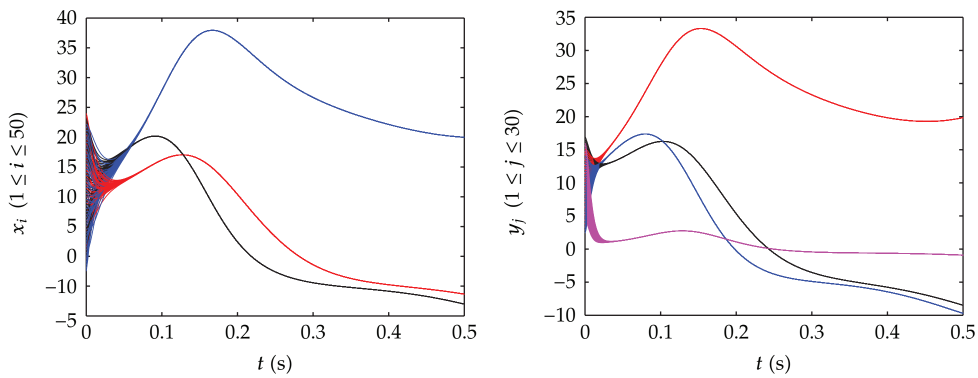
6. Conclusions

In this paper, we have further investigated the cluster synchronization of a complex dynamical network with given configuration which is connected by two groups of different oscillators. we present a linear coupling scheme and the corresponding sufficient condition is derived for the cluster synchronization. Moreover, an adaptive coupling scheme to lead

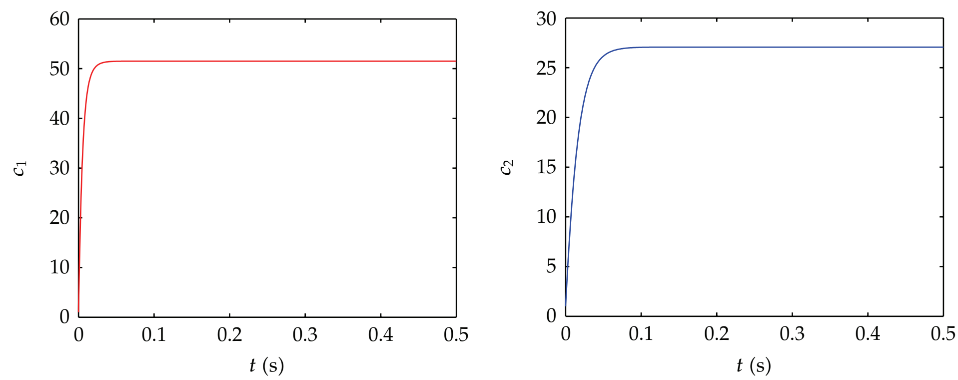


(a) The synchrony state x_i $(1 \le i \le 50)$ (b) The synchrony states y_j $(1 \le j \le 30)$

Figure 2: Cluster synchronization of the linearly coupled network (3.6) with 50 lorenz oscillators in the x-cluster and 30 hyperchaotic Lü oscillators in the y-cluster.

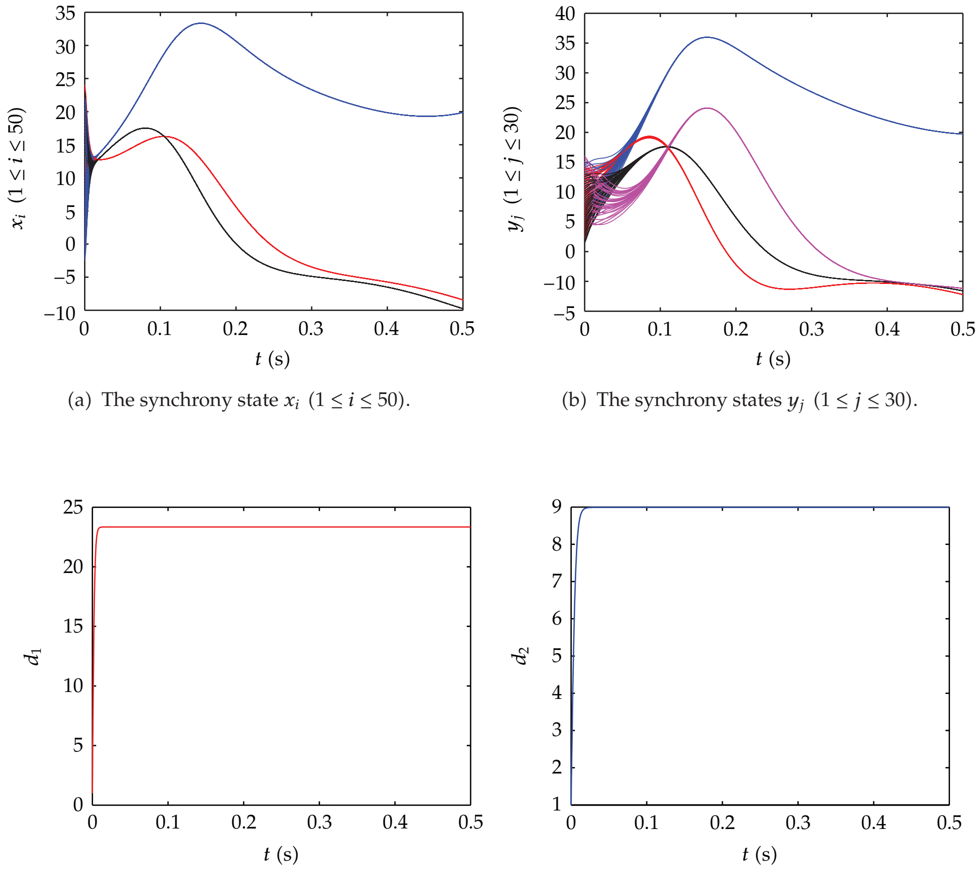


(a) The synchrony state x_i $(1 \le i \le 50)$ (b) The synchrony states y_j $(1 \le j \le 30)$



(c) The estimation of coupling strength c_1 (d) The estimation of coupling strength c_2

Figure 3: Cluster synchronization of the adaptively coupled network (2.1) with 50 lorenz oscillators in the x-cluster and 30 hyperchaotic Lü oscillators in the y-cluster.



(a) The synchrony state x_i $(1 \leq i \leq 50)$.

(b) The synchrony states y_j $(1 \leq j \leq 30)$.

(c) The estimation of coupling strength d_1.

(d) The estimation of coupling strength d_2.

Figure 4: Cluster synchronization of the adaptively coupled network (3.6) with 50 lorenz oscillators in the x-cluster and 30 hyperchaotic Lü oscillators in the y-cluster.

the cluster synchronization is proposed based on adaptive control technique. Our study shows that the global stability of the cluster synchronization can be guaranteed by increasing coupling strength in each cluster or enhancing the external excitation even if there are no connections insider a cluster. Chaos synchronization of delay systems [23] with adaptive impulsive control method [24] is widely discussed and applied, so we will study cluster synchronization in delay-coupling cluster networks with impulsive control in the future work.

Acknowledgments

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