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Research Article

A New Efficient Method for Nonlinear Fisher-Type Equations

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Laplace transform and new homotopy perturbation methods are adopted to study Fisher-type equations analytically. The solutions introduced in this study can be used to obtain the closed form of the solutions if they are required. The combined method needs less work in comparison with the other homotopy perturbation methods and decreases volume of calculations considerably. The method is tested on various examples, and results show that new method is more effective and convenient to use, and high accuracy of it is evident.

1. Introduction

Solving nonlinear partial differential equations is very important in mathematical sciences, and it is one of the most stimulating and particularly active areas of the research. In the recent years, an increasing interest of scientists and engineers has been devoted to the analytical asymptotic techniques for solving nonlinear problems. Many new numerical techniques have been widely applied to the nonlinear problems. Based on homotopy, which is a basic concept in topology, general analytical method, namely, the homotopy perturbation method (HPM) is established by He [1–8] in 1998 to obtain series solutions of nonlinear differential equations. The He's HPM has been already used to solve various functional equations. In this method, the nonlinear problem is transferred to an infinite number of subproblems and then the solution is approximated by the sum of the solutions of the first several subproblems. This simple method has been applied to solve linear and nonlinear equations of heat transfer [9–11], fluid mechanics [12], nonlinear Schrodinger equations [13], integral equations [14], boundary value problems [15], fractional KdV-Burgers equation [16], and nonlinear system of second-order boundary value problems [17]. Khan, et al. [18] solved the long porous slider problem by homotopy perturbation method which is coupled nonlinear ordinary differential equations resulting from the momentum equation. Also, Khan, et al. [19] studied the long porous slider problem in which the fluid is injected through the porous bottom by the Adomian decomposition method (ADM). This problem is similar to the problem we consider in this paper and has many application in chemical reactions, heat transfer, and chromatography. Recently, Moosaei et al. [20] suggest an alternative way to a similar problem to the problem we consider in this work. They solved the perturbed nonlinear Schrodinger's equation with Kerr law nonlinearity by using the first integral method.

In the present work, we construct the solution using a different approach. In this work, we obtain an analytical approximation to the solution of the nonlinear Fisher equation using combination of Laplace transform and new homotopy perturbation method (LTNHPM). The Fisher equation as a nonlinear model for a physical system involving linear diffusion and nonlinear growth takes the nondimensional form:

$$u_t = u_{xx} + \alpha\left(1 - u^{\beta}\right)(u - a). \tag{1.1}$$

A kink-like traveling wave solutions of (1.1) describe a constant-velocity front of transition from one homogeneous state to another. On the other hand, the solitons appear as a result of a balance between weak nonlinearity and dispersion. Therefore, in mathematics and physics, a soliton is described as a self-reinforcing solitary wave (a wave packet or pulse) that maintains its shape while it travels at constant speed. "Dispersive effects" refer to dispersion relation between the frequency and the speed of the waves. Solitons arise as the solutions of a widespread class of weakly nonlinear dispersive partial differential equations describing physical systems. However, when diffusion takes part, instead of dispersion, energy release by nonlinearity balances energy consumption by diffusion which results in traveling waves or fronts. Hence, traveling wave fronts are a great extent studied solution form for reaction-diffusion equations, with important applications to chemistry, biology, and medicine. Several studies in the literature, employing a large variety of methods, have been conducted to derive explicit solutions for the Fisher equation (1.1) and for the generalized Fisher equation. For more details about these investigations, the reader is referred to [21–26] and the references therein. Recently, Wazwaz and Gorguis [27] studied the Fisher equation, the general Fisher equation, and nonlinear diffusion equation of the Fisher type subject to initial conditions by using Adomian decomposition method.

The results obtained via LTNHPM confirm validity of the proposed method. The rest of this paper is organized as follows.

In Section 2, basic ideas of NHPM and the homotopy perturbation method are presented. In Section 3, the uses of NHPM for solving nonlinear Fisher-type equations are presented. Some examples are solved by the proposed method in Section 4. Conclusion will be appeared in Section 5.

2. Basic Ideas of the LTNHPM

To illustrate the basic ideas of this method, let us consider the following nonlinear differential equation:

$$A(u) - f(r) = 0, \quad r \in \Omega, \tag{2.1}$$

with the following initial conditions:

$$u(0) = \alpha_0, \qquad u'(0) = \alpha_1, \dots, \qquad u^{(n-1)}(0) = \alpha_{n-1}, \tag{2.2}$$

where A is a general differential operator and $f(r)$ is a known analytical function. The operator A can be divided into two parts, L and N, where L is a linear and N is a nonlinear operator. Therefore, (2.1) can be rewritten as

$$L(u) + N(u) - f(r) = 0. \tag{2.3}$$

Based on NHPM [28], we construct a homotopy $U(r,p) : \Omega \times [0,1] \to \mathbb{R}$, which satisfies

$$H(U,p) = (1-p)[L(U) - u_0] + p\big[A(U) - f(r)\big] = 0, \quad p \in [0,1], \ r \in \Omega, \tag{2.4}$$

or equivalently:

$$H(U,p) = L(U) - u_0 + pu_0 + p\big[N(U) - f(r)\big] = 0, \tag{2.5}$$

where $p \in [0,1]$ is an embedding parameter, u_0 is an initial approximation for the solution of (2.1). Clearly, (2.4) and (2.5) give

$$\begin{aligned} H(U,0) &= L(U) - u_0 = 0, \\ H(U(x),1) &= A(U) - f(r) = 0. \end{aligned} \tag{2.6}$$

Applying Laplace transform to the both sides of (2.5), we arrive at

$$\mathrm{L}\big\{L(U) - u_0 + pu_0 + p\big[N(U) - f(r)\big]\big\} = 0. \tag{2.7}$$

Using the differential property of Laplace transform we have

$$s^n \mathrm{L}\{U\} - s^{n-1}U(0) - s^{n-2}U'(0) - \cdots - U^{(n-1)}(0) = \mathrm{L}\big\{u_0 - pu_0 + p\big[N(U) - f(r)\big]\big\} \tag{2.8}$$

or

$$\mathrm{L}\{U\} = \frac{1}{s^n}\Big\{s^{n-1}U(0) + s^{n-2}U'(0) + \cdots + U^{(n-1)}(0) + \mathrm{L}\big\{u_0 - pu_0 + p\big[N(U) - f(r)\big]\big\}\Big\}. \tag{2.9}$$

Finally, applying the inverse Laplace transform to the both sides of (2.9), one can successfully reach to the following:

$$U = \mathrm{L}^{-1}\left\{\frac{1}{s^n}\Big\{s^{n-1}U(0) + s^{n-2}U'(0) + \cdots + U^{(n-1)}(0) + \mathrm{L}\big\{u_0 - pu_0 + p\big[N(U) - f(r)\big]\big\}\Big\}\right\}. \tag{2.10}$$

According to the HPM, we can first use the embedding parameter p as a small parameter and assume that the solutions of (2.10) can be represented as a power series in p as

$$U(x) = \sum_{n=0}^{\infty} p^n U_n. \tag{2.11}$$

Now let us rewrite (2.10) using (2.11) as

$$\begin{aligned}\sum_{n=0}^{\infty} p^n U_n = \mathrm{L}^{-1}\Bigg\{ \frac{1}{s^n}\Bigg\{ & s^{n-1}U(0) + s^{n-2}U'(0) + \cdots + U^{(n-1)}(0) \\ & + \mathrm{L}\Bigg\{ u_0 - pu_0 + p\Bigg[N\Bigg(\sum_{n=0}^{\infty} p^n U_n\Bigg) - f(r)\Bigg]\Bigg\}\Bigg\}\Bigg\}.\end{aligned} \tag{2.12}$$

Therefore, equating the coefficients of p with the same power leads to

$$\begin{aligned}
p^0 &: U_0 = \mathrm{L}^{-1}\left\{ \frac{1}{s^n}\left(s^{n-1}U(0) + s^{n-2}U'(0) + \cdots + U^{(n-1)}(0) + \mathrm{L}\{u_0\}\right)\right\}, \\
p^1 &: U_1 = \mathrm{L}^{-1}\left\{ \frac{1}{s^n}(\mathrm{L}\{N(U_0) - u_0 - f(r)\})\right\}, \\
p^2 &: U_2 = \mathrm{L}^{-1}\left\{ \frac{1}{s^n}(\mathrm{L}\{N(U_0, U_1)\})\right\}, \\
p^3 &: U_3 = \mathrm{L}^{-1}\left\{ \frac{1}{s^n}(\mathrm{L}\{N(U_0, \mathrm{U}_1, U_2)\})\right\}, \\
&\ \ \vdots \\
p^j &: U_j = \mathrm{L}^{-1}\left\{ \frac{1}{s^n}(\mathrm{L}\{N(U_0, U_1, U_2, \ldots, U_{j-1})\})\right\}. \\
&\ \ \vdots
\end{aligned} \tag{2.13}$$

Suppose that the initial approximation has the form $U(0) = u_0 = \alpha_0$, $U'(0) = \alpha_1, \ldots,$ $U^{(n-1)}(0) = \alpha_{n-1}$, therefore the exact solution may be obtained as following:

$$u = \lim_{p \to 1} U = U_0 + U_1 + U_2 + \cdots. \tag{2.14}$$

To show the capability of the method, we apply the NHPM to some examples in Section 4.

3. Analysis of the Method

Consider the nonlinear Fisher (1.1):

$$u_t = u_{xx} + \alpha\left(1 - u^{\beta}\right)(u - a). \tag{3.1}$$

For solving this equation by applying the new homotopy perturbation method, we construct the following homotopy:

$$H(U,p) = U_t - u_0 + p\left[u_0 - U_{xx} - \alpha\left(1 - U^{\beta}\right)(U - a)\right] = 0, \tag{3.2}$$

where $p \in [0,1]$ is an embedding parameter, u_0 is an initial approximation of solution of equation. Clearly, we have from (3.2)

$$\begin{gathered} H(U,0) = U_t - u_0 = 0, \\ H(U,1) = U_t - U_{xx} - \alpha\left(1 - U^{\beta}\right)(U - a) = 0. \end{gathered} \tag{3.3}$$

By applying Laplace transform on both sides of (3.2), we have

$$\mathrm{L}\{H(U,p)\} = \mathrm{L}\left\{U_t - u_0 + p\left[u_0 - U_{xx} - \alpha\left(1 - U^{\beta}\right)(U - a)\right]\right\}. \tag{3.4}$$

Using the differential property of Laplace transform we have

$$s\mathrm{L}\{U(x,t)\} - U(x,0) = \mathrm{L}\left\{u_0 - p\left[u_0 - U_{xx} - \alpha\left(1 - U^{\beta}\right)(U - a)\right]\right\} \tag{3.5}$$

or

$$\mathrm{L}\{U(x,t)\} = \frac{1}{s}\left(U(x,0) + \mathrm{L}\left\{u_0 - p\left[u_0 - U_{xx} - \alpha\left(1 - U^{\beta}\right)(U - a)\right]\right\}\right). \tag{3.6}$$

By applying inverse Laplace transform on both sides of (3.6), we have

$$U(x,t) = \mathrm{L}^{-1}\left\{\frac{1}{s}\left(U(x,0) + \mathrm{L}\left\{u_0 - p\left[u_0 - U_{xx} - \alpha\left(1 - U^{\beta}\right)(U - a)\right]\right\}\right)\right\}. \tag{3.7}$$

According to the HPM, we use the embedding parameter p as a small parameter and assume that the solutions of (3.7) can be represented as a power series in p as

$$U(x,t) = \sum_{n=0}^{\infty} p^n U_n(x,t). \tag{3.8}$$

Substituting (3.8) into (3.7) and equating the terms with the identical powers of p, leads to calculate $U_j(x,t),\ j = 0,1,2,\ldots$

$$\begin{aligned}
p^0 : U_0(x,t) &= \mathrm{L}^{-1}\left\{\frac{1}{s}(U(x,0) + \mathrm{L}\{u_0(x,t)\})\right\},\\
p^1 : U_1(x,t) &= \mathrm{L}^{-1}\left\{\frac{-1}{s}\mathrm{L}\left\{u_0(x,t) - (U_0)_{xx} + \alpha a - \alpha U_0 + \alpha U_0^{\beta+1} - \alpha a U_0^{\beta}\right\}\right\},\\
p^2 : U_2(x,t) &= \mathrm{L}^{-1}\left\{\frac{-1}{s}\mathrm{L}\left\{-(U_1)_{xx} - \alpha U_1 + (\beta+1)\alpha U_0^{\beta}U_1 - (\beta)\alpha a U_0^{\beta-1}U_1\right\}\right\},\\
&\vdots\\
p^j : U_j(x,t) &= \mathrm{L}^{-1}\left\{\frac{-1}{s}\mathrm{L}\left\{-(U_{j-1})_{xx} - \alpha U_{j-1} + \alpha \sum_{k_1+k_2+\cdots+k_{\beta+1}=j-1} U_{k_1}U_{k_2}\cdots U_{k_{\beta+1}}\right.\right.\\
&\qquad \left.\left. - a\alpha \sum_{k_1+k_2+\cdots+k_{\beta}=j-1} U_{k_1}U_{k_2}\cdots U_{k_{\beta}}\right\}\right\}.
\end{aligned} \tag{3.9}$$

Suppose that the initial approximation has the form $U(x,0) = u_0(x,t)$, therefore the exact solution may be obtained as following:

$$u(X,t) = \lim_{p\to 1} U(x,t) = U_0(x,t) + U_1(x,t) + U_2(x,t) + \cdots. \tag{3.10}$$

4. Examples

Example 4.1. Consider the following Fisher equation for $\alpha = \beta = 1$ and $a = 0$ taken from [29] such that

$$u_t = u_{xx} + u(1-u) \tag{4.1}$$

subject to a constant initial condition

$$u(x,0) = \lambda. \tag{4.2}$$

To solve (4.2) by the LTNHPM, we construct the following homotopy:

$$H(U,p) = U_t - \lambda + p[\lambda - U_{xx} - U(1-U)] = 0 \tag{4.3}$$

or

$$H(U,p) = U_t - \lambda + p\left[\lambda - U_{xx} - U + U^2\right] = 0. \tag{4.4}$$

Applying Laplace transform on both sides of (4.4), we have

$$\mathrm{L}\{H(U,p)\} = \mathrm{L}\Big\{U_t - \lambda + p\Big[\lambda - U_{xx} - U + U^2\Big]\Big\}. \tag{4.5}$$

Using the differential property of Laplace transform we have

$$s\mathrm{L}\{U(x,t)\} - \lambda = \mathrm{L}\Big\{\lambda - p\Big[\lambda - U_{xx} - U + U^2\Big]\Big\} \tag{4.6}$$

or

$$\mathrm{L}\{U(x,t)\} = \frac{1}{s}\Big(\lambda + \mathrm{L}\Big\{\lambda - p\Big[\lambda - U_{xx} - U + U^2\Big]\Big\}\Big). \tag{4.7}$$

By applying inverse Laplace transform on both sides of (4.7), we have

$$U(x,t) = \mathrm{L}^{-1}\Big\{\frac{1}{s}\Big(\lambda + \mathrm{L}\Big\{\lambda - p\Big[\lambda - U_{xx} - U + U^2\Big]\Big\}\Big)\Big\}. \tag{4.8}$$

Suppose the solution of (4.8) to have the following form:

$$U(x,t) = \sum_{n=0}^{\infty} p^n U_n(x,t), \tag{4.9}$$

where $U_i(x,t)$ are unknown functions which should be determined. Substituting (4.9) into (4.8), and equating the terms with the identical powers of p, leads to calculate $U_j(x,t),\ j = 0,1,2,\ldots$

$$\begin{aligned}
p^0 &: U_0(x,t) = \mathrm{L}^{-1}\Big\{\frac{1}{s}(\lambda + \mathrm{L}\{\lambda\})\Big\},\\
p^1 &: U_1(x,t) = \mathrm{L}^{-1}\Big\{\frac{-1}{s}\mathrm{L}\Big\{\lambda - (U_0)_{xx} - U_0 + U_0^2\Big\}\Big\},\\
p^2 &: U_2(x,t) = \mathrm{L}^{-1}\Big\{\frac{-1}{s}\mathrm{L}\{-(U_1)_{xx} - U_1 + 2U_0U_1\}\Big\},\\
&\vdots\\
p^j &: U_j(x,t) = \mathrm{L}^{-1}\Big\{\frac{-1}{s}\mathrm{L}\Big\{-(U_{j-1})_{xx} - U_{j-1} + \Big(\sum_{k=0}^{j-1} U_k U_{k-j-1}\Big)\Big\}\Big\}.
\end{aligned} \tag{4.10}$$

Assuming $u_0(x,t) = U(x,0) = \lambda$, and solving the above equation for $U_j(x,t)$, $j = 0,1,\ldots$ leads to the result

$$
\begin{aligned}
U_0(x,t) &= \lambda(1+t),\\
U_1(x,t) &= -\frac{1}{3}\lambda^2 t^3 + \left(\frac{1}{2}\lambda - \lambda^2\right)t^2 - t\lambda^2,\\
U_2(x,t) &= \frac{2}{15}\lambda^3 t^5 + \left(-\frac{1}{3}\lambda^2 + \frac{2}{3}\lambda^3\right)t^4 + \left(\frac{1}{6}\lambda + \frac{4}{3}\lambda^3 - \frac{2}{3}\lambda^2\right)t^3 + \left(\lambda^3 - \frac{1}{2}\lambda^2\right)t^2,\\
U_3(x,t) &= -\frac{17}{315}\lambda^4 t^7 + \left(-\frac{17}{45}\lambda^4 + \frac{17}{90}\lambda^3\right)t^6 + \left(\frac{11}{15}\lambda^3 - \frac{17}{15}\lambda^4 - \frac{11}{60}\lambda^2\right)t^5\\
&\quad + \left(\frac{7}{6}\lambda^3 + \frac{1}{24}\lambda - \frac{5}{3}\lambda^4 - \frac{1}{4}\lambda^2\right)t^4 + \left(\frac{2}{3}\lambda^3 - \frac{1}{6}\lambda^2 - \lambda^4\right)t^3,\\
U_4(x,t) &= \frac{62}{2835}\lambda^5 t^9 + \left(-\frac{31}{315}\lambda^4 + \frac{62}{315}\lambda^5\right)t^8 + \left(\frac{1}{7}\lambda^3 + \frac{248}{315}\lambda^5 - \frac{4}{7}\lambda^4\right)t^7\\
&\quad + \left(-\frac{43}{30}\lambda^4 - \frac{13}{180}\lambda^2 + \frac{77}{45}\lambda^5 + \frac{13}{30}\lambda^3\right)t^6 + \frac{1}{120}\lambda\left(-208\lambda^3 + 68\lambda^2 + 240\lambda^4 - 8\lambda + 1\right)t^5\\
&\quad + \frac{1}{24}\lambda^2(2\lambda - 1)\left(12\lambda^2 - 4\lambda + 1\right)t^4.\\
&\ \ \vdots
\end{aligned}
\tag{4.11}
$$

Therefore using some algebra with the aid of symbolic computation tool, we gain the solution of (4.1) as

$$
\begin{aligned}
u(x,t) &= U_0(x,t) + U_1(x,t) + U_2(x,t) + U_3 + (x,t) + \cdots\\
&= \lambda + \lambda(1-\lambda)t + \lambda(1-\lambda)(1-2\lambda)\frac{t^2}{2!} + \lambda(1-\lambda)\left(1 - 6\lambda + 6\lambda^2\right)\frac{t^3}{3!} + \cdots\\
&= \frac{\lambda e^t}{1 - \lambda + \lambda e^t},
\end{aligned}
\tag{4.12}
$$

which is exact solution of problem.

Example 4.2. Consider the following Fisher equation for $\alpha = 6,\ \beta = 1,\ a = 0$, [29] such that

$$u_t = u_{xx} + 6u(1-u) \tag{4.13}$$

subject to a initial condition:

$$u(x,0) = (1+e^x)^{-2}. \tag{4.14}$$

To solve (4.13) by the LTNHPM, we construct the following homotopy:

$$H(U,p) = U_t - u_0 + p[u_0 - U_{xx} - 6U(1-U)] = 0 \tag{4.15}$$

or

$$H(U,p) = U_t - u_0 + p\left[u_0 - U_{xx} - 6U + 6U^2\right] = 0. \tag{4.16}$$

Applying Laplace transform on both sides of (4.16), we have

$$\mathrm{L}\{H(U,p)\} = \mathrm{L}\left\{U_t - u_0 + p\left[u_0 - U_{xx} - 6U + 6U^2\right]\right\}. \tag{4.17}$$

Using the differential property of Laplace transform we have

$$s\mathrm{L}\{U(x,t)\} - U(x,0) = \mathrm{L}\left\{u_0 - p\left[u_0 - U_{xx} - 6U + 6U^2\right]\right\} \tag{4.18}$$

or

$$\mathrm{L}\{U(x,t)\} = \frac{1}{s}\left(U(x,0) + \mathrm{L}\left\{u_0 - p\left[u_0 - U_{xx} - 6U + 6U^2\right]\right\}\right). \tag{4.19}$$

By applying inverse Laplace transform on both sides of (4.19), we have

$$U(x,t) = \mathrm{L}^{-1}\left\{\frac{1}{s}\left((1+e^x)^{-2} + \mathrm{L}\left\{u_0 - p\left[u_0 - U_{xx} - 6U + 6U^2\right]\right\}\right)\right\}. \tag{4.20}$$

Suppose the solution of (4.20) to have the following form:

$$U(x,t) = \sum_{n=0}^{\infty} p^n U_n(x,t), \tag{4.21}$$

where $U_i(x,t)$ are unknown functions which should be determined. Substituting (4.21) into (4.20), and equating the terms with the identical powers of p, leads to calculate $U_j(x,t),\ j = 0,1,2,\ldots$

$$\begin{aligned}
p^0 &: U_0(x,t) = \mathrm{L}^{-1}\left\{\frac{1}{s}\left(U(x,0) + \mathrm{L}\{u_0(x,t)\}\right)\right\},\\
p^1 &: U_1(x,t) = \mathrm{L}^{-1}\left\{\frac{-1}{s}\mathrm{L}\left\{u_0(x,t) - (U_0)_{xx} - 6U_0 + 6U_0^2\right\}\right\},\\
p^2 &: U_2(x,t) = \mathrm{L}^{-1}\left\{\frac{-1}{s}\mathrm{L}\{-(U_1)_{xx} - 6U_1 + 12U_0U_1\}\right\},\\
&\vdots\\
p^j &: U_j(x,t) = \mathrm{L}^{-1}\left\{\frac{-1}{s}\mathrm{L}\left\{-\left(U_{j-1}\right)_{xx} - 6U_{j-1} + 6\left(\sum_{k=0}^{j-1} U_k U_{k-j-1}\right)\right\}\right\}.
\end{aligned} \tag{4.22}$$

Assuming $u_0(x,t) = U(x,0) = \lambda$, and solving the above equation for $U_j(x,t),\ j = 0,1,\ldots$ leads to the result

$$\begin{aligned}
U_0(x,t) &= \frac{1+t}{(1+e^x)^2},\\
U_1(x,t) &= -2\frac{t^3}{(1+e^x)^4} + \frac{(-3+5e^{2x}+5e^x)t^2}{(1+e^x)^4} + \frac{(-1+9e^{2x}+8e^x)t}{(1+e^x)^4},\\
U_2(x,t) &= \frac{24}{5}\frac{t^5}{(1+e^x)^6} + \frac{1}{15}\frac{(180-390e^{2x}-285e^x)t^4}{(1+e^x)^6}\\
&\quad + \frac{1}{15}\frac{(250e^{4x}-870e^{2x}+375e^{3x}-725e^x+150)t^3}{(1+e^x)^6}\\
&\quad + \frac{1}{15}\frac{(45+675e^{4x}+900e^{3x}-360e^x-180e^{2x})t^2}{(1+e^x)^6},
\end{aligned}$$

$$
\begin{aligned}
&U_3(x,t)\\
&= -\frac{408}{35}\frac{t^7}{(1+e^x)^8} + \frac{1}{420}\frac{\left(-17136+44352e^{2x}+26376e^x\right)t^6}{(1+e^x)^8}\\
&\quad + \frac{1}{420}\frac{\left(-22680-78036e^{3x}+132216e^{2x}-77448e^{4x}+93996e^x\right)t^5}{(1+e^x)^8}\\
&\quad + \frac{1}{420}\frac{\left(-261380e^{4x}+21875e^{5x}-271250e^{3x}+17500e^{6x}+81620e^{2x}+98455e^x-13860\right)t^4}{(1+e^x)^8}\\
&\quad + \frac{1}{420}\frac{\left(-3360+63000e^{6x}+32760e^x+63000e^{5x}-183120e^{3x}-142800e^{4x}-4200e^{2x}\right)t^3}{(1+e^x)^8}.\\
&\qquad\vdots
\end{aligned}
\tag{4.23}
$$

Therefore using some algebra with the aid of symbolic computation tool, we gain the solution of (4.13) as

$$
\begin{aligned}
u(x,t) &= U_0(x,t)+U_1(x,t)+U_2(x,t)+U_3(x,t)+\cdots\\
&= \frac{1}{(1+e^x)^2}+\frac{10e^x}{(1+e^x)^3}t+\frac{25e^x(-1+2e^x)}{(1+e^x)^4}t^2+\frac{-125e^x\left(-1+7e^x-4e^{2x}\right)}{3(1+e^x)^5}t^3+\cdots\\
&= \frac{1}{(1+e^{x-5t})^2},
\end{aligned}
\tag{4.24}
$$

which is exact solution of problem.

Example 4.3. Consider the following generalized Fisher equation for $\alpha = 1,\ \beta = 6$ and $a = 0$ taken from [29] such that

$$u_t = u_{xx} + u\left(1-u^6\right) \tag{4.25}$$

subject to a constant initial condition:

$$u(x,0) = \frac{1}{\sqrt[3]{1+e^{(3/2)x}}}. \tag{4.26}$$

To solve (4.26) by the LTNHPM, we construct the following homotopy:

$$H(U,p) = U_t - u_0 + p\left[u_0 - U_{xx} - U\left(1-U^6\right)\right] = 0 \tag{4.27}$$

or

$$H(U,p) = U_t - u_0 + p\left[u_0 - U_{xx} - U + U^7\right] = 0. \tag{4.28}$$

Applying Laplace transform on both sides of (4.28), we have

$$\mathrm{L}\{H(U,p)\} = \mathrm{L}\left\{U_t - u_0 + p\left[u_0 - U_{xx} - U + U^7\right]\right\}. \tag{4.29}$$

Using the differential property of Laplace transform we have

$$s\mathrm{L}\{U(x,t)\} - U(x,0) = \mathrm{L}\left\{u_0 - p\left[u_0 - U_{xx} - U + U^7\right]\right\} \tag{4.30}$$

or

$$\mathrm{L}\{U(x,t)\} = \frac{1}{s}\left(U(x,0) + \mathrm{L}\left\{u_0 - p\left[u_0 - U_{xx} - U + U^7\right]\right\}\right). \tag{4.31}$$

By applying inverse Laplace transform on both sides of (4.31), we have

$$U(x,t) = \mathrm{L}^{-1}\left\{\frac{1}{s}\left(\frac{1}{\sqrt[3]{1+\mathrm{e}^{(3/2)x}}} + \mathrm{L}\{u_0 - p[u_0 - U_{xx} - U + U^7]\}\right)\right\}. \tag{4.32}$$

Suppose the solution of (4.32) to have the following form:

$$U(x,t) = \sum_{n=0}^{\infty} p^n U_n(x,t), \tag{4.33}$$

where $U_i(x,t)$ are unknown functions which should be determined. Substituting (4.33) into (4.32), and equating the terms with the identical powers of p, leads to calculate $U_j(x,t)$, $j = 0,1,2,\ldots$

$$\begin{aligned}
p^0 &: U_0(x,t) = \mathrm{L}^{-1}\left\{\frac{1}{s}(U(x,0) + \mathrm{L}\{u_0(x,t)\})\right\},\\
p^1 &: U_1(x,t) = \mathrm{L}^{-1}\left\{\frac{-1}{s}\mathrm{L}\left\{u_0(x,t) - (U_0)_{xx} - U_0 + U_0^7\right\}\right\},\\
p^2 &: U_2(x,t) = \mathrm{L}^{-1}\left\{\frac{-1}{s}\mathrm{L}\left\{-(U_1)_{xx} - U_1 + 7U_0^6 U_1\right\}\right\},\\
&\quad\vdots\\
p^j &: U_j(x,t) = \mathrm{L}^{-1}\left\{\frac{-1}{s}\mathrm{L}\left\{-(U_{j-1})_{xx} - U_{j-1} + \left(\sum_{k_1+k_2+\cdots+k_7=j-1} U_{k_1}U_{k_2}\cdots U_{k_7}\right)\right\}\right\}.
\end{aligned} \tag{4.34}$$

Assuming $u_0(x,t) = U(x,0) = 1/\sqrt[3]{1+\mathrm{e}^{(3/2)x}}$, and solving the above equation for $U_{\mathrm{j}}(x,t)$, $j = 0,1,2,\ldots$ leads to the result

$$U_0(x,t) = \frac{1+t}{\sqrt[3]{1+\mathrm{e}^{(3/2)x}}},$$

$$U_1(x,t) = \frac{-t^8 - 8t^7 - 28t^6 - 56t^5 - 70t^4 - 56t^3 + \left(5\mathrm{e}^{(3/2)x} + 5\mathrm{e}^{3x} - 24\right)t^2 + \left(-8 - 6\mathrm{e}^{(3/2)x} + 2\mathrm{e}^{3x}\right)t}{8\left(1+\mathrm{e}^{(3/2)x}\right)^{7/3}},$$

$$\begin{aligned}U_2(x,t) &= \frac{1}{\left(1+\mathrm{e}^{(3/2)x}\right)^{13/3}}\\&\quad\times\left(\frac{7}{120}t^{15} + \frac{7}{8}t^{14} + \frac{49}{8}t^{13} + \frac{637}{24}t^{12} + \cdots\right.\\&\qquad + \frac{1}{1440}\left(-28800 + 25995\mathrm{e}^{3x} - 18675\mathrm{e}^{(3/2)x} - 375\mathrm{e}^{6x} + 375\mathrm{e}^{(9/2)x}\right)t^3\\&\qquad\left. + \frac{1}{1440}\left(-4320 - 225\mathrm{e}^{6x} + 4005\mathrm{e}^{3x} - 4365\mathrm{e}^{(3/2)x} + 3825\mathrm{e}^{(9/2)x}\right)t^2\right),\end{aligned}$$

$$\begin{aligned}U_3(x,t) &= \frac{1}{\left(1+\mathrm{e}^{(3/2)x}\right)^{19/3}}\\&\quad\times\left(-8484t^{22} - 186648t^{21} - 1959804t^{20} - 13065360t^{19} - 62060460t^{18}\right.\\&\qquad + \cdots + \left(-30222720 - 35272215\mathrm{e}^{(3/2)x} + 20625\mathrm{e}^{9x} - 15482610\mathrm{e}^{6x}\right.\\&\qquad\qquad\left. + 44573265\mathrm{e}^{3x} + 64407750\mathrm{e}^{(9/2)x} - 309375\mathrm{e}^{(15/2)x}\right)t^4\\&\qquad + \left(-3294720 - 5585580\mathrm{e}^{(3/2)x} + 2950200\mathrm{e}^{6x} + 10880760\mathrm{e}^{(9/2)x}\right.\\&\qquad\qquad\left.\left. + 4319700\mathrm{e}^{3x} - 1303500\mathrm{e}^{(15/2)x} + 16500\mathrm{e}^{9x}\right)t^3\right).\\&\qquad\vdots\end{aligned} \tag{4.35}$$

Therefore using some algebra with the aid of symbolic computation tool, we gain the solution of (4.25) as

$$\begin{aligned}u(x,t) &= U_0(x,t) + U_1(x,t) + U_2(x,t) + U_3(x,t) + \cdots\\&= \frac{1}{\sqrt[3]{1+\mathrm{e}^{(3/2)x}}} + \frac{5e^{(3/2)x}}{4\left(\sqrt[3]{1+\mathrm{e}^{(3/2)x}}\right)^4}t + \frac{25e^{(3/2)x}\left(e^{(3/2)x} - 3\right)}{32\left(\sqrt[3]{1+\mathrm{e}^{(3/2)x}}\right)^7}t^2 + \cdots\\&= \sqrt[3]{0.5 + 0.5\tanh[0.75(x - 2.5t)]},\end{aligned} \tag{4.36}$$

which is exact solution of problem.

Example 4.4. Consider the following nonlinear diffusion equation of the Fisher type for $\alpha = \beta = 1$ taken from [29] such that

$$u_t = u_{xx} + u(1-u)(u-a), \quad 0 < a < 1 \tag{4.37}$$

subject to an initial condition:

$$u(x,0) = \frac{1}{1+e^{-x/\sqrt{2}}}. \tag{4.38}$$

To solve (4.38) by the LTNHPM, we construct the following homotopy:

$$H(U,p) = U_t - u_0 + p[u_0 - U_{xx} - U(1-U)(U-a)] = 0 \tag{4.39}$$

or

$$H(U,p) = U_t - u_0 + p\left[u_0 - U_{xx} + U^3 + aU - (1+a)U^2\right] = 0. \tag{4.40}$$

Applying Laplace transform on both sides of (4.40), we have

$$\mathrm{L}\{H(U,p)\} = \mathrm{L}\left\{U_t - u_0 + p\left[u_0 - U_{xx} + U^3 + aU - (1+a)U^2\right]\right\}. \tag{4.41}$$

Using the differential property of Laplace transform we have

$$s\mathrm{L}\{U(x,t)\} - U(x,0) = \mathrm{L}\left\{u_0 - p\left[u_0 - U_{xx} + U^3 + aU - (1+a)U^2\right]\right\} \tag{4.42}$$

or

$$\mathrm{L}\{U(x,t)\} = \frac{1}{s}\left(U(x,0) + \mathrm{L}\left\{u_0 - p\left[u_0 - U_{xx} - U + U^2\right]\right\}\right). \tag{4.43}$$

By applying inverse Laplace transform on both sides of (4.43), we have

$$U(x,t) = \mathrm{L}^{-1}\left\{\frac{1}{s}\left(\lambda + \mathrm{L}\left\{u_0 - p\left[u_0 - U_{xx} + U^3 + aU - (1+a)U^2\right]\right\}\right)\right\}. \tag{4.44}$$

Suppose the solution of (4.44) to have the following form:

$$U(x,t) = \sum_{n=0}^{\infty} p^n U_n(x,t), \tag{4.45}$$

where $U_i(x,t)$ are unknown functions which should be determined. Substituting (4.45) into (4.43), and equating the terms with the identical powers of p, leads to calculate $U_j(x,t),\ j = 0,1,2,\ldots$

$$\begin{aligned}
p^0 &: U_0(x,t) = \mathrm{L}^{-1}\left\{\frac{1}{s}(U(x,0) + \mathrm{L}\{u_0(x,t)\})\right\},\\
p^1 &: U_1(x,t) = \mathrm{L}^{-1}\left\{\frac{-1}{s}\mathrm{L}\left\{u_0(x,t) - (U_0)_{xx} + U_0^3 + aU_0 - (1+a)U_0^2\right\}\right\},\\
p^2 &: U_2(x,t) = \mathrm{L}^{-1}\left\{\frac{-1}{s}\mathrm{L}\left\{-(U_1)_{xx} + 3U_0^2U_1 + aU_1 - 2(1+a)U_0U_1\right\}\right\},\\
&\ \vdots\\
p^j &: U_j(x,t) = \mathrm{L}^{-1}\left\{\frac{-1}{s}\mathrm{L}\left\{-(U_{j-1})_{xx} + aU_{j-1} + \left(\sum_{k=0}^{j-1}\sum_{i=0}^{k}U_{j-k-1}U_iU_{k-i}\right)\right.\right.\\
&\qquad\left.\left. -(1+a)\sum_{k=0}^{j-1}U_kU_{j-k-1}\right\}\right\}.
\end{aligned} \tag{4.46}$$

Assuming $u_0(x,t) = U(x,0) = 1/(1+\mathrm{e}^{-x/\sqrt{2}})$, and solving the above equation for $U_j(x,t),\ j = 0,1,2,\ldots$ leads to the result

$$\begin{aligned}
U_0(x,t) &= \frac{1+t}{1+\mathrm{e}^{-x/\sqrt{2}}},\\
U_1(x,t) &= \frac{1}{12\left(1+\mathrm{e}^{-x/\sqrt{2}}\right)^3}\left(-3t^4 + \left(4\mathrm{e}^{\mathrm{e}^{-x/\sqrt{2}}} - 8 + 4a + 4\mathrm{e}^{\mathrm{e}^{-x/\sqrt{2}}}a\right)t^3\right.\\
&\qquad + \left(3\mathrm{e}^{-\sqrt{2}x} - 6 + 9\mathrm{e}^{\mathrm{e}^{-x/\sqrt{2}}} - 6\mathrm{e}^{-\sqrt{2}x}a + 6a\right)t^2\\
&\qquad \left. + \left(-18\mathrm{e}^{\mathrm{e}^{-x/\sqrt{2}}} - 12\mathrm{e}^{-\sqrt{2}x}a - 12 - 12\mathrm{e}^{\mathrm{e}^{-x/\sqrt{2}}}a - 6\mathrm{e}^{-\sqrt{2}x}\right)t\right),\\
U_2(x,t) &= \frac{1}{840\left(1+\mathrm{e}^{-x/\sqrt{2}}\right)^5}\\
&\quad\times\left[90t^7 + \left(-210\mathrm{e}^{-x/\sqrt{2}} - 210a - 210\mathrm{e}^{-x/\sqrt{2}}a + 420\right)t^6\right.\\
&\qquad + \left(-203\mathrm{e}^{-\sqrt{2}x} + 112\mathrm{e}^{-\sqrt{2}x}a^2 + 518\mathrm{e}^{-\sqrt{2}x}a - 742a\right.\\
&\qquad\quad \left.\left. + 112a^2 + 742 - 847\mathrm{e}^{-x/\sqrt{2}} + 224\mathrm{e}^{-x/\sqrt{2}}a^2 - 224\mathrm{e}^{-x/\sqrt{2}}a\right)t^5 + \cdots\right],
\end{aligned}$$

$$U_3(x,t) = \frac{1}{20160\left(1+e^{-x/\sqrt{2}}\right)^7}$$
$$\Big[1026t^{10} + \left(-3420a - 3420e^{-1/2\sqrt{2}x} - 3420e^{-1/2\sqrt{2}x}a + 6840\right)t^9$$
$$+ \left(3528e^{-\sqrt{2}x}a^2 + 3528a^2 - 20403e^{-x/\sqrt{2}} + 18918 - 7056e^{-x/\sqrt{2}}a\right.$$
$$\left. +11862e^{-\sqrt{2}x}a + 7056e^{-x/\sqrt{2}}a^2 - 3627e^{-\sqrt{2}x} - 18918a\right)t^8 + \cdots\Big].$$
$$\vdots \tag{4.47}$$

Therefore using some algebra with the aid of symbolic computation tool, we gain the solution of (4.37) as

$$\begin{aligned} u(x,t) &= U_0(x,t) + U_1(x,t) + U_2(x,t) + U_3(x,t) + \cdots \\ &= \frac{1}{1+e^{-x/\sqrt{2}}} - \frac{e^{-x/\sqrt{2}}(-1+2\,a)}{2\left(1+e^{-x/\sqrt{2}}\right)^2} \\ &\quad + \frac{e^{-x/\sqrt{2}}\left(e^{-x/\sqrt{2}} - 4\,e^{-x/\sqrt{2}}a + 4\,e^{-x/\sqrt{2}}a^2 - 1 + 4\,a - 4\,a^2\right)}{8\left(1+e^{-x/\sqrt{2}}\right)^3} + \cdots \\ &= \frac{1}{1+e^{(-x-\sqrt{2}(0.5-a)t)/\sqrt{2}}}, \end{aligned} \tag{4.48}$$

which is exact solution of problem.

5. Summary and Conclusion

In the present work, we proposed a combination of Laplace transform method and homotopy perturbation method to solve Fisher-type Equations. This method is simple and finds exact solution of the equations using the initial condition only. This method unlike the most numerical techniques provides a closed form of the solution. The new method developed in the current paper was tested on several examples. The obtained results show that this approach does not require specific algorithms and complex calculations, such as, ADM or construction of correction functionals using general Lagrange multipliers, such as, VIM and is much easier and more convenient than ADM and VIM, and this approach can solve the problem very fast and effectively.

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