Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2012, Article ID 575014, 13 pages
doi:10.1155/2012/575014

Research Article

Strong Convergence Theorems for the Generalized Split Common Fixed Point Problem

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Received 9 January 2012; Accepted 17 February 2012

Academic Editor: Rudong Chen

We introduce the generalized split common fixed point problem (GSCFPP) and show that the GSCFPP for nonexpansive operators is equivalent to the common fixed point problem. Moreover, we introduce a new iterative algorithm for finding a solution of the GSCFPP and obtain some strong convergence theorems under suitable assumptions.

1. Introduction

Let H_1 and H_2 be real Hilbert spaces and let $A : H_1 \to H_2$ be a bounded linear operator. Given intergers $p, r \geq 1$, let us recall that the multiple-set split feasibility problem (MSSFP) was recently introduced [1] and is to find a point:

$$x^* \in \bigcap_{i=1}^{p} C_i, \quad Ax^* \in \bigcap_{j=1}^{r} Q_j, \tag{1.1}$$

where $\{C_i\}_{i=1}^{p}$ and $\{Q_j\}_{j=1}^{r}$ are nonempty closed convex subsets of H_1 and H_2, respectively. If $p = r = 1$, the MSSFP (1.1) becomes the so-called split feasibility problem (SFP) [2] which is to find a point:

$$x^* \in C, \quad Ax^* \in Q, \tag{1.2}$$

where C and Q are nonempty closed convex subsets of H_1 and H_2, respectively. Recently, the SFP (1.2) and MSSFP (1.1) have been investigated by many researchers; see, [3–10].

Since every closed convex subset in a Hilbert space is looked as the fixed point set of its associating projection, the MSSFP (1.1) becomes a special case of the split common fixed point problem (SCFPP), which is to find a point:

$$x^* \in \bigcap_{i=1}^{p} \mathrm{Fix}(U_i), \quad Ax^* \in \bigcap_{j=1}^{r} \mathrm{Fix}(T_j), \tag{1.3}$$

where $U_i : H_1 \to H_1$ $(i = 1, 2, \ldots, p)$ and $T_j : H_2 \to H_2$ $(j = 1, 2, \ldots, r)$ are nonlinear operators. If $p = r = 1$, the problem (1.3) reduces to the so-called two-set SCFPP, which is to find a point:

$$x^* \in \mathrm{Fix}(U), \quad Ax^* \in \mathrm{Fix}(T). \tag{1.4}$$

Censor and Segal in [11] firstly introduced the concept of SCFPP in finite-dimensional Hilbert spaces and considered the following iterative algorithm for the two-set SCFPP (1.4) for Class-$\mathfrak{T}$ operators:

$$x_{n+1} = U\left(x_n - \gamma A^*(I - T)Ax_n\right), \quad n \geq 0, \tag{1.5}$$

where $x_0 \in H_1$, $0 < \gamma < 2/\|A\|^2$ and I is the identity operator. They proved the convergence of the algorithm (1.5) to a solution of problem (1.4). Moreover, they introduced a parallel iterative algorithm, which converges to a solution of the SCFPP (1.3). However, the parallel iterative algorithm does not include the algorithm (1.5) as a special case.

Very recently, Wang and Xu in [12] considered the SCFPP (1.3) for Class-$\mathfrak{T}$ operators and introduced the following iterative algorithm for solving the SCFPP (1.3):

$$x_{n+1} = U_{[n]}\left(x_n - \gamma A^*\left(I - T_{[n]}\right)Ax_n\right), \quad n \geq 0. \tag{1.6}$$

Under some mild conditions, they proved some weak and strong convergence theorems. Their iterative algorithm (1.6) includes Censor and Segal's algorithm (1.5) as a special case for the two-set SCFPP (1.4). Moreover, they prove that the SCFPP (1.3) for the Class-$\mathfrak{T}$ operators is equivalent to a common fixed point problem. This is also a classical method. Many problems eventually converted to a common fixed point problem; see [13–15].

Motivated and inspired by the aforementioned research works, we introduce a generalized split common fixed point problem (GSCFPP) which is to find a point:

$$x^* \in \bigcap_{i=1}^{\infty} \mathrm{Fix}(U_i), \quad Ax^* \in \bigcap_{j=1}^{\infty} \mathrm{Fix}(T_j). \tag{1.7}$$

Then, we show that the GSCFPP (1.7) for nonexpansive operators is equivalent to the following common fixed point problem:

$$x^* \in \bigcap_{i=1}^{\infty} \mathrm{Fix}(U_i), \qquad x^* \in \bigcap_{j=1}^{\infty} \mathrm{Fix}(V_j), \tag{1.8}$$

where $V_j = I - \gamma A^*(I - T_j)A$ $(0 < \gamma \leq 1/\|A\|^2)$ for every $j \in \mathbb{N}$. Moreover, we give a new iterative algorithm for solving the GSCFPP (1.7) for nonexpansive operators and obtain some strong convergence theorems.

2. Preliminaries

Throughout this paper, we write $x_n \rightharpoonup x$ and $x_n \to x$ to indicate that $\{x_n\}$ converges weakly to x and converges strongly to x, respectively.

An operator $T : H \to H$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in H$. The set of fixed points of T is denoted by $F(T)$. It is known that $F(T)$ is closed and convex. An operator $f : H \to H$ is called contraction if there exists a constant $\rho \in [0,1)$ such that $\|f(x) - f(y)\| \leq \rho\|x - y\|$ for all $x, y \in H$. Let C be a nonempty closed convex subset of H. For each $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that $\|x - P_C x\| \leq \|x - y\|$ for every $y \in C$. P_C is called a metric projection of H onto C. It is known that for each $x \in H$,

$$\langle x - P_C x, y - P_C x \rangle \leq 0 \tag{2.1}$$

for all $y \in C$.

Let $\{T_n\}$ be a sequence of operators of H into itself. The set of common fixed points of $\{T_n\}$ is denoted by $F(\{T_n\})$, that is, $F(\{T_n\}) = \bigcap_{n=1}^{\infty} F(T_n)$. A sequence $\{T_n\}$ is said to be strongly nonexpansive if each $\{T_n\}$ is nonexpansive and

$$x_n - y_n - (T_n x_n - T_n y_n) \longrightarrow 0 \tag{2.2}$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in C such that $\{x_n - y_n\}$ is bounded and $\|x_n - y_n\| - \|T_n x_n - T_n y_n\| \to 0$; see [16, 17]. A sequence $\{z_n\}$ in H is said to be an approximate fixed point sequence of $\{T_n\}$ if $z_n - T_n z_n \to 0$. The set of all bounded approximate fixed point sequences of $\{T_n\}$ is denoted by $\widetilde{F}(\{T_n\})$; see [16, 17]. We know that if $\{T_n\}$ has a common fixed point, then $\widetilde{F}(\{T_n\})$ is nonempty; that is, every bounded sequence in the common fixed point set is an approximate fixed point sequence. A sequence $\{T_n\}$ with a common fixed point is said to satisfy the condition (Z) if every weak cluster point of $\{x_n\}$ is a common fixed point whenever $\{x_n\} \in \widetilde{F}(\{T_n\})$. A sequence $\{T_n\}$ of nonexpansive mappings of H into itself is said to satisfy the condition (R) if

$$\lim_{n \to \infty} \sup_{y \in D} \|T_{n+1} y - T_n y\| = 0 \tag{2.3}$$

for every nonempty bounded subset D of H; see [18].

In order to prove our main results, we collect the following lemmas in this section.

Lemma 2.1 (see [16]). *Let C be a nonempty subset of a Hilbert space H. Let $\{T_n\}$ be a sequence of nonexpansive mappings of C into H. Let $\{\lambda_n\}$ be a sequence in $[0,1]$ such that* $\liminf_{n\to\infty}\lambda_n > 0$. *Let $\{U_n\}$ be a sequence of mappings of C into H defined by $U_n = \lambda_n I + (1 - \lambda_n)T_n$ for $n \in \mathbb{N}$, where I is the identity mapping on C. Then $\{U_n\}$ is a strongly nonexpansive sequence.*

Lemma 2.2 (see [16]). *Let H be a Hilbert space, C a nonempty subset of H, and $\{S_n\}$ and $\{T_n\}$ sequences of nonexpansive self-mappings of C. Suppose that $\{S_n\}$ or $\{T_n\}$ is a strongly nonexpansive sequence and $\widetilde{F}(\{S_n\}) \cap \widetilde{F}(\{T_n\})$ is nonempty. Then $\widetilde{F}(\{S_n\}) \cap \widetilde{F}(\{T_n\}) = \widetilde{F}(\{S_n T_n\})$.*

Lemma 2.3 (see [17]). *Let H be a Hilbert space, and C a nonempty subset of H. Both $\{S_n\}$ and $\{T_n\}$ satisfy the condition (R) and $\{T_n y : n \in \mathbb{N},\ y \in D\}$ is bounded for any bounded subset D of C. Then $\{S_n T_n\}$ satisfies the condition (R).*

Lemma 2.4 (see [19]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)\ y_n + \beta_n\ x_n$ for all integers $n \ \ge\ 0$ and*

$$\limsup_{n\to\infty}\left(\|y_{n+1} - y_n\| \ - \ \|x_{n+1} - x_n\|\right) \le 0. \tag{2.4}$$

Then $\lim_{n\to\infty}\|y_n - x_n\| = 0$.

Lemma 2.5 (see [20]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1-\gamma_n)a_n + \delta_n, \quad n \ge 0, \tag{2.5}$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(i) $\sum_{n=1}^{\infty}\gamma_n = \infty$,

(ii) $\limsup_{n\to\infty}\delta_n/\gamma_n \le 0$ *or* $\sum_{n=1}^{\infty}|\delta_n| < \infty$.

Then $\lim_{n\to\infty}a_n = 0$.

3. Main Results

Now we state and prove our main results of this paper.

Lemma 3.1. *Let $A : H_1 \ \to\ H_2$ be a given bounded linear operator and let $T_n : H_2 \ \to\ H_2$ be a sequence of nonexpansive operators. Assume*

$$A^{-1}(\mathrm{Fix}(\{T_n\})) = \{x \in H_1 : Ax \in \mathrm{Fix}(\{T_n\})\} \neq \emptyset. \tag{3.1}$$

For each constant $\gamma > 0$, V_n is defined by the following:

$$V_n = I - \gamma A^*(I - T_n)A. \tag{3.2}$$

Then $\mathrm{Fix}(\{V_n\}) = A^{-1}(\mathrm{Fix}(\{T_n\}))$. Moreover, for $0 < \gamma \le 1/\|A\|^2$, V_n is nonexpansive on H_1 for $n \in \mathbb{N}$.

Proof. Since the inclusion $A^{-1}(\mathrm{Fix}(\{T_n\})) \subseteq \mathrm{Fix}(\{V_n\})$ is evident, now we only need to show the converse inclusion. If $z \in \mathrm{Fix}(\{V_n\})$, then we have $A^*(I - T_n)Az = 0$. Since $A^{-1}(\mathrm{Fix}(\{T_n\})) \neq \emptyset$, we take an arbitrary $p \in A^{-1}(\mathrm{Fix}(\{T_n\}))$. Hence

$$\begin{aligned}
\|Az - T_nAz\|^2 &= \langle Az - T_nAz, Az - T_nAz\rangle \\
&= \left\langle Az - T_nAz, Az - Ap + Ap - T_nAz\right\rangle \\
&= \left\langle A^*(I - T_n)Az, z - p\right\rangle + \left\langle Az - T_nAz, Ap - T_nAz\right\rangle \\
&= -\frac{1}{2}\left\|Az - Ap\right\|^2 + \frac{1}{2}\|Az - T_nAz\|^2 + \frac{1}{2}\left\|Ap - T_nAz\right\|^2 \\
&\leq \frac{1}{2}\|Az - T_nAz\|^2.
\end{aligned} \tag{3.3}$$

It follows that $(1/2)\|Az - T_nAz\|^2 \leq 0$, then $Az = T_nAz$ for every $n \in \mathbb{N}$, hence $z \in A^{-1}(\mathrm{Fix}(\{T_n\}))$. Next we turn to show that V_n is a nonexpansive operator for $n \in \mathbb{N}$. Since T_n is nonexpansive, we have

$$\begin{aligned}
\left\|(I - T_n)Ax - (I - T_n)Ay\right\|^2 &= \left\|Ax - Ay\right\|^2 + \left\|T_nAx - T_nAy\right\|^2 - 2\left\langle Ax - Ay, T_nAx - T_nAy\right\rangle \\
&\leq 2\left\|Ax - Ay\right\|^2 - 2\left\langle Ax - Ay, T_nAx - T_nAy\right\rangle \\
&\leq 2\left\langle Ax - Ay, Ax - Ay - \left(T_nAx - T_nAy\right)\right\rangle.
\end{aligned} \tag{3.4}$$

Hence

$$\begin{aligned}
\left\|V_nx - V_ny\right\|^2 &= \left\|(I - \gamma A^*(I - T_n)A)x - (I - \gamma A^*(I - T_n)A)y\right\|^2 \\
&= \left\|x - y\right\|^2 + \gamma^2\|A\|^2\left\|(I - T_n)Ax - (I - T_n)Ay\right\|^2 \\
&\quad - 2\gamma\left\langle Ax - Ay, (I - T_n)Ax - (I - T_n)Ay\right\rangle \\
&\leq \left\|x - y\right\|^2 + \gamma\left(\gamma\|A\|^2 - 1\right)\left\|(I - T_n)Ax - (I - T_n)Ay\right\|^2.
\end{aligned} \tag{3.5}$$

For $0 < \gamma \leq 1/\|A\|^2$, we can immediately obtain that V_n is a nonexpansive operator for every $n \in \mathbb{N}$. □

From Lemma 3.1, we can obtain that the solution set of GSCFPP (1.7) is identical to the solution set of problem (1.8).

Theorem 3.2. *Let $\{U_n\}$ and $\{V_n\}$ be sequences of nonexpansive operators on Hilbert space H_1. Both $\{U_n\}$ and $\{V_n\}$ satisfy the conditions (R) and (Z). Let $f : H_1 \to H_1$ be a contraction with coefficient*

$\rho \in [0,1)$. *Suppose* $\Omega = \mathrm{Fix}(U_n) \bigcap \mathrm{Fix}(V_n) \neq \emptyset$. *Take an initial guess* $x_1 \in H_1$ *and define a sequence* $\{x_n\}$ *by the following algorithm:*

$$\begin{aligned} y_n &= \lambda_n x_n + (1-\lambda_n) V_n x_n, \\ x_{n+1} &= \alpha_n f(x_n) + \beta_n x_n + \gamma_n U_n y_n, \end{aligned} \tag{3.6}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, *and* $\{\lambda_n\}$ *are sequences in* $[0,1]$. *If the following conditions are satisfied:*

(i) $\alpha_n + \beta_n + \gamma_n = 1$, *for all* $n \geq 1$;

(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\Sigma_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(iv) $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 1$;

(v) $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$,

then $\{x_n\}$ *converges strongly to* $w \in \Omega$ *where* $w = P_\Omega f(w)$.

Proof. We proceed with the following steps.

Step 1. First show that there exists $w \in \Omega$ such that $w = P_\Omega f(w)$.

In fact, since f is a contraction with coefficient ρ, we have

$$\left\| P_\Omega f(x) - P_\Omega f(y) \right\| \leq \left\| f(x) - f(y) \right\| \leq \rho \left\| x - y \right\| \tag{3.7}$$

for every x, y. Hence $P_\Omega f$ is also a contraction. Therefore, there exists a unique $w \in \Omega$ such that $w = P_\Omega f(w)$.

Step 2. Now we show that $\{x_n\}$ is bounded.

Let $p \in \Omega$, then $p \in \mathrm{Fix}(\{U_n\})$ and $p \in \mathrm{Fix}(\{V_n\})$. Hence

$$\|U_n y_n - p\| \leq \|y_n - p\| \leq \lambda_n \|x_n - p\| + (1-\lambda_n)\|V_n x_n - p\| \leq \|x_n - p\|. \tag{3.8}$$

Then

$$\begin{aligned} \left\| x_{n+1} - p \right\| &\leq \alpha_n \left\| f(x_n) - p \right\| + \beta_n \left\| x_n - p \right\| + \gamma_n \left\| U_n y_n - p \right\| \\ &\leq \alpha_n \rho \left\| x_n - p \right\| + \alpha_n \left\| f(p) - p \right\| + \beta_n \left\| x_n - p \right\| + \gamma_n \left\| x_n - p \right\| \\ &\leq (1 - \alpha_n(1-\rho)) \left\| x_n - p \right\| + \alpha_n (1-\rho) \frac{1}{1-\rho} \left\| f(p) - p \right\| \\ &\leq \max\left\{ \left\| x_n - p \right\|, \frac{1}{1-\rho} \left\| f(p) - p \right\| \right\}. \end{aligned} \tag{3.9}$$

By induction on n,

$$\left\| V_n x_n - p \right\| \leq \left\| x_n - p \right\| \leq \max\left\{ \left\| x_1 - p \right\|, \frac{1}{1-\rho} \left\| f(p) - p \right\| \right\} \tag{3.10}$$

for every $n \in \mathbb{N}$. This shows that $\{x_n\}$ and $\{V_n x_n\}$ are bounded, and hence, $\{U_n y_n\}$, $\{y_n\}$, and $\{f(x_n)\}$ are also bounded.

Step 3. We claim that $\widetilde{F}(\{A_n\}) = \widetilde{F}(\{V_n\})$ and $\widetilde{F}(\{U_n A_n\}) = \widetilde{F}(\{U_n\}) \cap \widetilde{F}(\{V_n\})$, where $A_n = \lambda_n I + (1-\lambda_n)V_n$.

We first show the former equality. Let $\{z_n\}$ be a bounded sequence in H_1. If $\{z_n\} \in \widetilde{F}(\{V_n\})$, then

$$\|A_n z_n - z_n\| = \|\lambda_n z_n + (1-\lambda_n)V_n z_n - z_n\| = (1-\lambda_n)\|V_n z_n - z_n\| \longrightarrow 0. \tag{3.11}$$

Hence $\{z_n\} \in \widetilde{F}(\{A_n\})$. On the other hand, if $\{z_n\} \in \widetilde{F}(\{A_n\})$, combining (3.11) and $\limsup_{n\to\infty}\lambda_n < 1$, we obtain that $\|V_n z_n - z_n\| \to 0$. Hence $\{z_n\} \in \widetilde{F}(\{V_n\})$. Therefore, $\widetilde{F}(\{A_n\}) = \widetilde{F}(\{V_n\})$.

Next, we show the latter equality. Using Lemma 2.1, we know that $\{A_n\}$ is a strongly nonexpansive sequence. Thus, since $\widetilde{F}(\{U_n\}) \cap \widetilde{F}(\{A_n\}) = \widetilde{F}(\{U_n\}) \cap \widetilde{F}(\{V_n\}) \neq \emptyset$, from Lemma 2.2 we have

$$\widetilde{F}(\{U_n A_n\}) = \widetilde{F}(\{U_n\}) \cap \widetilde{F}(\{A_n\}) = \widetilde{F}(\{U_n\}) \cap \widetilde{F}(\{V_n\}). \tag{3.12}$$

Step 4. $\{S_n\}$ satisfies the condition (R), where $S_n = U_n A_n$.

Let D be a nonempty bounded subset of H_1. From the definition of $\{A_n\}$, we have, for all $y \in D$,

$$\begin{aligned}
\|A_{n+1}y - A_n y\| &= \|\lambda_{n+1}y + (1-\lambda_{n+1})V_{n+1}y - \lambda_n y - (1-\lambda_n)V_n y\| \\
&\le |\lambda_{n+1}-\lambda_n|\|y\| + \|V_{n+1}y - V_n y\| + \|\lambda_{n+1}V_{n+1}y - \lambda_n V_n y\| \\
&\le |\lambda_{n+1}-\lambda_n|\|y\| + \|V_{n+1}y - V_n y\| + \|\lambda_{n+1}V_{n+1}y - \lambda_n V_{n+1} y\| \\
&\quad + \|\lambda_n V_{n+1}y - \lambda_n V_n y\| \\
&= |\lambda_{n+1}-\lambda_n|\|y\| + \|V_{n+1}y - V_n y\| + |\lambda_{n+1}-\lambda_n|\|V_{n+1}y\| \\
&\quad + \lambda_n\|V_{n+1}y - V_n y\| \\
&= |\lambda_{n+1}-\lambda_n|(\|y\| + \|V_{n+1}y\|) + (1+\lambda_n)\|V_{n+1}y - V_n y\|.
\end{aligned} \tag{3.13}$$

It follows that

$$\sup_{y\in D}\|A_{n+1}y - A_n y\| \le |\lambda_{n+1}-\lambda_n|\sup_{y\in D}(\|y\| + \|V_{n+1}y\|) + (1+\lambda_n)\sup_{y\in D}\|V_{n+1}y - V_n y\|. \tag{3.14}$$

Since $\{V_n\}$ satisfies the condition (R) and $\lim_{n\to\infty}|\lambda_{n+1}-\lambda_n| = 0$, we have

$$\lim_{n\to\infty}\sup_{y\in D}\|A_{n+1}y - A_n y\| = 0, \tag{3.15}$$

that is, $\{A_n\}$ satisfies the condition (R). Since $\{A_n y : n \in \mathbb{N},\ y \in D\}$ is bounded for any bounded subset D of H_1, by using Lemma 2.3, we have that $\{V_n A_n\}$ satisfies the condition (R), that is, $\{S_n\}$ satisfies the condition (R).

Step 5. We show $\|x_{n+1} - x_n\| \to 0$.

We can write (3.6) as $x_{n+1} = \beta_n x_n + (1-\beta_n) z_n$ where $z_n = (\alpha_n f(x_n) + \gamma_n S_n x_n)/1-\beta_n$. It follows that

$$\begin{aligned}
z_{n+1} - z_n &= \frac{\alpha_{n+1} f(x_{n+1}) + \gamma_{n+1} S_{n+1} x_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n f(x_n) + \gamma_n S_n x_n}{1-\beta_n} \\
&= \frac{\alpha_{n+1}}{1-\beta_{n+1}} \left(f(x_{n+1}) - f(x_n) \right) + \left(\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n} \right) f(x_n) \\
&\quad + \frac{\gamma_{n+1}}{1-\beta_{n+1}} (S_{n+1} x_{n+1} - S_n x_n) + \left(\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n} \right) S_n x_n.
\end{aligned} \tag{3.16}$$

From Step 2, we may assume that $\{x_n\} \subset D'$, where D' is a bounded set of H_1. Then from (3.16), we obtain

$$\begin{aligned}
\|z_{n+1} - z_n\| &\le \left| \frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n} \right| \left(\|f(x_n)\| + \|S_n x_n\| \right) + \frac{\alpha_{n+1}}{1-\beta_{n+1}} \rho \|x_{n+1} - x_n\| \\
&\quad + \frac{\gamma_{n+1}}{1-\beta_{n+1}} \|S_{n+1} x_{n+1} - S_n x_{n+1}\| + \frac{\gamma_{n+1}}{1-\beta_{n+1}} \|S_n x_{n+1} - S_n x_n\| \\
&\le \left| \frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n} \right| \left(\|f(x_n)\| + \|S_n x_n\| \right) + \left[1 - \frac{\alpha_{n+1}}{1-\beta_{n+1}} (1-\rho) \right] \|x_{n+1} - x_n\| \\
&\quad + \frac{\gamma_{n+1}}{1-\beta_{n+1}} \sup_{y \in D'} \|S_{n+1} y - S_n y\|.
\end{aligned} \tag{3.17}$$

It follows that

$$\begin{aligned}
\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| &\le \left| \frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n} \right| \left(\|f(x_n)\| + \|S_n x_n\| \right) \\
&\quad + \frac{\gamma_{n+1}}{1-\beta_{n+1}} \sup_{y \in D'} \|S_{n+1} y - S_n y\| - \frac{\alpha_{n+1}}{1-\beta_{n+1}} (1-\rho) \|x_{n+1} - x_n\|.
\end{aligned} \tag{3.18}$$

Since $\{S_n\}$ satisfies the condition (R), combining $\alpha_n \to 0$ as $n \to \infty$, we have

$$\limsup_{n \to \infty} \left(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \right) \le 0. \tag{3.19}$$

Hence by Lemma 2.4, we get $\|z_n - x_n\| \to 0$ as $n \to \infty$. Consequently,

$$\lim_{n \to \infty} \|x_{n+1} - x_n\| = \lim_{n \to \infty} (1-\beta_n) \|z_n - x_n\| = 0. \tag{3.20}$$

Step 6. We claim that $\{x_n\} \in \widetilde{F}(\{U_n\}) \cap \widetilde{F}(\{V_n\})$.

From (3.6), we have

$$\begin{aligned}
\|S_n x_n - x_n\| &\le \|S_n x_n - x_{n+1}\| + \|x_{n+1} - x_n\| \\
&= \left\|S_n x_n - \alpha_n f(x_n) - \beta_n x_n - \gamma_n S_n x_n\right\| + \|x_{n+1} - x_n\| \\
&\le \alpha_n \left\|S_n x_n - f(x_n)\right\| + \beta_n \|S_n x_n - x_n\| + \|x_{n+1} - x_n\|,
\end{aligned} \tag{3.21}$$

and hence

$$(1-\beta_n)\|S_n x_n - x_n\| \le \alpha_n \left\|S_n x_n - f(x_n)\right\| + \|x_{n+1} - x_n\|. \tag{3.22}$$

Since $\|x_{n+1} - x_n\| \to 0$, $\alpha_n \to 0$ and $\limsup_{n\to\infty}\beta_n < 1$, we derive

$$\|S_n x_n - x_n\| \longrightarrow 0. \tag{3.23}$$

Thus (3.23) and Steps 2 and 3 imply that

$$\{x_n\} \in \widetilde{F}(\{S_n\}) = \widetilde{F}(\{U_n\}) \cap \widetilde{F}(\{V_n\}). \tag{3.24}$$

Step 7. Show $\limsup_{n\to\infty}\langle f(w) - w, x_n - w\rangle \le 0$, where $w = P_\Omega f(w)$.

Since $\{x_n\}$ is bounded, there exist a point $v \in H_1$ and a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\left\langle f(w) - w, x_n - w\right\rangle = \lim_{i\to\infty}\left\langle f(w) - w, x_{n_i} - w\right\rangle \tag{3.25}$$

and $x_{n_i} \rightharpoonup v$. Since $\{U_n\}$ and $\{V_n\}$ satisfy the condition (Z), from Step 6, we have $v \in F(\{U_n\}) \cap F(\{V_n\})$. Using (2.1), we get

$$\begin{aligned}
\limsup_{n\to\infty}\left\langle f(w) - w, x_n - w\right\rangle &= \lim_{i\to\infty}\left\langle f(w) - w, x_{n_i} - w\right\rangle \\
&= \left\langle f(w) - w, v - w\right\rangle \le 0.
\end{aligned} \tag{3.26}$$

Step 8. Show $x_n \to w = P_\Omega f(w)$.

Since $w \in \Omega$, using (3.8), we have

$$\begin{aligned}
\|x_{n+1} - w\|^2 &= \langle \alpha_n (f(x_n) - w) + \beta_n (x_n - w) + \gamma_n (U_n y_n - w), x_{n+1} - w \rangle \\
&\leq \alpha_n \langle f(x_n) - f(w), x_{n+1} - w \rangle + \alpha_n \langle f(w) - w, x_{n+1} - w \rangle \\
&\quad + \beta_n \|x_n - w\| \cdot \|x_{n+1} - w\| + \gamma_n \left\| y_n - w \right\| \cdot \|x_{n+1} - w\| \\
&\leq \frac{1}{2} \alpha_n \rho \left(\|x_n - w\|^2 + \|x_{n+1} - w\|^2 \right) + \alpha_n \langle f(w) - w, x_{n+1} - w \rangle \\
&\quad + \frac{1}{2} \beta_n \left(\|x_n - w\|^2 + \|x_{n+1} - w\|^2 \right) + \frac{1}{2} \gamma_n \left(\|x_n - w\|^2 + \|x_{n+1} - w\|^2 \right) \\
&\leq \frac{1}{2} [1 - \alpha_n (1 - \rho)] \|x_n - w\|^2 + \frac{1}{2} \|x_{n+1} - w\|^2 + \alpha_n \langle f(w) - w, x_{n+1} - w \rangle,
\end{aligned} \tag{3.27}$$

which implies that

$$\|x_{n+1} - w\|^2 \leq \left[1 - \alpha_n (1 - \rho)\right] \|x_n - w\|^2 + 2\alpha_n (1 - \rho) \frac{1}{1 - \rho} \langle f(w) - w, x_{n+1} - w \rangle, \tag{3.28}$$

for every $n \in \mathbb{N}$. Consequently, according to Step 7, $\rho \in [0, 1)$, and Lemma 2.5, we deduce that $\{x_n\}$ converges strongly to $w = P_\Omega(w)$. This completes the proof. □

Combining Lemma 3.1 and Theorem 3.2, we can obtain the following strong convergence theorem for solving the GSCFPP (1.7).

Theorem 3.3. *Let $\{U_n\}$ and $\{T_n\}$ be sequences of nonexpansive operators on Hilbert space H_1 and H_2, respectively. Both $\{U_n\}$ and $\{T_n\}$ satisfy the conditions (R) and (Z). Let $f : H_1 \to H_1$ be a contraction with coefficient $\rho \in [0, 1)$. Suppose that the solution set Ω of GSCFPP (1.7) is nonempty. Take an initial guess $x_1 \in H_1$ and define a sequence $\{x_n\}$ by the following algorithm:*

$$\begin{aligned}
y_n &= x_n - \gamma (1 - \lambda_n) A^* (I - T_n) A x_n, \\
x_{n+1} &= \alpha_n f(x_n) + \beta_n x_n + \gamma_n U_n y_n,
\end{aligned} \tag{3.29}$$

where $\gamma \in (0, 1/\|A\|^2)$, and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\lambda_n\}$ are sequences in $[0, 1]$. If the following conditions are satisfied:

(i) $\alpha_n + \beta_n + \gamma_n = 1$, *for all* $n \geq 1$;

(ii) $\lim_{n \to \infty} \alpha_n = 0$ *and* $\Sigma_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1$;

(iv) $0 < \liminf_{n \to \infty} \lambda_n \leq \limsup_{n \to \infty} \lambda_n < 1$;

(v) $\lim_{n \to \infty} |\lambda_{n+1} - \lambda_n| = 0$,

then $\{x_n\}$ converges strongly to $w \in \Omega$ where $w = P_\Omega f(w)$.

Proof. Set $V_n = I - \gamma A^*(I - T_n)A$. By Lemma 3.1, V_n is a nonexpansive operator for every $n \in \mathbb{N}$. We can rewrite (3.29) as

$$\begin{aligned} y_n &= \lambda_n x_n + (1 - \lambda_n) V_n x_n, \\ x_{n+1} &= \alpha_n f(x_n) + \beta_n x_n + \gamma_n U_n y_n. \end{aligned} \tag{3.30}$$

We only need to prove that $\{V_n\}$ satisfies the conditions (R) and (Z). Assume that D is a nonempty bounded subset of H_1. For every $y \in D$, we have

$$\begin{aligned} \left\| (I - \gamma A^*(I - T_{n+1})A)y - (I - \gamma A^*(I - T_n)A)y \right\| &\le \gamma \left\| A^*(I - T_{n+1})Ay - A^*(I - T_n)Ay \right\| \\ &\le \gamma \|A\| \left\| T_{n+1}(Ay) - T_n(Ay) \right\|. \end{aligned} \tag{3.31}$$

Since $\{T_n\}$ satisfies the condition (R), and $D' = \{Ay : y \in D\}$ is bounded, it follows from (3.31) that

$$\begin{aligned} \sup_{y \in D} \left\| (I - \gamma A^*(I - T_{n+1})A)y - (I - \gamma A^*(I - T_n)A)y \right\| &\le \gamma \|A\| \sup_{y \in D} \left\| T_{n+1}(Ay) - T_n(Ay) \right\| \\ &= \gamma \|A\| \sup_{z \in D'} \|T_{n+1}z - T_n z\| \longrightarrow 0. \end{aligned} \tag{3.32}$$

Therefore, $\{V_n\}$ satisfies the condition (R).

Assume that $x_n \rightharpoonup z$ and $x_n - V_n x_n \to 0$; we next show that $V_n z = z$. By using $x_n - V_n x_n \to 0$, we have $A^*(I - T_n)Ax_n \to 0$. Since $A^{-1}(\mathrm{Fix}(\{T_n\})) \neq \emptyset$, we choose an arbitrary point $p \in A^{-1}(\mathrm{Fix}(\{T_n\}))$; then for every $n \in \mathbb{N}$,

$$\begin{aligned} \|Ax_n - T_n Ax_n\|^2 &= \left\langle Ax_n - T_n Ax_n, Ax_n - Ap + Ap - T_n Ax_n \right\rangle \\ &= \left\langle A^*(I - T_n)Ax_n, x_n - p \right\rangle + \left\langle Ax_n - T_n Ax_n, Ap - T_n Ax_n \right\rangle \\ &= \left\langle A^*(I - T_n)Ax_n, x_n - p \right\rangle - \frac{1}{2} \left\| Ax_n - Ap \right\|^2 + \frac{1}{2} \|Ax_n - T_n Ax_n\|^2 \\ &\quad + \frac{1}{2} \left\| Ap - T_n Ax_n \right\|^2 \\ &\le \left\langle A^*(I - T_n)Ax_n, x_n - p \right\rangle + \frac{1}{2} \|Ax_n - T_n Ax_n\|^2. \end{aligned} \tag{3.33}$$

Hence

$$\frac{1}{2} \|Ax_n - T_n Ax_n\|^2 \le \left\langle A^*(I - T_n)Ax_n, x_n - p \right\rangle \longrightarrow 0. \tag{3.34}$$

Then we get $Ax_n \in \widetilde{F}(\{T_n\})$. Since $\{T_n\}$ satisfies the condition (Z) and $Ax_n \rightharpoonup Az$, we have $Az \in F(\{T_n\})$. From Lemma 3.1, we have $z \in \mathrm{Fix}(\{V_n\})$. □

Let $T : H \to H$ be a nonexpansive mapping with a fixed point, and define $T_n = T$ for all $n \in \mathbb{N}$. Then $\{T_n\}$ satisfies the conditions (R) and (Z). Thus, one obtains the algorithm for solving the two-set SCFPP (1.4).

Corollary 3.4. *Let U and T be nonexpansive operators on Hilbert space H_1 and H_2, respectively. Let $f : H_1 \to H_1$ be a contraction with coefficient $\rho \in [0,1)$. Suppose that the solution set Ω of SCFPP* (1.4) *is nonempty. Take an initial guess $x_1 \in H_1$ and define a sequence $\{x_n\}$ by the following algorithm in* (3.29)*, where $\gamma \in (0, 1/\|A\|^2)$, and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\lambda_n\}$ are sequences in $[0,1]$. If the following conditions are satisfied:*

(i) $\alpha_n + \beta_n + \gamma_n = 1$, *for all* $n \geq 1$;

(ii) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$;

(iii) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$;

(iv) $0 < \liminf_{n\to\infty}\lambda_n \leq \limsup_{n\to\infty}\lambda_n < 1$;

(v) $\lim_{n\to\infty}|\lambda_{n+1} - \lambda_n| = 0$.

Then $\{x_n\}$ converges strongly to $w \in \Omega$ where $w = P_\Omega f(w)$.

Remark 3.5. By adding more operators to the families $\{U_n\}$ and $\{T_n\}$ by setting $U_i = I$ for $i \geq p+1$ and $T_j = I$ for $j \geq r+1$, the SCFPP (1.3) can be viewed as a special case of the GSCFPP (1.7).

Acknowledgment

This research is supported by the science research foundation program in Civil Aviation University of China (07kys09), the Fundamental Research Funds for the Central Universities (Program No. ZXH2011D005), and the NSFC Tianyuan Youth Foundation of Mathematics of China (No. 11126136).

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