Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2012, Article ID 569313, 15 pages
doi:10.1155/2012/569313

Research Article

Positive Solutions to a Generalized Second-Order Difference Equation with Summation Boundary Value Problem

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Received 1 December 2011; Accepted 22 February 2012

Academic Editor: Yansheng Liu

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By using Krasnoselskii's fixed point theorem, we study the existence of positive solutions to the three-point summation boundary value problem $\Delta^2 u(t-1) + a(t) f(u(t)) = 0$, $t \in \{1, 2, \dots, T\}$, $u(0) = \beta \sum_{s=1}^{\eta} u(s)$, $u(T+1) = \alpha \sum_{s=1}^{\eta} u(s)$, where f is continuous, $T \geq 3$ is a fixed positive integer, $\eta \in \{1, 2, \dots, T-1\}$, $0 < \alpha < (2T+2)/\eta(\eta+1)$, $0 < \beta < (2T+2-\alpha\eta(\eta+1))/\eta(2T-\eta+1)$, and $\Delta u(t-1) = u(t) - u(t-1)$. We show the existence of at least one positive solution if f is either superlinear or sublinear.

1. Introduction

The study of the existence of solutions of multipoint boundary value problems for linear second-order ordinary differential and difference equations was initiated by Ilin and Moiseev [1]. Then Gupta [2] studied three-point boundary value problems for nonlinear second-order ordinary differential equations. Since then, nonlinear second-order three-point boundary value problems have also been studied by many authors; one may see the text books [3, 4] and the papers [5–10]. However, all these papers are concerned with problems with three-point boundary condition restrictions on the difference of the solutions and the solutions themselves, for example,

$$
\begin{aligned}
u(0) &= 0, \qquad u(T+1) = 0, \\
u(0) &= 0, \qquad au(s) = u(T+1), \\
u(0) &= 0, \qquad u(T+1) - au(s) = b,
\end{aligned}
$$

$$u(0) - \alpha \Delta u(0) = 0, \qquad u(T+1) = \beta u(s),$$
$$u(0) - \alpha \Delta u(0) = 0, \qquad \Delta u(T+1) = 0, \tag{1.1}$$

and so forth.

In [5], Leggett-Williams developed a fixed point theorem to prove the existence of three positive solutions for Hammerstein integral equations. Since then, this theorem has been reported to be a successful technique for dealing with the existence of three solutions for the two-point boundary value problems of differential and difference equations; see [6, 7]. In [8], X. Lin and W. Liu, using the properties of the associate Green's function and Leggett-Williams fixed point theorem, studied the existence of positive solutions of the problem.

In [9], Zhang and Medina studied the existence of positive solutions for second-order boundary value problems of difference equations by applying Krasnoselskii's fixed point theorem. In [10], Henderson and Thompson used lower and upper solution methods to study the existence of multiple solutions for second-order discrete boundary value problems.

We are interested in the existence of positive solutions of the following second-order difference equation with three-point summation boundary value problem (BVP):

$$\Delta^2 u(t-1) + a(t) f(u(t)) = 0, \quad t \in \{1, 2, \ldots, T\},$$
$$u(0) = \beta \sum_{s=1}^{\eta} u(s), \qquad u(T+1) = \alpha \sum_{s=1}^{\eta} u(s), \tag{1.2}$$

where f is continuous, $T \geq 3$ is a fixed positive integer, $\eta \in \{1, 2, \ldots, T-1\}$.

The aim of this paper is to give some results for existence of positive solutions to (1.2), assuming that $0 < \alpha < (2T+2)/\eta(\eta+1)$, $0 < \beta < (2T+2-\alpha\eta(\eta+1))/\eta(2T-\eta+1)$, and f is either superlinear or sublinear. Set

$$f_0 = \lim_{u \to 0^+} \frac{f(u)}{u}, \qquad f_\infty = \lim_{u \to \infty} \frac{f(u)}{u}. \tag{1.3}$$

Then $f_0 = 0$ and $f_\infty = \infty$ correspond to the superlinear case, and $f_0 = \infty$ and $f_\infty = 0$ correspond to the sublinear case. Let $\mathbb{N}$ be the nonnegative integer; we let $\mathbb{N}_{i,j} = \{k \in \mathbb{N} \mid i \leq k \leq j\}$ and $\mathbb{N}_p = \mathbb{N}_{0,p}$. By the positive solution of (1.2), we mean that a function $u(t) : \mathbb{N}_{T+1} \to [0, \infty)$ and satisfies the problem (1.2).

Recently, Sitthiwirattham [11] proved the existence of positive solutions for the boundary value problem with summation condition

$$\Delta^2 u(t-1) + a(t) f(u(t)) = 0, \quad t \in \{1, 2, \ldots, T\},$$
$$u(0) = 0, \qquad u(T+1) = \alpha \sum_{s=1}^{\eta} u(s), \tag{1.4}$$

where f is continuous, $T \geq 3$ is a fixed positive integer, $\eta \in \{1, 2, \ldots, T-1\}$, and $0 < \alpha < 2T + 2/\eta(\eta+1)$.

Throughout this paper, we suppose the following conditions hold:

$(A1)$ $f \in C([0, \infty), [0, \infty))$;

$(A2)$ $a \in C(\mathbb{N}_{T+1}, [0, \infty))$ and there exists $t_0 \in \mathbb{N}_{\eta, T+1}$ such that $a(t_0) > 0$.

The proof of the main theorem is based upon an application of the following Krasnoselskii's fixed point theorem in a cone.

Theorem 1.1 (see [12]). *Let E be a Banach space, and let $K \subset E$ be a cone. Assume Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let*

$$A : K \cap \left(\overline{\Omega}_2 \setminus \Omega_1\right) \longrightarrow K \tag{1.5}$$

be a completely continuous operator such that

(i) $\|Au\| \leqslant \|u\|$, $u \in K \cap \partial\Omega_1$, *and* $\|Au\| \geqslant \|u\|$, $u \in K \cap \partial\Omega_2$, *or*

(ii) $\|Au\| \geqslant \|u\|$, $u \in K \cap \partial\Omega_1$, *and* $\|Au\| \leqslant \|u\|$, $u \in K \cap \partial\Omega_2$.

Then A has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

2. Preliminaries

We now state and prove several lemmas before stating our main results.

Lemma 2.1. *Let $\beta \neq (2T+2-\alpha\eta(\eta+1))/\eta(2T-\eta+1)$. Then, for $y \in C(\mathbb{N}_{T+1}, [0,\infty))$, the problem*

$$\Delta^2 u(t-1) + y(t) = 0, \quad t \in N_{1,T}, \tag{2.1}$$

$$u(0) = \beta\sum_{s=1}^{\eta} u(s), \qquad u(T+1) = \alpha\sum_{s=1}^{\eta} u(s), \tag{2.2}$$

has a unique solution

$$\begin{aligned} u(t) = {} & \frac{\beta\eta(\eta+1) + 2t(1-\beta\eta)}{(2T+2-\alpha\eta(\eta+1)) - \beta\eta(2T-\eta+1)} \sum_{s=1}^{T}(T-s+1)y(s) \\ & - \frac{\beta(T+1) - (\beta-\alpha)t}{(2T+2-\alpha\eta(\eta+1)) - \beta\eta(2T-\eta+1)} \sum_{s=1}^{\eta-1}(\eta-s)(\eta-s-1)y(s) \\ & - \sum_{s=1}^{t-1}(t-s)y(s), \quad t \in \mathbb{N}_{T+1}. \end{aligned} \tag{2.3}$$

Proof. From (2.1), we get

$$\begin{gathered} \Delta u(t) - \Delta u(t-1) = -y(t), \\ \Delta u(t-1) - \Delta u(t-2) = -y(t-1), \\ \vdots \\ \Delta u(1) - \Delta u(0) = -y(1). \end{gathered} \tag{2.4}$$

We sum the above equations to obtain

$$\Delta u(t) = \Delta u(0) - \sum_{s=1}^{t} y(s), \quad t \in \mathbb{N}_T, \tag{2.5}$$

we denote $\sum_{s=p}^{q} y(s) = 0$, if $p > q$. Similarly, summing the above equation from $t = 0$ to $t = h$, we get

$$u(h+1) = u(0) + (h+1)\Delta u(0) - \sum_{s=1}^{h} (h+1-s)y(s), \quad h \in \mathbb{N}_T, \tag{2.6}$$

changing the variable from $h+1$ to t, we have

$$u(t) = u(0) + t\Delta u(0) - \sum_{s=1}^{t-1} (t-s)y(s) : A + Bt - \sum_{s=1}^{t-1} (t-s)y(s), \quad t \in \mathbb{N}_{T+1}. \tag{2.7}$$

We sum (2.7) from $s = 1, 2, \ldots, \eta$,

$$\begin{aligned} \sum_{s=1}^{\eta} u(s) &= \eta A + \frac{\eta(\eta+1)}{2} B - \sum_{s=1}^{\eta-1} \sum_{l=1}^{\eta-s} (l-s)y(s) \\ &= \eta A + \frac{\eta(\eta+1)}{2} B - \frac{1}{2} \sum_{s=1}^{\eta-1} (\eta-s)(\eta-s+1)y(s). \end{aligned} \tag{2.8}$$

By (2.2) from $u(0) = \beta \sum_{s=1}^{\eta} u(s)$, we get

$$(1-\beta\eta)A - \frac{\beta\eta(\eta+1)}{2} B = -\frac{\beta}{2} \sum_{s=1}^{\eta-1} (\eta-s)(\eta-s+1)y(s), \tag{2.9}$$

and from $u(T+1) = \alpha \sum_{s=1}^{\eta} u(s)$, we obtain

$$(1-\alpha\eta)A + \left(T+1-\frac{\alpha\eta(\eta+1)}{2}\right)B = \sum_{s=1}^{T} (T-s+1)y(s) - \frac{\alpha}{2} \sum_{s=1}^{\eta-1} (\eta-s)(\eta-s+1)y(s). \tag{2.10}$$

Therefore,

$$\begin{aligned}
A &= \frac{\beta\eta(\eta+1)}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\sum_{s=1}^{T}(T-s+1)y(s) \\
&\quad - \frac{\beta(T+1)}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\sum_{s=1}^{\eta-1}(\eta-s)(\eta-s+1)y(s), \\
B &= \frac{2(1-\beta\eta)}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\sum_{s=1}^{T}(T-s+1)y(s) \\
&\quad + \frac{\beta-\alpha}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\sum_{s=1}^{\eta-1}(\eta-s)(\eta-s+1)y(s).
\end{aligned} \tag{2.11}$$

Hence, (2.1)-(2.2) has a unique solution

$$\begin{aligned}
u(t) &= \frac{\beta\eta(\eta+1)+2t(1-\beta\eta)}{(2t+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\sum_{s=1}^{T}(T-s+1)y(s) \\
&\quad - \frac{\beta(T+1)-(\beta-\alpha)t}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\sum_{s=1}^{\eta-1}(\eta-s)(\eta-s+1)y(s) \\
&\quad - \sum_{s=1}^{t-1}(t-s)y(s), \quad t\in\mathbb{N}_{T+1}.
\end{aligned} \tag{2.12}$$

□

Lemma 2.2. *Let* $0<\alpha<(2T+2)/\eta(\eta+1)$, $0<\beta<(2T+2-\alpha\eta(\eta+1))/\eta(2T-\eta+1)$. *If* $y\in C(\mathbb{N}_{T+1},[0,\infty))$ *and* $y(t)\geqslant 0$ *for* $t\in\mathbb{N}_{1,T}$, *then the unique solution* u *of* (2.1)-(2.2) *satisfies* $u(t)\geqslant 0$ *for* $t\in\mathbb{N}_{T+1}$.

Proof. From the fact that $\Delta^2 u(t-1)=u(t+1)-2u(t)+u(t-1)=-y(t)\leq 0$, we know that $u(t)\geq(u(t+1)+u(t-1))/2$, so $u(t+1)/(t+1)<u(t)/t$.

Hence

$$\frac{u(T+1)-u(0)}{T+1}<\frac{u(\eta)-u(0)}{\eta+1}, \quad \eta\in\mathbb{N}_{1,T}, \tag{2.13}$$

since $u(T)\geqslant 0$ and $u(0)\geq 0$ imply that $u(t)\geqslant 0$ for $t\in\mathbb{N}_{T+1}$.

Moreover, from $u(i) > (i/\eta)u(\eta) + ((\eta - i)/\eta)u(0)$, we get

$$\begin{aligned}\sum_{s=1}^{\eta} u(s) &> \left[\frac{1}{\eta}u(\eta) + \frac{\eta-1}{\eta}u(0)\right] + \left[\frac{2}{\eta}u(\eta) + \frac{\eta-2}{\eta}u(0)\right] + \cdots + \left[\frac{\eta}{\eta}u(\eta) + \frac{\eta-\eta}{\eta}u(0)\right] \\ &= \frac{1}{\eta}u(\eta)\left[1+2+\cdots+\eta\right] + \frac{1}{\eta}u(0)\left[(\eta-1)+(\eta-2)+\cdots+0\right] \\ &= \frac{1}{\eta}u(\eta)\left[\frac{1}{2}\eta(\eta+1)\right] + \frac{1}{\eta}u(0)\left[\eta^2 - \frac{1}{2}\eta(\eta+1)\right] \\ &= \frac{1}{2}(\eta+1)u(\eta) + \frac{1}{2}(\eta-1)u(0).\end{aligned} \tag{2.14}$$

Combining (2.14) with (2.2), we can get

$$u(0) > \frac{\beta(\eta+1)}{2-\beta(\eta-1)}u(\eta), \tag{2.15}$$

again combining (2.2), (2.14), and (2.15), we obtain

$$u(T+1) > \frac{\alpha(\eta+1)}{2-\beta(\eta-1)}u(\eta), \tag{2.16}$$

such that

$$2-\beta(\eta-1) > 2-\beta\eta > 2 - \frac{2T+2-\alpha\eta(\eta+1)}{2T-\eta+1} = \frac{2(T-\eta)+\alpha\eta(\eta+1)}{2T-\eta+1} > 0. \tag{2.17}$$

By using (2.13), (2.15), and (2.16), we obtain

$$\frac{2-2\beta\eta}{\eta}u(\eta) \geqslant \frac{(\alpha-\beta)(\eta+1)}{T+1}u(\eta). \tag{2.18}$$

If $u(0) < 0$, then $u(\eta) < 0$. It implies that $(2T+2-\alpha\eta(\eta+1))/\eta(2T-\eta+1) \leqslant \beta$, a contradiction to $\beta < (2T+2-\alpha\eta(\eta+1))/\eta(2T-\eta+1)$. If $u(T) < 0$, then $u(\eta) < 0$, and the same contradiction emerges. Thus, it is true that $u(0) \geqslant 0$, $u(T) \geqslant 0$, together with (2.13), we have

$$u(t) \geqslant 0, \quad t \in \mathbb{N}_{T+1}. \tag{2.19}$$

This proof is complete. □

Lemma 2.3. *Let* $\alpha\eta(\eta+1) > 2T+2$, $\beta > \max\{(2T+2-\alpha\eta(\eta+1))/\eta(2T-\eta+1), 0\}$. *If* $y \in C(\mathbb{N}_{T+1}, [0,\infty))$ *and* $y(t) \geqslant 0$ *for* $t \in \mathbb{N}_{1,T}$, *then problem* (2.1)-(2.2) *has no positive solutions.*

Proof. Suppose that problem (2.1)-(2.2) has a positive solution u satisfying $u(t) \geqslant 0$, $t \in \mathbb{N}_{T+1}$, and there is a $\tau_0 \in \mathbb{N}_{1,T}$ such that $u(\tau_0) > 0$.

If $u(T+1) > 0$, then $\sum_{s=1}^{\eta} u(s) > 0$. It implies

$$u(0) = \beta\sum_{s=1}^{\eta}u(s) > \frac{2T+2-\alpha\eta(\eta+1)}{\eta(2T-\eta+1)}\sum_{s=1}^{\eta}u(s) \geqslant \frac{\eta(T+1)(u(0)+u(\eta))-\eta(\eta+1)u(T+1)}{\eta(2T-\eta+1)}, \tag{2.20}$$

that is,

$$\frac{u(T+1)-u(0)}{T+1} > \frac{u(\eta)-u(0)}{\eta+1}, \tag{2.21}$$

which is a contradiction to (2.13).

If $u(T+1) = 0$, then $\sum_{s=1}^{\eta} u(s)ds = 0$. When $\tau_0 \in \mathbb{N}_{1,\eta-1}$, we get $u(\tau_0) > u(T) = 0 > u(\eta)$, which contradicts to (2.13). When $\tau_0 \in \mathbb{N}_{\eta+1,T}$, we get $u(\eta) \leqslant 0 = u(0) < u(\tau_0)$, which contradicts to (2.13) again. Therefore, no positive solutions exist. □

Let $E = C(\mathbb{N}_{T+1}, [0,\infty))$, then E is a Banach space with respect to the norm

$$\|u\| = \sup_{t\in\mathbb{N}_{T+1}} |u(t)|. \tag{2.22}$$

Lemma 2.4. *Let* $0 < \alpha < (2T+2)/\eta(\eta+1)$, $0 < \beta < (2T+2-\alpha\eta(\eta+1))/\eta(2T-\eta+1)$. *If* $y \in C(\mathbb{N}_{T+1}, [0,\infty))$ *and* $y(t) \geqslant 0$, *then the unique solution to problem* (2.1)-(2.2) *satisfies*

$$\inf_{t\in\mathbb{N}_{T+1}} u(t) \geqslant \gamma\|u\|, \tag{2.23}$$

where

$$\gamma := \min\left\{\frac{\alpha(\eta+1)(T+1-\eta)}{(T+1)(2-\beta(\eta-1))-\alpha\eta(\eta+1)}, \frac{\alpha\eta(\eta+1)}{(2-\beta(\eta-1))(T+1)}, \frac{\beta(\eta+1)(T+1-\eta)}{(2-\beta(\eta-1))(T+1)}, \frac{\beta\eta(\eta+1)}{(2-\beta(\eta-1))(T+1)}\right\}. \tag{2.24}$$

Proof. Let $u(t)$ be maximal at $t = \tau_1$, when $\tau_1 \in \mathbb{N}_{1,T}$ and $\|u\| = u(\tau_1)$. We divide the proof into two cases.

Case i. If $u(0) \geqslant u(T+1)$ and $\inf_{t\in\mathbb{N}_{T+1}} u(t) = u(T+1)$, then either $0 \leqslant \tau_1 \leqslant \eta < T+1$ or $0 < \eta < \tau_1 < T+1$, if $0 \leqslant \tau_1 \leqslant \eta < T+1$, then

$$\begin{aligned} u(\tau_1) &\leqslant u(T+1) + \frac{u(T+1) - u(\eta)}{T+1-\eta}(\tau_1 - (T+1)) \\ &\leqslant u(T+1) + \frac{u(T+1) - u(\eta)}{T+1-\eta}(0 - (T+1)) \\ &\leqslant u(T+1)\left[1 - \left(\frac{(T+1) - (T+1)(2-\beta(\eta-1))/\alpha(\eta+1)}{T+1-\eta}\right)\right] \\ &\leqslant u(T+1)\left[\frac{(T+1)(2-\beta(\eta-1)) - \alpha\eta(\eta+1)}{\alpha(T+1)(T+1-\eta)}\right]. \end{aligned} \tag{2.25}$$

This implies

$$\inf_{t\in\mathbb{N}_{T+1}} u(t) \geqslant \frac{\alpha(T+1)(T+1-\eta)}{(T+1)(2-\beta(\eta-1)) - \alpha\eta(\eta+1)}\|u\|. \tag{2.26}$$

Similarly, if $0 < \eta < \tau_1 < T+1$, from

$$\frac{u(\eta)}{\eta} \geqslant \frac{u(\tau_1)}{\tau_1} \geqslant \frac{u(\tau_1)}{T+1}, \tag{2.27}$$

together with (2.16), we have

$$u(T+1) \geqslant \frac{\alpha\eta(\eta+1)}{(2-\beta(\eta-1))(T+1)}\ u(\tau_1). \tag{2.28}$$

This implies

$$\inf_{t\in\mathbb{N}_{T+1}} u(t) \geqslant \frac{\alpha\eta(\eta+1)}{(2-\beta(\eta-1))(T+1)}\ \|u\|. \tag{2.29}$$

Case ii. If $u(0) \leqslant u(T+1)$ and $\inf_{t\in\mathbb{N}_{T+1}} u(t) = u(0)$, then either $0 < \tau_1 < \eta < T+1$ or $0 < \eta \leqslant \tau_1 \leqslant T+1$, by (2.13). If $0 < \tau_1 < \eta < T+1$, from

$$\frac{u(\eta)}{T+1-\eta} \geqslant \frac{u(\tau_1)}{T+1-\tau_1} \geqslant \frac{u(\tau_1)}{T+1}, \tag{2.30}$$

together with (2.15), we have

$$u(0) > \frac{\beta(\eta+1)(T+1-\eta)}{(2-\beta(\eta-1))(T+1)}\ u(\tau_1). \tag{2.31}$$

Hence,

$$\inf_{t\in\mathbb{N}_{T+1}} u(t) > \frac{\beta(\eta+1)(T+1-\eta)}{(2-\beta(\eta-1))(T+1)} \|u\|. \tag{2.32}$$

If $0 < \eta \leqslant \tau_1 \leqslant T+1$, from

$$\frac{u(\tau_1)}{T+1} \leqslant \frac{u(\tau_1)}{\tau_1} \leqslant \frac{u(\eta)}{\eta}, \tag{2.33}$$

together with (2.15), we have

$$u(0) < \frac{\beta\eta(\eta+1)}{(2-\beta(\eta-1))(T+1)} u(\tau_1). \tag{2.34}$$

This implies

$$\inf_{t\in N_{T+1}} u(t) < \frac{\beta\eta(\eta+1)}{(2-\beta(\eta-1))(T+1)} \|u\|. \tag{2.35}$$

This completes the proof. □

In the rest of the paper, we assume that $0 < \alpha < (2T+2)/\eta(\eta+1)$, $T \in \mathbb{N}_{1,T}$; $0 < \beta < (2T+2-\alpha\eta(\eta+1))/\eta(2T-\eta+1)$. It is easy to see that the BVP (1.2) has a solution $u = u(t)$ if and only if u is a solution of the operator equation

$$\begin{aligned} Au(t) \triangleq{}& \frac{\beta\eta(\eta+1)+2t(1-\beta\eta)}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)} \sum_{s=1}^{T}(T-s+1)u(s)f(u(s)) \\ &- \frac{\beta(T+1)-(\beta-\alpha)t}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)} \sum_{s=1}^{\eta-1}(\eta-s)(\eta-s+1)u(s)f(u(s)) \\ &- \sum_{s=1}^{t-1}(t-s)u(s)f(u(s)). \end{aligned} \tag{2.36}$$

Denote

$$K = \left\{u \in E : u \geqslant 0, \min_{t\in\mathbb{N}_{T+1}} u(t) \geqslant \gamma\|u\|\right\}, \tag{2.37}$$

where γ is defined in (2.24).

It is obvious that K is a cone in E. Since $Au = u$ and from Lemmas 2.2 and 2.4, then $A(K) \subset K$. It is also easy to check that $A : K \to K$ is completely continuous. In the following, for the sake of convenience, set

$$\Lambda_1 = \frac{(2T+2)(1-\beta\eta)+\beta\eta(\eta+1)}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\sum_{s=1}^{T}(T-s+1)a(s),$$
$$\Lambda_2 = \frac{\gamma(2-\beta\eta+\beta)(T-\eta)}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\sum_{s=1}^{T}sa(s). \tag{2.38}$$

3. Main Results

Now we are in the position to establish the main result.

Theorem 3.1. *The BVP* (1.2) *has at least one positive solution in the case*

(H_1) $f_0 = 0$ *and* $f_\infty = \infty$ *(superlinear) or*

(H_2) $f_0 = \infty$ *and* $f_\infty = 0$ *(sublinear).*

Proof. Superlinear Case

Let (H_1) hold. Since $f_0 = \lim_{u\to 0^+}(f(u)/u) = 0$ for any $\varepsilon \in (0, \Lambda_1^{-1}]$, there exists ρ_* such that

$$f(u) \leqslant \varepsilon u \quad \text{for } u \in [0, \rho_*]. \tag{3.1}$$

Let $\Omega_{\rho_*} = \{u \in E : \|u\| < \rho_*\}$ for any $u \in K \cap \partial\Omega_{\rho_*}$. From (3.1), we get

$$\begin{aligned}
Au(t) &= \frac{2t(1-\beta\eta)+\beta\eta(\eta+1)}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\sum_{s=1}^{T}(T-s+1)a(s)f(u(s)) \\
&\quad - \frac{\beta(T+1)-(\beta-\alpha)t}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\sum_{s=1}^{\eta-1}(\eta-s)(\eta-s+1)a(s)f(u(s)) \\
&\quad - \sum_{s=1}^{t-1}(t-s)a(s)f(u(s)) \\
&\leqslant \frac{2t(1-\beta\eta)+\beta\eta(\eta+1)}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\sum_{s=1}^{T}(T-s+1)a(s)f(u(s)) \\
&\leqslant \frac{2(T+1)(1-\beta\eta)+\beta\eta(\eta+1)}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\sum_{s=1}^{T}(T-s+1)a(s)f(u(s)) \\
&\leqslant \varepsilon\rho_*\frac{(2T+2)(1-\beta\eta)+\beta\eta(\eta+1)}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\sum_{s=1}^{T}(T-s+1)a(s) \\
&= \varepsilon\Lambda_1\rho_* \leqslant \rho_* = \|u\|,
\end{aligned} \tag{3.2}$$

which yields

$$\|Au\| \leqslant \|u\| \quad \text{for } u \in K \cap \partial\Omega_{\rho_*}. \tag{3.3}$$

Further, since $f_\infty = \lim_{u\to\infty}(f(u)/u) = \infty$, then, for any $M^* \in [\Lambda_2^{-1}, \infty)$, there exists $\rho^* > \rho_*$ such that

$$f(u) \geqslant M^* u \quad \text{for } u \geqslant \gamma\rho^*. \tag{3.4}$$

Set $\Omega_{\rho^*} = \{u \in E : \|u\| < \rho^*\}$ for $u \in K \cap \partial\Omega_{\rho^*}$.

Since $u \in K$, $\min_{t\in N_T} u(t) \geqslant \gamma\|u\| = \gamma\rho^*$. Hence, for any $u \in K \cap \Omega_{\rho^*}$, from (3.4) and (2.23), we get

$$\begin{aligned}
Au(\eta) &= \frac{(2-\beta\eta+\beta)\eta}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\sum_{s=1}^{T}(T-s+1)a(s)f(u(s)) \\
&\quad - \frac{\beta(T+1)-(\beta-\alpha)\eta}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\sum_{s=1}^{\eta}(\eta-s)(\eta-s+1)a(s)f(u(s)) \\
&\quad - \sum_{s=1}^{\eta-1}(\eta-s)a(s)f(u(s)) \\
&= \frac{(2-\beta\eta+\beta)\eta}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\sum_{s=1}^{T}(T-s+1)a(s)f(u(s)) \\
&\quad + \frac{1}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)} \\
&\quad \times \sum_{s=1}^{\eta-1}(\eta-s)\left[-(2-\beta\eta+\beta)T+(\beta(T-\eta)+\alpha\eta+1)s+(\eta-1)\beta\right]a(s)f(u(s)) \\
&= \frac{(2-\beta\eta+\beta)\eta}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\sum_{s=1}^{T}(T-s+1)a(s)f(u(s)) \\
&\quad - \frac{T(2-\beta\eta+\beta)}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\sum_{s=1}^{\eta-1}(\eta-s)a(s)f(u(s)) \\
&\quad + \frac{\beta(t-\eta)+\alpha\eta+1}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\sum_{s=1}^{\eta-1}\left(\eta s-s^2\right)a(s)f(u(s)) \\
&\quad + \frac{(\eta-1)\beta}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\sum_{s=1}^{\eta-1}(\eta-s)a(s)f(u(s)) \\
&\geqslant \frac{(2-\beta\eta+\beta)\eta}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\sum_{s=1}^{T}(T-s+1)a(s)f(u(s))
\end{aligned}$$

$$
\begin{aligned}
&- \frac{T(2-\beta\eta+\beta)}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\sum_{s=1}^{T}(\eta-s)a(s)f(u(s)) \\
&= \frac{(2-\beta\eta+\beta)(T-\eta)}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\sum_{s=1}^{T}sa(s)f(u(s)) \\
&\geqslant \gamma\rho^{*}M^{*}\frac{(2-\beta\eta+\beta)(T-\eta)}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\sum_{s=1}^{T}sa(s) = M^{*}\Lambda_2\rho^{*} \\
&\geqslant \rho^{*} = \|u\|,
\end{aligned}
\tag{3.5}
$$

which implies

$$
\|Au\| \geqslant \|u\| \quad \text{for } u \in K \cap \partial\Omega_{\rho^{*}}. \tag{3.6}
$$

Therefore, from (3.3), (3.6), and Theorem 1.1, it follows that A has a fixed point in $K \cap (\overline{\Omega}_{\rho^{*}} \setminus \Omega_{\rho_{*}})$ such that $\rho_{*} \leqslant \|u\| \leqslant \rho^{*}$.

Sublinear Case

Let (H_2) hold. In view of $f_0 = \lim_{u\to 0^{+}}(f(u)/u) = \infty$ for any $M_{*} \in [\Lambda_2^{-1}, \infty)$, there exists $r_{*} > 0$ such that

$$
f(u) \geqslant M_{*}u \quad \text{for } 0 \leqslant u \leqslant r_{*}. \tag{3.7}
$$

Set $\Omega_{r_{*}} = \{u \in E : \|u\| < r_{*}\}$ for $u \in K \cap \partial\Omega_{r_{*}}$. Since $u \in K$, then $\min_{t\in\mathbb{N}_{T+1}} u(t) \geqslant \gamma\|u\| = \gamma r_{*}$. Thus, from (3.7) for any $u \in K \cap \partial\Omega_{r_{*}}$, we can get

$$
\begin{aligned}
Au(\eta) &= \frac{(2-\beta\eta+\beta)\eta}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\sum_{s=1}^{T}(T-s+1)a(s)f(u(s)) \\
&\quad - \frac{\beta(T+1)-(\beta-\alpha)\eta}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\sum_{s=1}^{\eta-1}(\eta-s)(\eta-s+1)a(s)f(u(s)) \\
&\quad - \sum_{s=1}^{\eta-1}(\eta-s)a(s)f(u(s)) \\
&\geqslant \gamma r_{*}M_{*}\frac{(2-\beta\eta+\beta)(T-\eta)}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\sum_{s=1}^{T}sa(s) = M_{*}\Lambda_2 r_{*} \geqslant r_{*} = \|u\|,
\end{aligned}
\tag{3.8}
$$

which yields

$$
\|Au\| \geqslant \|u\| \quad \text{for } u \in K \cap \partial\Omega_{r_{*}}. \tag{3.9}
$$

Since $f_\infty = \lim_{u\to\infty}(f(u)/u) = 0$, then, for any $\varepsilon_1 \in (0, \Lambda_1^{-1}]$, there exists $r_0 > r_*$ such that

$$f(u) \leqslant \varepsilon_1 u \quad \text{for } u \in [r_0, \infty). \tag{3.10}$$

We have the following two cases.

Case i. Suppose that $f(u)$ is unbounded, then, from $f \in C([0,\infty),[0,\infty))$, we know that there is $r^* > r_0$ such that

$$f(u) \leqslant f(r^*) \quad \text{for } u \in [0, r^*]. \tag{3.11}$$

Since $r^* > r_0$, then, from (3.10) and (3.11), one has

$$f(u) \leqslant f(r^*) \leqslant \varepsilon_1 r^* \quad \text{for } u \in [0, r^*]. \tag{3.12}$$

For $u \in K$, $\|u\| = r^*$, from (3.12), we obtain

$$\begin{aligned} Au(t) &\leqslant \frac{(2T+2)(1-\beta\eta)+\beta\eta(\eta+1)}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\sum_{s=1}^{T}(T-s+1)a(s)f(u(s)) \\ &\leqslant \varepsilon_1 r^* \frac{(2T+2)(1-\beta\eta)+\beta\eta(\eta+1)}{(2T+2-\alpha(\eta+1))-\beta\eta(2T-\eta+1)}\sum_{s=1}^{T}(T-s+1)a(s) \\ &= \varepsilon_1\Lambda_1 r^* \leqslant r^* = \|u\|. \end{aligned} \tag{3.13}$$

Case ii. Suppose that $f(u)$ is bounded, say $f(u) \leqslant N$ for all $u \in [0,\infty)$. Taking $r^* \geqslant \max\{N/\varepsilon_1, r_*\}$, for $u \in K$, $\|u\| = r^*$, we have

$$\begin{aligned} Au(t) &\leqslant \frac{(2T+2)(1-\beta\eta)+\beta\eta(\eta+1)}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\sum_{s=1}^{T}(T-s+1)a(s)f(u(s)) \\ &\leqslant N\frac{(2T+2)(1-\beta\eta)+\beta\eta(\eta+1)}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\sum_{s=1}^{T}(T-s+1)a(s) \\ &\leqslant \varepsilon_1 r^* \frac{(2T+2)(1-\beta\eta)+\beta\eta(\eta+1)}{(2T-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\sum_{s=1}^{T}(T-s+1)a(s) \\ &= \varepsilon_1\Lambda_1 r^* \leqslant r^* = \|u\|. \end{aligned} \tag{3.14}$$

Hence, in either case, we may always set $\Omega_{r^*} = \{u \in E : \|u\| < r^*\}$ such that

$$\|Au\| \leqslant \|u\| \quad \text{for } u \in K \cap \partial\Omega_{r^*}. \tag{3.15}$$

Hence, from (3.9), (3.15), and Theorem 1.1, it follows that A has a fixed point in $K \cap (\overline{\Omega}_{\rho^*} \setminus \Omega_{\rho_*})$ such that $r_* \leqslant \|u\| \leqslant r^*$. The proof is complete. □

4. Some Examples

In this section, in order to illustrate our result, we consider some examples.

Example 4.1. Consider the BVP

$$\Delta^2 u(\mathrm{t}-1) + t^2 u^k = 0, \quad t \in N_{1,4},$$
$$u(0) = \frac{1}{3}\sum_{s=1}^{2} u(s), \qquad u(5) = \frac{2}{3}\sum_{s=1}^{2} u(s). \tag{4.1}$$

Set $\alpha = 2/3$, $\beta = 1/3$, $\eta = 2$, $T = 4$, $a(t) = t^2$, and $f(u) = u^k$.

We can show that

$$0 < \alpha = \frac{2}{3} < \frac{5}{3} = \frac{2T+2}{\eta(\eta+1)}, \qquad 0 < \beta = \frac{1}{3} < \frac{3}{7} = \frac{2T+2-\alpha\eta(\eta+1)}{\eta(2T-\eta+1)}. \tag{4.2}$$

Case I. $k \in (1, \infty)$. In this case, $f_0 = 0$, $f_\infty = \infty$, and H_1 of Theorem 3.1 holds. Then BVP (4.1) has at least one positive solution.

Case II. $k \in (0, 1)$. In this case, $f_0 = \infty$, $f_\infty = 0$, and H_2 of Theorem 3.1 holds. Then BVP (4.1) has at least one positive solution.

Example 4.2. Consider the BVP

$$\Delta^2 u(t-1) + e^t t^e \left(\frac{\pi \sin u + 2\cos u}{u^2}\right) = 0, \quad t \in N_{1,4},$$
$$u(0) = \frac{1}{4}\sum_{s=1}^{3} u(s), \qquad u(5) = \frac{1}{3}\sum_{s=1}^{3} u(s). \tag{4.3}$$

Set $\alpha = 1/3$, $\beta = 1/4$, $\eta = 3$, $T = 4$, $a(t) = e^t t^e$, $f(u) = (\pi \sin u + 2\cos u)/u^2$.

We can show that

$$0 < \alpha = \frac{1}{3} < \frac{5}{6} = \frac{2T+2}{\eta(\eta+1)}, \qquad 0 < \beta = \frac{1}{4} < \frac{1}{3} = \frac{2T+2-\alpha\eta(\eta+1)}{\eta(2T-\eta+1)}. \tag{4.4}$$

Through a simple calculation we can get $f_0 = \infty$, $f_\infty = 0$. Thus, by H_2 of Theorem 3.1, we can get BVP (4.3) has at least one positive solution.

Acknowledgment

This research is supported by the Centre of Excellence in Mathematics, Commission on Higher Education, Thailand.

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