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*Research Article*

# Carleson Measure and Tent Spaces on the Siegel Upper Half Space

**Kai Zhao**

*College of Mathematics, Qingdao University, Qingdao 266071, China*

Correspondence should be addressed to Kai Zhao, zkzc@yahoo.com.cn

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The Hausdorff capacity on the Heisenberg group is introduced. The Choquet integrals with respect to the Hausdorff capacity on the Heisenberg group are defined. Then the fractional Carleson measures on the Siegel upper half space are discussed. Some characterized results and the dual of the fractional Carleson measures on the Siegel upper half space are studied. Therefore, the tent spaces on the Siegel upper half space in terms of the Choquet integrals are introduced and investigated. The atomic decomposition and the dual spaces of the tent spaces are obtained at the last.

## 1. Introduction

It is well known that harmonic analysis plays an important role in partial differential equations. The theory of function spaces constitutes an important part of harmonic analysis. Heisenberg group, just as its name coming from the physicists Heisenberg, is very useful in quantum mechanics. Therefore, to discuss new function or distribution spaces and some characterizations of them is very significant in modern harmonic analysis and partial differential equations. Especially, the function spaces related to the Heisenberg group will be used in partial differential equations and physics. It is precisely the reason in which we are interested.

The Hausdorff capacity on $\mathbb{R}^n$ is introduced by Adams in [1]. Some limiting form is the classical Hausdorff measure. Adams also discussed some boundedness of Hardy-Littlewood maximal functions related to it. The capacity and Choquet integrals, in some sense, are from and applied to partial differential equations (see [2, 3]). As we know, the tent spaces have been considered by many authors and play an important role in harmonic analysis on $\mathbb{R}^n$ (cf. [4, 5]). By using the Choquet integrals with respect to Hausdorff capacity on $\mathbb{R}^n$, the new tent spaces and their applications on the duality results for fractional Carleson measures,

$Q$ spaces, and Hardy-Hausdorff spaces were discussed (see [6]). The theory of $Q$ spaces can be found in [7–11]). Inspired by the literature [6], in this paper, we will discuss the fractional Carleson measures and the tent spaces on the Siegel upper half space. In order to get the results in Section 3, some boundedness of Hardy-Littlewood maximal functions on the Choquet integral space $L^q_{\Lambda_p^\infty}(\mathbb{H}^n)$ with Hausdorff capacity on the Heisenberg group are discussed in Section 2. In Section 3, by the Choquet integrals with respect to Hausdorff capacity on the Heisenberg group, we will introduce the fractional Carleson measures on the Siegel upper half space. The characterizations of the fractional Carleson measures on the Siegel upper half space are obtained. And the dual of the fractional Carleson measures on the Siegel upper half space is also obtained. In the last section, the tent spaces on the Siegel upper half space in terms of Choquet integrals with respect to the Hausdorff capacity on the Heisenberg group are introduced. Then, the atomic decomposition and the dual spaces of the tent spaces are obtained. The fractional Carleson measures and the tent spaces on the Siegel upper half space will be used for $Q$ spaces and the Hardy-Hausdorff spaces on the Heisenberg group (which will be discussed in another paper by us). On the other hand, they may be used in partial differential equations and quantum mechanics.

As we know, Heisenberg group was discussed by many authors, such as [5, 12–15]. For convenience, let us recall some basic knowledge for the Heisenberg group.

Let $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ be the sets of all integers, real numbers, and complex numbers, and $\mathbb{Z}^n$, $\mathbb{R}^n$, and $\mathbb{C}^n$ be $n$-dimensional $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$, respectively. The Siegel upper half space $\mathbb{U}^n$ in $\mathbb{C}^{n+1}$ is defined by

$$\mathbb{U}^n = \left\{ z = (z', z_{n+1}) \in \mathbb{C}^{n+1} : \operatorname{Im} z_{n+1} > \left|z'\right|^2 \right\}, \tag{1.1}$$

where $z' = (z_1, z_2, \ldots, z_n) \in C^n$, $|z'|^2 = \sum_{k=1}^n |z_k|^2$. The boundary of $\mathbb{U}^n$ is

$$\partial\mathbb{U}^n = \left\{ z = (z', z_{n+1}) \in \mathbb{C}^{n+1} : \operatorname{Im} z_{n+1} = \left|z'\right|^2 \right\}. \tag{1.2}$$

The Heisenberg group on $\mathbb{C}^n \times \mathbb{R}$, denoted by $\mathbb{H}^n$, is a noncommutative nilpotent Lie group with the underlying manifold $\mathbb{C}^n \times \mathbb{R}$. The group law is given by

$$zw = (z', t)(w', s) = \left(z' + w', t + s + 2 \operatorname{Im} z'\overline{w'}\right), \quad \text{for } z = (z', t),\ w = (w', s) \in \mathbb{H}^n, \tag{1.3}$$

where $z'\overline{w'} = z_1\overline{w_1} + \cdots + z_n\overline{w_n}$ is the Hermitean product on $\mathbb{C}^n$.

It is easy to check that the inverse of the element $z = (z', t)$ is $z^{-1} = (-z', -t)$, and the unitary element is $0 = (0, 0)$. The Haar measure $dz$ on $\mathbb{H}^n$ coincides with the Lebesgue measure $dz'dt$ on $\mathbb{C}^n \times \mathbb{R}$.

For each element $\zeta = (\zeta', t)$ of $\mathbb{H}^n$, the following affine self-mapping of $\mathbb{U}^n$:

$$(\zeta', t) : (z', z_{n+1}) \longmapsto \left(z' + \zeta', z_{n+1} + t + 2iz'\overline{\zeta'} + i\left|\zeta'\right|^2\right) \tag{1.4}$$

is an action of the group $\mathbb{H}^n$ on the space $\mathbb{U}^n$. Observe that the mapping (1.4) gives us a realization of $\mathbb{H}^n$ as a group of affine holomorphic bijections of $\mathbb{U}^n$ (see [5]).

The dilations on $\mathbb{H}^n$ are defined by $\delta z = \delta(z', t) = (\delta z', \delta^2 t)$, $\delta > 0$, and the rotation on $\mathbb{H}^n$ is defined by $\sigma z = \sigma(z', t) = (\sigma z', t)$ with a unitary map $\sigma$ of $\mathbb{C}^n$. The conjugation of $z$ is $\overline{z} = \overline{(z', t)} = (\overline{z'}, -t)$. The norm function is given by

$$|z| = \left(\left|z'\right|^4 + |t|^2\right)^{1/4}, \qquad z = (z', t) \in \mathbb{H}^n, \tag{1.5}$$

which is homogeneous of degree 1 and satisfies $|z^{-1}| = |z|$ and $|zw| \leq C(|z| + |w|)$ for some absolute constant $C$. The distance function $d(z, w)$ of point $z$ and $w$ in $\mathbb{H}^n$ is defined by $d(z, w) = |w^{-1}z|$.

For $z = (z', t) \in \mathbb{H}^n$ (or $z = (x, y, t) \in \mathbb{H}^n$ for $x = (x_1, x_2, , \ldots, x_n)$, $y = (y_1, y_2, , \ldots, y_n)$), we define

$$|z|_\infty = \max\left\{|x_1|, \ldots, |x_n|, \left|y_1\right|, \ldots, \left|y_n\right|, \sqrt{|t|}\right\}, \tag{1.6}$$

where $z' = (z_1, \ldots, z_n)$, and $z_k = x_k + iy_k$, $1 \leq k \leq n$.

To define the cube in $\mathbb{H}^n$, let $[x] = ([x_1], [x_2], \ldots, [x_n])$, where $[a]$ denotes the largest integer less than or equal to $a$, and $\langle x \rangle = x - [x]$. We define the function $F(x, y)$ by

$$F(x, y) = \sum_{j=1}^{\infty} \frac{2}{4^j}\left(\left\langle 2^j x\right\rangle\left(\left[2^j y\right] \bmod (2\mathbb{Z}^n)\right) - \left\langle 2^j y\right\rangle\left(\left[2^j x\right] \bmod (2\mathbb{Z}^n)\right)\right), \tag{1.7}$$

and set

$$I_0 = \left\{(x, y, t) \in \mathbb{H}^n : 0 \leq x_k \leq 1, 0 \leq y_k \leq 1,\ 1 \leq k \leq n,\ 0 \leq t - F(x, y) \leq 1\right\}, \tag{1.8}$$

$\Gamma = \{(m, n, l) \in \mathbb{H}^n : m, n \in \mathbb{Z}^n; l \in \mathbb{Z}\}$. Then

$$\mathbb{H}^n = \sum_{\gamma \in \Gamma} \gamma^{-1} I_0', \quad \text{(disjoint union)}, \tag{1.9}$$

where $I_0' = \{(x, y, t) \in \mathbb{H}^n : 0 \leq x_k < 1,\ 0 \leq y_k < 1,\ k = 1, 2, \ldots, n,\ 0 \leq t - F(x, y) < 1\}$ (see [14]). Now a "cube" (in fact, called "tile" more precisely) with center $w = (u, v, s)$ and edge sidelength $l$ is defined by $I = lw^{-1}I_0$. Let $|I|$ (with the Lebesgue measure) be the volume of $I$ with length $l(I)$. It is easy to see that $|I| = [l(I)]^{2n+2}$. Obviously, the diameter of $I$ denoted by $\mathrm{diam}(I)$ is equal to $c_n l(I)$, where $c_n$ is a constant depending only on $n$. The "dyadic cubes" on $\mathbb{H}^n$ can be defined by $I = 2^{-j}\gamma^{-1}I_0$, $\gamma \in \Gamma$, $j \in \mathbb{Z}$. We also call the "cube" and "dyadic cube" as cube and dyadic cube, respectively.

A ball in $\mathbb{H}^n$ with center $w$ and radius $r$ is denoted as $B = B(w, r) = \{z : |w^{-1}z| < r\}$. The tent based on the set $E \subset \mathbb{H}^n$ is defined by

$$T(E) = \left\{(\xi, \rho) \in \mathbb{U}^n : B(\xi, \rho) \subset E\right\}. \tag{1.10}$$

The Schwartz class of rapidly decreasing smooth function on $\mathbb{H}^n$ will be denoted by $\mathcal{S}(\mathbb{H}^n)$. The dual of $\mathcal{S}(\mathbb{H}^n)$ is $\mathcal{S}'(\mathbb{H}^n)$, the space of tempered distributions on $\mathbb{H}^n$.

## 2. Hausdorff Capacity on Heisenberg Group

In order to discuss the fractional Carleson measures on the Siegel upper half space, we introduce the $p$-dimensional Hausdorff capacity and Choquet integral on the Heisenberg group in the following.

*Definition 2.1.* For $p \in (0,1]$ and $E \subset \mathbb{H}^n$, the $p$-dimensional Hausdorff capacity of $E$ is defined by

$$\Lambda_p^{\infty}(E) = \inf\left\{ \sum_j |I_j|^p : E \subset \bigcup_{j=1}^{\infty} I_j \right\}, \tag{2.1}$$

where the infimum ranges only over covers of $E$ by dyadic cubes.

*Remark 2.2.* It is easy to check that $\Lambda_p^{\infty}$ satisfies the following properties.

(i) If $E_j$ is nondecreasing, then $\Lambda_p^{\infty}(\bigcup_j E_j) = \lim_{j\to\infty}\Lambda_p^{\infty}(E_j)$.

(ii) If $E_j$ is nonincreasing, then $\Lambda_p^{\infty}(\bigcap_j E_j) = \lim_{j\to\infty}\Lambda_p^{\infty}(E_j)$.

(iii) $\Lambda_p^{\infty}$ has the strong subadditivity condition.

$$\Lambda_p^{\infty}\left(E_1 \bigcup E_2\right) + \Lambda_p^{\infty}(E_1 \cap E_2) \le \Lambda_p^{\infty}(E_1) + \Lambda_p^{\infty}(E_2). \tag{2.2}$$

For a nonnegative function $f$ on $\mathbb{H}^n$, the Choquet integral with respect to the Hausdorff capacity on the Heisenberg group is defined by

$$\int_{\mathbb{H}^n} f(z)\,d\Lambda_p^{\infty} = \int_0^{\infty} \Lambda_p^{\infty}\left(\{z \in \mathbb{H}^n : f(z) > \lambda\}\right) d\lambda. \tag{2.3}$$

Since $\Lambda_p^{\infty}$ is monotonous, the integrand on the right hand side in (2.3) is nonincreasing and then is Lebesgue measurable on $[0,\infty)$. It is easy to see that the equality (2.3) with the Hausdorff measure is similar to the equality of the distribution function with Lebesgue measure.

In order to characterize the fractional Carleson measure in the Siegel upper half space in Section 3, we need to discuss the Hardy-Littlewood maximal operator on the Heisenberg group.

Let $f \in L^1_{\mathrm{loc}}(\mathbb{H}^n)$. $M(f)$, the dyadic Hardy-Littlewood maximal operator of $f$, is defined by

$$M(f)(z) = \sup_{I \ni z} \frac{1}{|I|} \int_I |f(\xi)|\,d\xi, \tag{2.4}$$

where the supremum is taken over all dyadic cubes $I$ containing $z$.

The boundedness of the dyadic Hardy-Littlewood maximal operators on the Choquet integral spaces $L^q_{\Lambda_p^{\infty}}(\mathbb{H}^n)$ is as follows.

**Theorem 2.3.** *Suppose $0 < p \le 1$. There exists a constant $C$ depending only on $p$ and $n$ such that*

$$\int_{\mathbb{H}^n} M(f)^q d\Lambda_p^\infty \le C\int_{\mathbb{H}^n} |f|^q d\Lambda_p^\infty, \quad \forall f \in L^q_{\Lambda_p^\infty}(\mathbb{H}^n),\ q > p, \tag{2.5}$$

$$\Lambda_p^\infty(\{\xi \in \mathbb{H}^n : M(f)(\xi) > \lambda\}) \le \frac{C}{\lambda^p}\int_{\mathbb{H}^n} |f|^p d\Lambda_p^\infty, \quad \forall f \in L^p_{\Lambda_p^\infty}(\mathbb{H}^n). \tag{2.6}$$

To prove Theorem 2.3, we need to prove the following lemmas first.

**Lemma 2.4.** *For $p \in (0,1]$, let $\{I_j\}$ be a sequence of dyadic cubes in $\mathbb{H}^n$ such that $\sum_j |I_j|^p < \infty$. Then there is a sequence of disjoint dyadic cubes $\{J_k\}$ so that $\bigcup_k J_k = \bigcup_j I_j$ and $\sum_k |J_k|^p \le \sum_j |I_j|^p$. Moreover, if $E \subset \bigcup_j I_j$, then the following tent inclusion $T(E) \subset \bigcup_k T(J_k^*)$ holds, where $J_k^*$ is the cube with the same center as $J_k$ and $\delta_n$ times the sidelength ($\delta_n$ is some constant depending only on $n$).*

*Proof.* Since $\sum_j |I_j|^p < \infty$, $p \in (0,1]$, we have that $\sum_j |I_j| < \infty$. Thus the union of $I_j$ cannot form arbitrarily large dyadic cubes. Thus, each $I_j$ must be included in some maximal dyadic cube $J_k \in \mathbb{J}$, where $\mathbb{J}$ denotes the collection of all dyadic cubes $J = \bigcup\{I_j : I_j \subset J\}$. If we write these maximal cubes as a sequence $\{J_k\}$, then $J_k$ is disjoint and $\bigcup_k J_k = \bigcup_j I_j$. By the definition of $J_k$ in $\mathbb{J}$, we know that $|J_k| \le \sum_{I_j \subset J_k} |I_j|$. Jensen's inequality gives $|J_k|^p \le \sum_{I_j \subset J_k} |I_j|^p$ for $0 < p \le 1$. Consequently,

$$\sum_k |J_k|^p \le \sum_k \sum_{I_j \subset J_k} |I_j|^p \le \sum_j |I_j|^p. \tag{2.7}$$

Suppose that $E \subset \bigcup_j I_j$ and $(\xi, \rho) \in T(E)$. Then $\xi \in E \subset \bigcup_k J_k$ and $\xi \in J_k$ for some fixed $k$. For this $J_k$, we consider the parent dyadic cube $J_k'$, namely, the unique dyadic cube containing $J_k$ whose sidelength is double of that of $J_k$. Since $J_k$ is maximal in $\mathbb{J}$, we have that $J_k'$ is not a union of the $I_j$'s, that is, $J_k'$ contains a point $\eta \in \mathbb{H}^n \setminus (\bigcup_j I_j) \subset \mathbb{H}^n \setminus E$. Denoting the boundary of $E$ by $\partial E$, there is

$$\rho < \operatorname{dist}(\xi, \partial E) \le \operatorname{diam}(J_k') = c_n l(J_k') = 2c_n l(J_k). \tag{2.8}$$

If $J_k^*$ is the cube with the same center as $J_k$, and $5c_n$ times the sidelength, then

$$\operatorname{dist}(\xi, \partial J_k^*) \ge \frac{1}{2}(5c_n - 1) l(J_k) \ge 2c_n l(J_k) > \rho, \quad (\text{since } c_n > 1), \tag{2.9}$$

which means that $(\xi, \rho) \in T(J_k^*)$. The proof is completed. □

**Lemma 2.5.** *Let $\chi_I$ be the characteristic function on the cube $I$. Then*

$$\int_{\mathbb{H}^n} M(\chi_I)^q d\Lambda_p^\infty \le C|I|^p, \quad q > p. \tag{2.10}$$

*Proof.* Let $z_I$ be the center of $I$. By the definition of the maximal function $M$, we obtain

$$M(\chi_I)(\xi) \leq C \inf\left\{1, \frac{|I|}{\left|z_I^{-1}\xi\right|^{(2n+2)}}\right\}. \tag{2.11}$$

Since $p/q < 1$, there is

$$\int_{\mathbb{H}^n} M(\chi_I)^q d\Lambda_p^\infty \leq C|I|^p + C\int_0^1 |I|^p \lambda^{-p/q} d\lambda = C|I|^p. \tag{2.12}$$

The proof of Lemma 2.5 is complete. □

**Lemma 2.6.** *Let $\{I_j\}$ be a family of nonoverlapping dyadic cubes. Then there is a maximal subfamily $\{I_{j_k}\}$ such that for every dyadic cube $I$,*

$$\sum_{I_{j_k}\subset I} \left|I_{j_k}\right|^p \leq 2|I|^p, \tag{2.13}$$

$$\Lambda_p^\infty\left(\bigcup I_j\right) \leq 2\sum_k \left|I_{j_k}\right|^p. \tag{2.14}$$

*Proof.* Similar to the proof in [16], if $I_{j_1} = I_1$, then obviously $I_{j_1}$ satisfies (2.13). If $j_1, \ldots, j_k$ have been chosen so that (2.13) holds, then we define $j_{k+1}$ as the first index such that the family $\{I_{j_1}, \ldots, I_{j_k}, I_{j_{k+1}}\}$ satisfies (2.13). Continuing this proceeding, therefore, we have that $\{I_{j_k}\}$ is a maximal subfamily of $\{I_j\}$ satisfying (2.13). Hence (2.13) holds.

To prove (2.14), let $j$ be an index such that $j_m < j < j_{m+1}$ for some $m \in \mathbb{Z}$. Then by the proof of (2.13), there exists a dyadic cube $I_j^* \supset I_j$ such that

$$\sum_{I_{j_k}\subset I_j^*, k\leq m} \left|I_{j_k}\right|^p + \left|I_j\right|^p > 2\left|I_j^*\right|^p. \tag{2.15}$$

Therefore,

$$\left|I_j^*\right|^p \leq \sum_{I_{j_k}\subset I_j^*, k\leq m} \left|I_{j_k}\right|^p. \tag{2.16}$$

We can assume that $\sum_k |I_{j_k}|^p < \infty$. Otherwise (2.14) is obviously correct. Then the sequence $\{|I_j^*|\}$ is bounded. Thus, we can consider the family $\{\widetilde{I}_l\}$ of maximal cubes of the family $\{I_j^*\}$. Hence

$$\bigcup_j I_j \subset \left(\bigcup_k I_{j_k}\right) \bigcup \left(\bigcup_l \widetilde{I}_l\right). \tag{2.17}$$

Consequently, by the definition of $\Lambda_p^\infty$, we obtain

$$\Lambda_p^\infty\left(\bigcup I_j\right) \le 2\sum_k |I_{j_k}|^p. \tag{2.18}$$

The proof is complete. □

*Proof of Theorem 2.3.* By the definition of $\Lambda_p^\infty$, for any $\varepsilon > 0$, there exist $\{I_j\}_j$ such that

$$\sum_j |I_j|^p < \Lambda_p^\infty(E_k) + \varepsilon, \tag{2.19}$$

where $E_k = \{\xi \in \mathbb{H}^n : 2^k < |f(\xi)| \le 2^{k+1}\}$. If $\Lambda_p^\infty(E_k) = 0$, we can choose $\sum_j |I_j|^p = 0$. If $\Lambda_p^\infty(E_k) > 0$, then

$$\sum_j |I_j|^p \le 2\Lambda_p^\infty(E_k). \tag{2.20}$$

By Lemma 2.4, for each integer $k$, there is a family of nonoverlapping dyadic cubes $\{I_j^{(k)}\}$ such that

$$\begin{gathered}\left\{\xi \in \mathbb{H}^n : 2^k < \left|f(\xi)\right| \le 2^{k+1}\right\} \subset \bigcup_j I_j^{(k)}, \\ \sum_j \left|I_j^{(k)}\right|^p \le 2\Lambda_p^\infty\left(\left\{\xi \in \mathbb{H}^n : 2^k < \left|f(\xi)\right| \le 2^{k+1}\right\}\right).\end{gathered} \tag{2.21}$$

Set $g = \sum_k 2^{(k+1)q}\chi_{F_k}$, where $\chi_{F_k}$ is the characteristic function of $F_k = \bigcup_j I_j^{(k)}$. Then $|f|^q \le g$.

First, assume that $q > 1$. Then the Hölder inequality tells us

$$M(f)^q \le M(|f|^q) \le M(g) \le \sum_k 2^{(k+1)q}\sum_j M\left(\chi_{I_j^{(k)}}\right). \tag{2.22}$$

Therefore, by Lemma 2.5, we have

$$\begin{aligned}\int_{\mathbb{H}^n} M(f)^q d\Lambda_p^\infty &\le C\sum_k 2^{(k+1)q}\sum_j \int_{\mathbb{H}^n} M\left(\chi_{I_j^{(k)}}\right) d\Lambda_p^\infty \\ &\le C\sum_k 2^{(k+1)q}\sum_j \left|I_j^{(k)}\right|^p \le C\sum_k 2^{(k+1)q}\Lambda_p^\infty(E_k) \\ &\le C\sum_k \frac{2^{2q}}{2^q - 1}\int_{2^{(k-1)q}}^{2^{kq}} \Lambda_p^\infty(\{\xi \in \mathbb{H}^n : |f(\xi)|^q > \lambda\})d\lambda \\ &\le C\int_0^\infty \Lambda_p^\infty(\{\xi \in \mathbb{H}^n : |f(\xi)|^q > \lambda\})d\lambda = C\int_{\mathbb{H}^n} |f|^q d\Lambda_p^\infty.\end{aligned} \tag{2.23}$$

Now, assume that $p < q \le 1$. Since $|f| \le \sum_k 2^{k+1}\chi_{F_k}$, there is

$$M(f) \le \sum_k 2^{k+1} \sum_j M\left(\chi_{I_j^{(k)}}\right). \tag{2.24}$$

By using Jensen's inequality, we obtain

$$M(f)^q \le \sum_k 2^{(k+1)q} \sum_j M\left(\chi_{I_j^{(k)}}\right)^q. \tag{2.25}$$

Thus, also by Lemma 2.5, there is

$$\begin{aligned}\int_{\mathbb{H}^n} M(f)^q d\Lambda_p^\infty &\le C\sum_k 2^{(k+1)q} \sum_j \int_{\mathbb{H}^n} M\left(\chi_{I_j^{(k)}}\right) d\Lambda_p^\infty \\ &\le C\sum_k 2^{(k+1)q} \sum_j \left|I_j^{(k)}\right|^p \le C\sum_k 2^{(k+1)q}\Lambda_p^\infty(E_k) \\ &\le C\int_0^\infty \Lambda_p^\infty\left(\left\{\xi \in \mathbb{H}^n : \left|f(\xi)\right|^q > \lambda\right\}\right) d\lambda = C\int_{\mathbb{H}^n} |f|^q d\Lambda_p^\infty.\end{aligned} \tag{2.26}$$

That means that (2.5) holds.

For a given $\lambda > 0$, let $\{I_j\}$ be the family of maximal dyadic cubes $I_j$ such that

$$\frac{1}{|I_j|}\int_{I_j} |f(\xi)| d\xi > \lambda. \tag{2.27}$$

Note that $M(f)$ is dyadic and $\{z \in \mathbb{H}^n : M(f)(z) > \lambda\} = \bigcup_j I_j$. Since $0 < p \le 1$, there is

$$\sum_j |I_j| \le \left(\sum_j |I_j|^p\right)^{1/p}, \qquad \Lambda_1^\infty(E) \le \Lambda_p^\infty(E)^{1/p}. \tag{2.28}$$

Hence

$$\begin{aligned}
\int_{\mathbb{H}^n} |f(z)|dz &= \int_0^\infty \Lambda_1^\infty(\{\xi \in \mathbb{H}^n : |f(\xi)| > \lambda\})d\lambda \\
&= \frac{1}{p}\int_0^\infty \Lambda_1^\infty\left(\left\{\xi \in \mathbb{H}^n : |f(\xi)| > \rho^{1/p}\right\}\right)\rho^{(1/p)-1}d\rho \\
&\leq \frac{1}{p}\int_0^\infty \Lambda_p^\infty\left(\left\{\xi \in \mathbb{H}^n : |f(\xi)| > \rho^{1/p}\right\}\right)^{1/p}\rho^{(1/p)-1}d\rho \\
&= \frac{1}{p}\int_0^\infty \left[\Lambda_p^\infty(\{\xi \in \mathbb{H}^n : f(\xi)|^p > \rho\})\rho\right]^{(1/p)-1}\Lambda_p^\infty(\{\xi \in \mathbb{H}^n : |f(\xi)|^p > \rho\})d\rho \\
&\leq \frac{1}{p}\left(\int_{\mathbb{H}^n} |f|^p \, d\Lambda_p^\infty\right)^{(1/p)-1}\int_0^\infty \Lambda_p^\infty(\{\xi \in \mathbb{H}^n : |f(\xi)|^p > \rho\})d\rho \\
&= \frac{1}{p}\left(\int_{\mathbb{H}^n} |f|^p \, d\Lambda_p^\infty\right)^{1/p},
\end{aligned} \tag{2.29}$$

where the last inequality is due to the fact that

$$\Lambda_p^\infty(\{\xi \in \mathbb{H}^n : |f(\xi)|^p > \rho\})\rho \leq \int_{\{\xi\in\mathbb{H}^n:|f(\xi)|^p>\rho\}} |f|^p d\Lambda_p^\infty, \qquad \rho > 0. \tag{2.30}$$

Therefore, by (2.27) and (2.29), we obtain

$$|I_j|^p \leq \left(\frac{1}{\lambda}\int_{I_j} |f|d\xi\right)^p \leq C\lambda^{-p}\int_{I_j} |f|^p d\Lambda_p^\infty. \tag{2.31}$$

For the above $\{I_j\}$, by Lemma 2.6 and (2.31), there exists a subfamily $\{I_{j_k}\}$ such that

$$\begin{aligned}
\Lambda_p^\infty(\{z \in \mathbb{H}^n : M(f)(z) > \lambda\}) = \Lambda_p^\infty\left(\bigcup_j I_j\right) &\leq 2\sum_k |I_{j_k}|^p \\
&\leq C\lambda^{-p}\sum_k \int_{I_{j_k}} |f|^p d\Lambda_p^\infty \leq C\lambda^{-p}\int_{\mathbb{H}^n} |f|^p d\Lambda_p^\infty.
\end{aligned} \tag{2.32}$$

The proof of Theorem 2.3 is complete. □

## 3. $p$-Carleson Measure on Siegel Upper Half Space

Let $I$ be a cube in $\mathbb{H}^n$ with center $\eta$. The Carleson box based on $I$ is defined by

$$S(I) = \left\{(\xi, \rho) \in \mathbb{U}^n : \left|\eta^{-1}\xi\right|_\infty \leq \frac{l(I)}{2}, \ \rho \leq l(I)\right\}. \tag{3.1}$$

For $p > 0$, a positive Borel measure $\mu$ on the Siegel upper half space $\mathbb{U}^n$ is called a $p$-Carleson measure if there exists a constant $C > 0$ such that

$$\mu(S(I)) \leq C|I|^p, \quad \forall \text{cubes } I \subset \mathbb{H}^n. \tag{3.2}$$

For $z \in \mathbb{H}^n$, let $\Gamma(z) = \{(\xi, \rho) \in \mathbb{U}^n : |z^{-1}\xi| < \rho\}$ be the cone at $z$. Suppose that $f$ is a measurable function on $\mathbb{U}^n$. The nontangential maximal function $N(f)$ of $f$ is defined by

$$N(f)(z) = \sup_{(\xi,\rho)\in\Gamma(z)} |f(\xi, \rho)|. \tag{3.3}$$

Since the nontangential maximal function and Hausdorff capacity are defined, we are paying attention on the characterizations of the $p$-Carleson measures.

**Theorem 3.1.** *For $p \in (0, 1]$, let $\mu$ be a positive Borel measure on $\mathbb{U}^n$. Then the following three conclusions are equivalent.*

(i) *$\mu$ is a $p$-Carleson measure.*

(ii) *There exists a constant $C > 0$ such that*

$$\int_{\mathbb{U}^n} |f(\xi, \rho)| d\mu \leq C \int_{\mathbb{H}^n} N(f) d\Lambda_p^{(\infty)} \tag{3.4}$$

*holds for all Borel measurable functions $f$ on $\mathbb{U}^n$.*

(iii) *For every $q > 0$, there exists a constant $C > 0$ such that*

$$\int_{\mathbb{U}^n} |f(\xi, \rho)|^q d\mu \leq C \int_{\mathbb{H}^n} (N(f))^q d\Lambda_p^{(\infty)} \tag{3.5}$$

*holds for all Borel measurable functions $f$ on $\mathbb{U}^n$.*

*Proof.* (i)$\Rightarrow$(ii). Assume that $\mu$ is a $p$-Carleson measure, and $f$ is a Borel measurable function on $\mathbb{U}^n$. For $\lambda > 0$, let $E_\lambda = \{z \in \mathbb{H}^n : N(f)(z) > \lambda\}$. If the integral on the right hand side of (3.4) is finite, we may assume that $\Lambda_p^{(\infty)}(E_\lambda) < \infty$. Let $\{I_j\}$ be any of the dyadic cubes covering of $E_\lambda$ with $\sum_j |I_j|^p < \infty$. Then Lemma 2.4 tells us that there is a sequence $\{J_k\}$ of dyadic cubes with mutually disjointed so that $\bigcup_k J_k = \bigcup_j I_j$, $\sum_k |J_k|^p \leq \sum_j |I_j|^p$ and $T(E_\lambda) \subset \bigcup_k T(J_k^*)$, where $J_k^*$ is the cube with the same center and $5c_n$ times sidelength of $J_k$.

If $(\xi, \rho) \in \mathbb{U}^n$ satisfies $|f(\xi, \rho)| > \lambda$, then $N(f)(z) > \lambda$ for all $z \in B(\xi, \rho)$. Thus $(\xi, \rho) \in T(E_\lambda)$, and hence

$$\{(\xi, \rho) \in \mathbb{U}^n : |f(\xi, \rho)| > \lambda\} \subset T(E_\lambda) \subset \bigcup_k T(J_k^*). \tag{3.6}$$

By Lemma 2.4, we obtain

$$\begin{aligned}\mu(\{(\xi,\rho)\in\mathbb{U}^n : |f(\xi,\rho)|>\lambda\}) &\leq \mu\left(\bigcup_k T(J_k^*)\right) \leq \sum_k \mu(T(J_k^*)) \leq C\|\mu\|\sum_k |J_k^*|^p \\ &\leq C\|\mu\|\sum_j \left|J_j^*\right|^p.\end{aligned} \tag{3.7}$$

Taking an infimum over all such dyadic cube coverings, we have

$$\mu(\{(\xi,\rho)\in\mathbb{U}^n : |f(\xi,\rho)|>\lambda\}) \leq C\|\mu\|\Lambda_p^{(\infty)}(E_\lambda). \tag{3.8}$$

Therefore,

$$\begin{aligned}\int_{\mathbb{U}^n}|f(\xi,\rho)|d\mu &= \int_0^\infty \mu(\{(\xi,\rho)\in\mathbb{U}^n : |f(\xi,\rho)|>\lambda\})d\lambda \\ &\leq C\int_0^\infty \|\mu\|\Lambda_p^{(\infty)}(E_\lambda)d\lambda = C\|\mu\|\int_{\mathbb{H}^n} N(f)d\Lambda_p^{(\infty)}.\end{aligned} \tag{3.9}$$

(ii)⇒(i). Suppose that (3.4) is valid for all Borel measurable functions on $\mathbb{U}^n$. For a cube $I\subset\mathbb{H}^n$, let $\phi(\xi,\rho)=\chi_{T(I)}$. Note that $(\xi,\rho)\in T(I)\cap\Gamma(z)$ if and only if $z\in B(\xi,\rho)\subset I$, there is $N(\phi)=\chi_I$. Then, by (3.4), we have

$$\mu(T(I)) \leq C\int_0^\infty \Lambda_p^{(\infty)}(\{N(\phi)>\lambda\})d\lambda = C\int_0^1 \Lambda_p^{(\infty)}(I)d\lambda \leq C|I|^p. \tag{3.10}$$

It means that $\mu$ is a $p$-Carleson measure.

(ii)⇒(iii). Replacing $|f|$ with $|f|^q$ in (3.4) for $q>0$, we immediately obtain

$$\int_{\mathbb{U}^n}|f(\xi,\rho)|^q d\mu \leq C\int_{\mathbb{H}^n}(N(f))^q d\Lambda_p^{(\infty)}. \tag{3.11}$$

(iii)⇒(ii). If we set $q=1$ in (3.5), then (3.4) holds.

The proof of Theorem 3.1 is complete. □

To continue the characterization of the $p$-Carleson measures on the Siegel upper half space, we need to prove the following Lemma 3.2. It is said that the nontangential maximal functions are dominated by the dyadic Hardy-Littlewood maximal operators on the Heisenberg group.

**Lemma 3.2.** *Let $f$ be a locally integrable function on $\mathbb{H}^n$, and $\phi \in \mathcal{S}(\mathbb{H}^n)$ be a nonnegative radial and decreasing function with $\int_{\mathbb{H}^n} \phi(z)\, dz = 1$. Then*

$$N(F)(z) = \sup_{(\xi,\rho)\in\Gamma(z)} \left|F(\xi,\rho)\right| = \sup_{(\xi,\rho)\in\Gamma(z)} \left|f * \phi_\rho(\xi)\right| \le CM(f)(z), \tag{3.12}$$

*where $\phi_\rho(\xi) = \rho^{-(2n+2)}\phi(\rho^{-1}\xi)$ and $\rho > 0$ is real.*

*Proof.* Since $\phi \in \mathcal{S}(\mathbb{H}^n)$, we can suppose that $|\phi(z)| \le C/(1+|z|)^{2n+2+\varepsilon}$ for some $\varepsilon > 0$. Then

$$\begin{aligned}
\left|f * \phi_\rho(\xi)\right| &= \left|\int_{\mathbb{H}^n} \rho^{-(2n+2)}\phi\left(\rho^{-1}\left(\eta^{-1}\xi\right)\right) f(\eta) d\eta\right| \\
&\le \int_{|\eta^{-1}\xi|\le\rho} \rho^{-(2n+2)}\left|\phi\left(\rho^{-1}\left(\eta^{-1}\xi\right)\right)\right| |f(\eta)| d\eta \\
&\quad + \sum_{k=0}^{\infty} \int_{2^k\rho<|\eta^{-1}\xi|\le 2^{k+1}\rho} \rho^{-(2n+2)}\left|\phi\left(\rho^{-1}\left(\eta^{-1}\xi\right)\right)\right| |f(\eta)| d\eta \\
&\le \frac{C}{\rho^{2n+2}} \int_{|\eta^{-1}\xi|\le\rho} |f(\eta)| d\eta + C\sum_{k=0}^{\infty} \int_{2^k\rho<|\eta^{-1}\xi|\le 2^{k+1}\rho} \frac{\rho^{-(2n+2)}|f(\eta)|}{\left(1+\left(\left(\eta^{-1}\xi\right)/\rho\right)\right)^{2n+2+\varepsilon}} d\eta \\
&\le CM(f)(\xi) + C\sum_{k=0}^{\infty} \frac{1}{\rho^{2n+2}} \int_{|\eta^{-1}\xi|\le 2^{k+1}\rho} \frac{1}{2^{k(2n+2+\varepsilon)}} |f(\eta)| d\eta \\
&\le C' M(f)(\xi) \sum_{k=0}^{\infty} 2^{-(k+1)\varepsilon} = C'' M(f)(\xi).
\end{aligned} \tag{3.13}$$

Hence

$$N(F)(z) = \sup_{(\xi,\rho)\in\Gamma(z)} \left|f * \phi_\rho(\xi)\right| \le CM(f)(z). \tag{3.14}$$

Lemma 3.2 is proved. □

**Theorem 3.3.** *For $0 < p \le 1$, $q > p$, let $\mu$ be a positive Borel measure on $\mathbb{U}^n$. Then $\mu$ is a $p$-Carleson measure if and only if*

$$\int_{\mathbb{U}^n} |G(\xi,\rho)|^q d\mu \le C \int_{\mathbb{H}^n} |g|^q d\Lambda_p^{(\infty)} \tag{3.15}$$

*holds for all functions $G$ on $\mathbb{U}^n$ which can be written as $G(\xi,\rho) = g * \phi_\rho(\xi)$, where $g$ is a locally integrable function on $\mathbb{H}^n$, and $\phi$ is the function in Lemma 3.2.*

*Proof.* Suppose that $\mu$ is a $p$-Carleson measure. By the conclusions of Theorems 3.1 and 2.3 and Lemma 3.2, there is

$$\int_{\mathbb{U}^n} |G(\xi,\rho)|^q d\mu \leq C\int_{\mathbb{H}^n} (N(G))^q d\Lambda_p^{(\infty)} \leq C\int_{\mathbb{H}^n} (M(g))^q d\Lambda_p^{(\infty)} \leq C\int_{\mathbb{H}^n} |g|^q d\Lambda_p^{(\infty)}. \quad (3.16)$$

Conversely, for any cube $I \subset \mathbb{H}^n$, if we set $g = \chi_I$, then for $(\xi,\rho) \in T(I)$, there is

$$\begin{aligned} G(\xi,\rho) &= \int_I \phi_\rho\left(\eta^{-1}\xi\right)d\eta \geq \int_{B(\xi,\rho)} \phi_\rho\left(\eta^{-1}\xi\right)d\eta = \int_{B(\xi,1)} \phi_1\left(\eta^{-1}\xi\right)d\eta \\ &= \int_{B(0,1)} \phi(z)dz = c. \end{aligned} \quad (3.17)$$

Thus, by using the inequality (3.15), we obtain

$$\begin{aligned} c^q\mu(T(I)) = c^q\int_{T(I)} d\mu &\leq \int_{T(I)} |G(\xi,\rho)|^q d\mu \leq \int_{\mathbb{U}^n} |G(\xi,\rho)|^q d\mu \leq C\int_{\mathbb{H}^n} |g|^q d\Lambda_p^{(\infty)} \\ &= C\int_{\mathbb{H}^n} \chi_I d\Lambda_p^{(\infty)} = C\Lambda_p^{(\infty)}(I) \leq C|I|^p. \end{aligned} \quad (3.18)$$

This ends the proof of Theorem 3.3. □

Let $\mathbb{N}(\Lambda_p^{(\infty)})$ be the space of all Borel measurable functions $f$ on $\mathbb{U}^n$ satisfying

$$\|f\|_{\mathbb{N}(\Lambda_p^{(\infty)})} = \int_{\mathbb{H}^n} N(f)d\Lambda_p^{(\infty)} < \infty. \quad (3.19)$$

Then $\|\cdot\|_{\mathbb{N}(\Lambda_p^{(\infty)})}$ gives a (quasi) norm and $\mathbb{N}(\Lambda_p^{(\infty)})$ is complete.

The following result says that $\mathbb{N}(\Lambda_p^{(\infty)})$ is the dual of $p$-Carleson measure.

**Theorem 3.4.** *Let $p \in (0,1]$. Then there exists a duality between the space of $p$-Carleson measures and $\mathbb{N}(\Lambda_p^{(\infty)})$ in the following sense.*

(i) *Every $p$-Carleson measure $\mu$ on $\mathbb{U}^n$ defines a bounded linear functional on $\mathbb{N}(\Lambda_p^{(\infty)})$ via the pairing*

$$\langle \mu, f\rangle = \int_{\mathbb{U}^n} f(\xi,\rho)d\mu. \quad (3.20)$$

(ii) *Let $\mathbb{N}_0(\Lambda_p^{(\infty)})$ be the closure in $\mathbb{N}(\Lambda_p^{(\infty)})$ of the continuous functions with compact support in $\mathbb{U}^n$. Then every bounded linear functional on $\mathbb{N}_0(\Lambda_p^{(\infty)})$ given via the pairing* (3.20) *by a Borel measure $\mu$ on $\mathbb{U}^n$ is a $p$-Carleson measure.*

*Proof.* Assume that $\mu$ is a $p$-Carleson measure and $f \in \mathbb{N}(\Lambda_p^{(\infty)})$. By (3.19) and (ii) in Theorem 3.1, we have

$$\int_{\mathbb{U}^n} |f(\xi,\rho)| d\mu \leq C \int_{\mathbb{H}^n} N(f) d\Lambda_p^{(\infty)} < \infty. \tag{3.21}$$

Thus, $\langle \mu, f \rangle = \int_{\mathbb{U}^n} f(\xi,\rho) d\mu$ is well defined. Hence, every $p$-Carleson measure $\mu$ defines a bounded linear functional on $\mathbb{N}(\Lambda_p^{(\infty)})$. The part (i) is proved.

For part (ii), let $L$ be a bounded linear functional on $\mathbb{N}_0(\Lambda_p^{(\infty)})$. By the continuity and the closure of $\mathbb{N}_0(\Lambda_p^{(\infty)})$, applying the Riesz representation theorem, we obtain a Borel measure $\mu$ on $\mathbb{U}^n$ having

$$L(f) = \int_{\mathbb{U}^n} f(\xi,\rho) d\mu = \langle \mu, f \rangle. \tag{3.22}$$

If $f = \chi_{T(I)}$ for any cube $I \in \mathbb{H}^n$, then

$$\begin{aligned} \mu(T(I)) &= \int_{\mathbb{U}^n} \chi_{T(I)} d\mu = L(\chi_{T(I)}) \leq \|L\| \cdot \|\chi_{T(I)}\|_{\mathbb{N}(\Lambda_p^{(\infty)})} \\ &= \|L\| \int_{\mathbb{H}^n} N(\chi_{T(I)}) d\Lambda_p^{(\infty)} = C\|L\| \int_I d\Lambda_p^{(\infty)} = C\|L\|\Lambda_p^{(\infty)}(I) \leq C\|L\| \cdot |I|^p. \end{aligned} \tag{3.23}$$

Hence, $\mu$ is a $p$-Carleson measure with $\|\mu\| \leq C\|L\|$. This ends the proof of Theorem 3.4. □

## 4. Tent Spaces with Hausdorff Capacity

With Hausdorff capacity on the Heisenberg group discussed above, in this section, we introduce the tent spaces on the Siegel upper half space, an analogy of the Coifman-Meyer-Stein tent space on $\mathbb{R}^n$ (cf. [4, 6]). Then the atomic decomposition of the tent spaces and the duality of the tent space are discussed.

*Definition 4.1.* Let $p \in (0,1]$. A Lebesgue measurable function $f$ on $\mathbb{U}^n$ is said to belong to $T_p^\infty$ if

$$\|f\|_{T_p^\infty} = \sup_{B \subset \mathbb{H}^n} \left( \frac{1}{|B|^p} \int_{T(B)} |f(\xi,\rho)|^2 \frac{d\xi d\rho}{\rho^{1+2(n+1)(1-p)}} \right)^{1/2} < \infty, \tag{4.1}$$

where $B$ runs over all balls in $\mathbb{H}^n$, and $T(B)$ is a tent based on $B$.

*Definition 4.2.* Let $p \in (0,1]$. The tent space $T_p^1$ consists of all measurable functions $f$ on $\mathbb{U}^n$ for which

$$\|f\|_{T_p^1} = \inf_w \left( \int_{\mathbb{U}^n} |f(\xi,\rho)|^2 w(\xi,\rho)^{-1} \frac{d\xi d\rho}{\rho^{1-2(n+1)(1-p)}} \right)^{1/2} < \infty, \tag{4.2}$$

where the infimum is taken over all nonnegative Borel measurable functions $w$ on $\mathbb{U}^n$ satisfying

$$\int_{\mathbb{H}^n} N(w)\,d\Lambda_p^{(\infty)} \leq 1, \tag{4.3}$$

and $w$ is allowed to vanish only where $f$ vanishes.

We will identify $T_p^{\infty}$ with a dual space of $T_p^1$. In order to do this, we first introduce the $T_p^1$-atom as follows.

*Definition 4.3.* A function $a$ on $\mathbb{U}^n$ is said to be a $T_p^1$-atom, if there exists a ball $B \subset \mathbb{H}^n$ such that $a$ is supported in the tent $T(B)$ and satisfies

$$\int_{T(B)} \left|a(\xi,\rho)\right|^2 \frac{d\xi d\rho}{\rho^{1-2(n+1)(1-p)}} \leq \frac{1}{|B|^p}. \tag{4.4}$$

For the tent space $T_p^1$ on $\mathbb{U}^n$ and $\|\cdot\|_{T_p^1}$, we have the triangle inequality with a constant in the following lemma.

**Lemma 4.4.** *Let $p \in (0,1]$. If $\sum_j \|g_j\|_{T_p^1} < \infty$, then $g = \sum_j g_j \in T_p^1$ and*

$$\|g\|_{T_p^1} \leq C \sum_j \|g_j\|_{T_p^1}. \tag{4.5}$$

*Proof.* Without loss of generality, we assume that $\lambda_j = \|g_j\|_{T_p^1} > 0$ for all $j$. Set $f_j = \|g\|_{T_p^1}^{-1} g_j$. Then $\|f_j\|_{T_p^1} \leq 1$ and $g = \sum_j \lambda_j f_j$. Suppose that $w_j \geq 0$ for all $j$ such that $\int_{\mathbb{H}^n} N(w_j)\,d\Lambda_p^{(\infty)} \leq 1$ and

$$\int_{\mathbb{U}^n} \left|g_j(\xi,\rho)\right|^2 w_j(\xi,\rho)^{-1} \frac{d\xi d\rho}{\rho^{1-2(n+1)(1-p)}} \leq 2\|g_j\|_{T_p^1}^2. \tag{4.6}$$

According to the definition of $T_p^1$, the above inequality holds obviously. By using the Cauchy-Schwarz inequality, we have

$$|g|^2 \leq \left(\sum_j \lambda_j w_j\right)\left(\sum_j \lambda_j \left|f_j\right|^2 w_j^{-1}\right). \tag{4.7}$$

Let $w = \left(\sum_j \lambda_j\right)^{-1} \sum_j \lambda_j w_j$. Notice that the vanishing of $w$ implies the vanishing of all $w_j$, which can only happen whenever all the $g_j$ vanish, that is, $g = 0$, then

$$\int_{\mathbb{H}^n} N(w)\,d\Lambda_p^{(\infty)} \leq \left(\sum_j \lambda_j\right)^{-1} \sum_j \lambda_j \int_{\mathbb{H}^n} N(w_j)\,d\Lambda_p^{(\infty)} \leq 1. \tag{4.8}$$

Thus, by the inequality (4.7), we obtain

$$\begin{aligned}
&\int_{\mathbb{U}^n} |g(\xi,\rho)|^2 w(\xi,\rho)^{-1} \frac{d\xi d\rho}{\rho^{1-2(n+1)(1-p)}} \\
&\quad \le C\left(\sum_j \lambda_j\right)\sum_j \lambda_j \int_{\mathbb{H}^n} |f_j(\xi,\rho)|^2 w_j(\xi,\rho)^{-1} \frac{d\xi d\rho}{\rho^{1-2(n+1)(1-p)}} \\
&\quad \le 2C\left(\sum_j \lambda_j\right)\sum_j \lambda_j \|f_j\|_{T_p^1}^2 = C\left(\sum_j \lambda_j\right)^2 = C\left(\sum_j \|g_j\|_{T_p^1}\right)^2.
\end{aligned} \tag{4.9}$$

Taking the infimum on the left above inequality, we have

$$\|g\|_{T_p^1} \le C \sum_j \|g_j\|_{T_p^1}. \tag{4.10}$$

The proof of Lemma 4.4 is complete. □

*Remark 4.5.* By Lemma 4.4, one can show that $\|\cdot\|_{T_p^1}$ is a quasinorm and the tent space $T_p^1$ is complete.

The main result of this section is the atomic decomposition of the tent space $T_p^1$ as follows.

**Theorem 4.6.** *Let $p \in (0,1]$. Then a function $f$ on $\mathbb{U}^n$ belongs to $T_p^1$ if and only if there exist a sequence of $T_p^1$-atoms $\{a_j\}$ and an $l^1$-sequence $\{\lambda_j\}$ such that*

$$f = \sum_j \lambda_j a_j. \tag{4.11}$$

*Moreover,*

$$\|f\|_{T_p^1} \approx \inf\left\{\sum_j |\lambda_j| : f = \sum_j \lambda_j a_j\right\}, \tag{4.12}$$

*where the infimum is taken over all possible atomic decompositions of $f \in T_p^1$.*

Note that the right hand side of (4.12) in fact defines a norm and then $T_p^1$ becomes a Banach space.

*Proof.* Suppose that $a$ is a $T_p^1$-atom on $\mathbb{U}^n$. Then there exists a ball $B = B(z,r) \subset \mathbb{H}^n$ such that supp $a \subset T(B)$ and

$$\int_{T(B)} |a(\xi,\rho)|^2 \frac{d\xi d\rho}{\rho^{1-2(n+1)(1-p)}} \le \frac{1}{|B|^p}. \tag{4.13}$$

Fix $\varepsilon > 0$, and let

$$w(\xi,\rho) = hr^{-(2n+2)p} \min\left\{1, \left(\frac{r}{D(\xi,z)}\right)^{(2n+2)p+\varepsilon}\right\}, \tag{4.14}$$

where $D(\xi,z)$ denotes the distance between $(\xi,\rho)$ and $(z,0)$ in $\mathbb{U}^n$, and $h$ is a suitable constant which will be chosen later. Obviously, $w(\xi,\rho)$ is identically equal to $hr^{-(2n+2)p}$ on the upper half ball of radius $r$ (center is $(z,0)$, and $D(\xi,z) \le r$) and decays radially outside the ball (outside the ball $D(\xi,z) > r$, and $D(\xi,z) \to +\infty$). For $\xi \in \mathbb{H}^n$, the distance in $\mathbb{U}^n$ from the cone $\Gamma(\xi)$ to $(z,0)$ is $c|z^{-1}\xi| = cd(z,\xi)$. Then

$$N(w)(z) \le hr^{-(2n+2)p} \min\left\{1, \left(\frac{r}{cd(\xi,z)}\right)^{(2n+2)p+\varepsilon}\right\}. \tag{4.15}$$

Therefore,

$$\begin{aligned} h^{-1}\int_{\mathbb{H}^n} N(w)d\Lambda_p^{(\infty)} &= \int_0^\infty \Lambda_p^{(\infty)}\left(\left\{\xi \in \mathbb{H}^n : h^{-1}N(w) > \lambda\right\}\right)d\lambda \\ &\le \int_0^{r^{-(2n+2)p}} \Lambda_p^{(\infty)}\left[B\left(z, \frac{1}{c}\left(\frac{r^\varepsilon}{\lambda}\right)^{1/(2n+2)p+\varepsilon}\right)\right]d\lambda \\ &\le C\int_0^{r^{-(2n+2)p}} \widetilde{\Lambda}_p^{(\infty)}\left[B\left(z, \frac{1}{c}\left(\frac{r^\varepsilon}{\lambda}\right)^{1/(2n+2)p+\varepsilon}\right)\right]d\lambda \\ &\le C\left(r^{\varepsilon/(2n+2)p+\varepsilon}\right)^{(2n+2)p}\int_0^{r^{-(2n+2)p}} \lambda^{-(2n+2)p/((2n+2)p+\varepsilon)}d\lambda = C. \end{aligned} \tag{4.16}$$

Thus, $\int_{\mathbb{H}^n} N(w)\, d\Lambda_p^{(\infty)} \le 1$ by choosing $h = C^{-1}$. On the other hand, let $w^{-1} = r^{(2n+2)p}$ on $T(B)$. We have

$$\begin{aligned} \int_{T(B)} \left|a(\xi,\rho)\right|^2 w(\xi,\rho)^{-1}\frac{d\xi d\rho}{\rho^{1-2(n+1)(1-p)}} &= r^{(2n+2)p}\int_{T(B)} \left|a(\xi,\rho)\right|^2\frac{d\xi d\rho}{\rho^{1-2(n+1)(1-p)}} \\ &\le r^{(2n+2)p}|B|^{-p} = C. \end{aligned} \tag{4.17}$$

Therefore, $a \in T_p^1$ and $\|a\|_{T_p^1} \le C$.

Now, taking a sum $\sum_j \lambda_j a_j$, where $\sum_j |\lambda_j| < \infty$ and every $a_j$ is $T_p^1$-atom on $\mathbb{U}^n$, by Lemma 4.4, the sum converges in the quasinorm to $f \in T_p^1$ with

$$\|f\|_{T_p^1} \le \sum_j |\lambda_j| \|a_j\|_{T_p^1} \le C\sum_j |\lambda_j| < \infty. \tag{4.18}$$

That means $f = \sum_j \lambda_j a_j \in T_p^1$.

Conversely, let $f \in T_p^1$. We choose a Borel measurable function $w \geq 0$ on $\mathbb{U}^n$ such that (4.3) holds and

$$\int_{\mathbb{U}^n} \left|f(\xi,\rho)\right|^2 w(\xi,\rho)^{-1} \frac{d\xi d\rho}{\rho^{1-2(n+1)(1-p)}} \leq 2\|f\|_{T_p^1}^2. \tag{4.19}$$

For each $k \in \mathbb{Z}$, let $E_k = \{Z \in \mathbb{H}^n : N(w)(z) > 2^k\}$. By Lemma 2.4 and the definition of $\Lambda_p^{(\infty)}$, it follows that there exists a disjoint dyadic cubes sequence $\{I_{k,j}\}$ such that

$$\sum_j \left|I_{k,j}\right|^p \leq 2\Lambda_p^{(\infty)}(E_k), \qquad T(E_k) \subset \bigcup_j S^*(I_{k,j}), \tag{4.20}$$

where $S^*(I_{k,j}) = \{(\xi,\rho) \in \mathbb{U}^n : \xi \in I_{k,j}, \rho < 2\,\mathrm{diam}(I_{k,j})\}$. Define

$$T_{k,j} = S^*(I_{k,j}) \setminus \bigcup_{m>k} \bigcup_l S^*(I_{m,l}). \tag{4.21}$$

Then $T_{k,j}$ is disjoint in $\mathbb{U}^n$ for different $j,k$. Therefore,

$$\bigcup_{k=-K}^{K} \bigcup_j T_{k,j} = \bigcup_j S^*(I_{-K,j}) \setminus \bigcup_{m>k} \bigcup_l S^*(I_{m,l}) \supset T(E_{-K}) \setminus \bigcup_{m>k} \bigcup_l S^*(I_{m,l}). \tag{4.22}$$

Note that $\bigcup_k T(E_k) = \{(\xi,\rho) \in \mathbb{U}^n : w(\xi,\rho) > 0\}$, $(\xi,\rho) \notin T(E_k)$ implies $w(\xi,\rho) \leq 2^k$. And each $S^*(I_{m,l})$ is contained in a cube of sidelength $4\,\mathrm{diam}(I_{m,l})$ in $\mathbb{U}^n$. By using the subadditivity of the $p$-Hausdorff capacity and (4.3), there is

$$\Lambda_p^{(\infty)}\left(\bigcup_{m>k} \bigcup_l S^*(I_{m,l})\right) \leq C \sum_{m>k} \sum_l |I_{m,l}|^p \leq C \sum_{m>k} \Lambda_p^{(\infty)}(E_m) \longrightarrow 0 \tag{4.23}$$

as $k \to \infty$. Thus, by (4.21),

$$\bigcup_k \bigcup_j T_{k,j} \supset \bigcup_k T(E_k) \setminus \bigcap_k \bigcup_{m>k} \bigcup_l S^*(I_{m,l}) = \left\{(\xi,\rho) \in \mathbb{U}^n : w(\xi,\rho) > 0\right\} \setminus E_\infty, \tag{4.24}$$

where $E_\infty$ is a set of zero $p$-Hausdorff capacity, that is,

$$\Lambda_p^{(\infty)}(E_\infty) = \Lambda_p^{(\infty)}\left(\bigcap_k \bigcup_{m>k} \bigcup_l S^*(I_{m,l})\right) = 0. \tag{4.25}$$

Since $w$ is allowed to vanish only where $f$ vanishes, we have $f = \sum_{k,j} f \cdot \chi_{T_{k,j}}$ a.e. on $\mathbb{U}^n$. Let

$$\begin{aligned} a_{k,j} &= \left(\left|I_{k,j}^*\right|^p \int_{T_{k,j}} |f(\xi,\rho)|^2 \frac{d\xi d\rho}{\rho^{1-2(n+1)(1-p)}}\right)^{-1/2} f \cdot \chi_{T_{k,j}}, \\ \lambda_{k,j} &= \left(\left|I_{k,j}^*\right|^p \int_{T_{k,j}} |f(\xi,\rho)|^2 \frac{d\xi d\rho}{\rho^{1-2(n+1)(1-p)}}\right)^{1/2}, \end{aligned} \tag{4.26}$$

where the $I_{k,j}^*$ is a cube as Lemma 2.4, that is, $I_{k,j}^* = 5c_n I_{k,j}$.

Thus

$$f = \sum_{k,j} \lambda_{k,j} a_{k,j}, \quad \text{a.e. on } \mathbb{U}^n. \tag{4.27}$$

Note that, by (4.21), $T_{k,j} \subset S^*(I_{k,j}) \subset T(B_{k,j})$, where $B_{k,j}$ is the ball with the same center as $I_{k,j}$ and radius $cl(I_{k,j}^*)$ ($c > 1$ is a constant). Thus, $a_{k,j}$ is supported in $T(B_{k,j})$, and

$$\begin{aligned} &\int_{T(B_{k,j})} |a_{k,j}(\xi,\rho)|^2 \frac{d\xi d\rho}{\rho^{1-2(n+1)(1-p)}} \\ &\quad = \left(\left|I_{k,j}^*\right|^p \int_{T_{k,j}} |f(\xi,\rho)|^2 \frac{d\xi d\rho}{\rho^{1-2(n+1)(1-p)}}\right)^{-1} \int_{T(B_{k,j})} \left|f \cdot \chi_{T_{k,j}}\right|^2 \frac{d\xi d\rho}{\rho^{1-2(n+1)(1-p)}} \\ &\quad = \left|I_{k,j}^*\right|^{-p} = C\left|B_{k,j}\right|^{-p}. \end{aligned} \tag{4.28}$$

This means that every $a_{k,j}$ is a $T_p^1$-atom. The remaining is to estimate $\sum_{k,j} |\lambda_{k,j}|$.

Notice the fact that

$$w \leq 2^{k+1} \quad \text{on } T_{k,j} \subset \left(\bigcup_l S^*(I_{k+1,l})\right)^c \subset (T(E_{k+1}))^c. \tag{4.29}$$

By the Cauchy-Schwarz inequality, (4.19) and (4.20), we have

$$\begin{aligned}
\sum_{k,j}|\lambda_{k,j}| &\le \sum_{k,j}2^{(k+1)/2}\left|I_{k,j}^*\right|^{p/2}\left(\int_{T_{k,j}}|f(\xi,\rho)|^2 w(\xi,\rho)^{-1}\frac{d\xi d\rho}{\rho^{1-2(n+1)(1-p)}}\right)^{1/2} \\
&\le \left(\sum_{k,j}2^{k+1}\left|I_{k,j}^*\right|^{p}\right)^{1/2}\left(\sum_{k,j}\int_{T_{k,j}}|f(\xi,\rho)|^2 w(\xi,\rho)^{-1}\frac{d\xi d\rho}{\rho^{1-2(n+1)(1-p)}}\right)^{1/2} \\
&\le C\|f\|_{T_p^1}\left(\sum_k 2^k\sum_j|I_{k,j}|^p\right)^{1/2} \le C\|f\|_{T_p^1}\left(\sum_k 2^k\Lambda_p^{(\infty)}(E_k)\right)^{1/2} \\
&= C\|f\|_{T_p^1}\left(\int_0^{+\infty}\Lambda_p^{(\infty)}(\{\xi\in\mathbb{H}^n : N(w)>\lambda\})d\lambda\right)^{1/2} \\
&\le C\|f\|_{T_p^1}\left(\int_{\mathbb{H}^n}N(w)d\Lambda_p^{(\infty)}\right)^{1/2} \le C\|f\|_{T_p^1}.
\end{aligned} \tag{4.30}$$

If $f=\sum_k\lambda_k a_k$ and every $a_k$ is a $T_p^1$-atom, then, by Lemma 4.4, we obtain

$$\|f\|_{T_p^1} \le C\sum_k|\lambda_k|\|a_k\|_{T_p^1} \le C\sum_k|\lambda_k|. \tag{4.31}$$

Thus

$$\|f\|_{T_p^1} \approx \inf\left\{\sum_j|\lambda_j| : f=\sum_j\lambda_j a_j\right\}, \tag{4.32}$$

where the infimum is taken over all atomic decompositions of $f\in T_p^1$. The proof is complete. □

The dual result is as follows.

**Theorem 4.7.** *Let $p\in(0,1]$. Then the dual of $T_p^1$ can be identified with $T_p^\infty$ under the pairing*

$$\langle f,g\rangle = \int_{\mathbb{U}^n}f(\xi,\rho)g(\xi,\rho)\frac{d\xi d\rho}{\rho}. \tag{4.33}$$

*Proof.* We first show that

$$\int_{\mathbb{U}^n}|f(\xi,\rho)g(\xi,\rho)|\frac{d\xi d\rho}{\rho} \le C\|f\|_{T_p^1}\|g\|_{T_p^\infty} \tag{4.34}$$

holds for all $f\in T_p^1$ and $g\in T_p^\infty$.

In fact, assume that $w$ is a nonnegative Borel measurable function on $\mathbb{U}^n$ satisfying inequality (4.3) in Definition 4.2 and $g \in T_p^\infty$. Then there exists a constant $C$ for any ball $B \subset \mathbb{H}^n$ satisfying

$$\frac{1}{|B|^p}\int_{T(B)} |g(\xi,\rho)|^2 \frac{d\xi d\rho}{\rho^{1+2(n+1)(1-p)}} \le C. \tag{4.35}$$

It means that $|g(\xi,\rho)|^2\rho^{-2(n+1)(1-p)-1}\ d\xi d\rho$ is a $p$-Carleson measure, and $\|\mu\| \approx \|g\|^2_{T_p^\infty}$. Hence, by Theorem 3.1, we obtain

$$\int_{\mathbb{U}^n} w(\xi,\rho)|g(\xi,\rho)|^2\rho^{-2(n+1)(1-p)-1}d\xi d\rho \le C\|g\|^2_{T_p^\infty}\int_{\mathbb{H}^n} N(w)d\Lambda_p^{(\infty)} \le C\|g\|^2_{T_p^\infty}. \tag{4.36}$$

Thus, for $f \in T_p^1$, by using the Cauchy-Schwarz inequality, there is

$$\begin{aligned}
&\int_{\mathbb{U}^n}|f(\xi,\rho)g(\xi,\rho)|\frac{d\xi d\rho}{\rho}\\
&\quad\le\left(\int_{\mathbb{U}^n}|f|^2w^{-1}\frac{d\xi d\rho}{\rho^{1-2(n+1)(1-p)}}\right)^{1/2}\left(\int_{\mathbb{U}^n}|g|^2w\frac{d\xi d\rho}{\rho^{1+2(n+1)(1-p)}}\right)^{1/2}\\
&\quad\le C\left(\int_{\mathbb{U}^n}|f(\xi,\rho)|^2w(\xi,\rho)^{-1}\frac{d\xi d\rho}{\rho^{1-2(n+1)(1-p)}}\right)^{1/2}\cdot\|g\|_{T_p^\infty}\\
&\quad\le C\|f\|_{T_p^1}\|g\|_{T_p^\infty}.
\end{aligned} \tag{4.37}$$

This gives (4.34). Thus, every $g \in T_p^\infty$ induces a bounded linear functional on $T_p^1$ via the pairing (4.33). It suffices to prove the converse.

Let $L$ be a bounded linear functional on $T_p^1$ and fix a ball $B = B(z,r) \subset \mathbb{H}^n$. If $f$ is supported in $T(B)$ with $f \in L^2(T(B),\rho^{-1}d\xi d\rho)$, then

$$\begin{aligned}
\int_{T(B)}|f(\xi,\rho)|^2\frac{d\xi d\rho}{\rho^{1-2(n+1)(1-p)}} &\le r^{2(n+1)(1-p)}\int_{T(B)}|f(\xi,\rho)|^2\frac{d\xi d\rho}{\rho}\\
&= Cr^{2(n+1)}|B|^{-p}\|f\|^2_{L^2(T(B),\rho^{-1}\ d\xi d\rho)}.
\end{aligned} \tag{4.38}$$

Therefore, $f$ is a multiple of a $T_p^1$-atom with

$$\|f\|^2_{T_p^1} \le Cr^{2(n+1)}\|f\|^2_{L^2(T(B),\rho^{-1}\ d\xi d\rho)}. \tag{4.39}$$

Hence, $L$ induces a bounded linear functional on $L^2(T(B), \rho^{-1} d\xi d\rho)$. Thus, there exists a function $g$ which is locally in $L^2(\mathbb{U}^n, \rho^{-1} d\xi d\rho)$ such that

$$L(f) = \int_{\mathbb{U}^n} f(\xi, \rho) g(\xi, \rho) \frac{d\xi d\rho}{\rho}, \tag{4.40}$$

whenever $f \in T_p^1$ with support in some finite tent $T(B)$. By the atomic decomposition of tent function in Theorem 4.6, obviously, the subspace of such $f$ is dense in $T_p^1$. Therefore, the rest of the proof is to show that $g \in T_p^\infty$ and $\|g\|_{T_p^\infty} \le C\|L\|$.

Now, again fix a ball $B \subset \mathbb{H}^n$ and for every $\varepsilon > 0$, let

$$f_\varepsilon(\xi, \rho) = \rho^{-2(n+1)(1-p)} \overline{g(\xi, \rho)} \chi_{T^\varepsilon(B)}(\xi, \rho), \tag{4.41}$$

where $T^\varepsilon(B) = T(B) \cap \{(\xi, \rho) : \rho > \varepsilon\}$ is the truncated of $T(B)$.

Since $g \in L^2(T(B))$, there is

$$\int_{T(B)} |f_\varepsilon(\xi, \rho)|^2 \frac{d\xi d\rho}{\rho^{1-2(n+1)(1-p)}} = \int_{T^\varepsilon(B)} |g(\xi, \rho)|^2 \frac{d\xi d\rho}{\rho^{1+2(n+1)(1-p)}} < \infty. \tag{4.42}$$

Hence, $f_\varepsilon$ is a multiple of a $T_p^1$-atom with

$$\|f_\varepsilon\|_{T_p^1}^2 \le C r^{2(n+1)p} \int_{T^\varepsilon(B)} |g(\xi, \rho)|^2 \frac{d\xi d\rho}{\rho^{1+2(n+1)(1-p)}}, \tag{4.43}$$

where $C$ is independent of $\varepsilon$. By (4.40), we obtain

$$\begin{aligned} \int_{T^\varepsilon(B)} |g(\xi, \rho)|^2 \frac{d\xi d\rho}{\rho^{1+2(n+1)(1-p)}} &= L(f_\varepsilon) \le \|L\| \cdot \|f_\varepsilon\|_{T_p^1} \\ &\le C\|L\| \left( |B|^p \int_{T^\varepsilon(B)} |g(\xi, \rho)|^2 \frac{d\xi d\rho}{\rho^{1+2(n+1)(1-p)}} \right)^{1/2}. \end{aligned} \tag{4.44}$$

Thus

$$\left( |B|^{-p} \int_{T^\varepsilon(B)} |g(\xi, \rho)|^2 \frac{d\xi d\rho}{\rho^{1+2(n+1)(1-p)}} \right)^{1/2} \le C\|L\|. \tag{4.45}$$

It follows that

$$\left( |B|^{-p} \int_{T(B)} |g(\xi, \rho)|^2 \frac{d\xi d\rho}{\rho^{1+2(n+1)(1-p)}} \right)^{1/2} \le C\|L\|, \tag{4.46}$$

since (4.40) is true for all $\varepsilon > 0$. Therefore, $g \in T_p^\infty$ and $\|g\|_{T_p^\infty} \le C\|L\|$. In fact, in (4.40), we can replace the local function $g$ by ordinary. Thus, we obtain the representation of $L$ via the pairing (4.40) for all $f \in T_p^1$. This ends the proof of Theorem 4.7. □

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