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Research Article

A Note on the Eigenvalue Analysis of the SIMPLE Preconditioning for Incompressible Flow

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We consider the SIMPLE preconditioning for block two-by-two generalized saddle point problems; this is the general nonsymmetric, nonsingular case where the (1,2) block needs not to equal the transposed (2,1) block, and the (2,2) block may not be zero. The eigenvalue analysis of the SIMPLE preconditioned matrix is presented. The relationship between the two different formulations spectrum of the SIMPLE preconditioned matrix is established by using the theory of matrix eigenvalue, and some corresponding results in recent article by Li and Vuik (2004) are extended.

1. Introduction

Consider the two-by-two generalized saddle point problems

$$\mathcal{A}\begin{bmatrix} x \\ y \end{bmatrix} \equiv \begin{bmatrix} A & B^T \\ C & -D \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}, \tag{1.1}$$

where $A \in \mathbb{R}^{n\times n}$ is nonsingular, $B, C \in \mathbb{R}^{m\times n}$ $(m \le n)$, $D \in \mathbb{R}^{m\times m}$.

Systems of the form (1.1) arise in a variety of scientific and engineering applications, such as linear elasticity, fluid dynamics, electromagnetics, and constrained quadratic programming [1–4]. We refer the reader to [5] for more applications and numerical solution techniques of (1.1).

Since the coefficient matrix of (1.1) is often large and sparse, it may be attractive to use iterative methods. In particular, Krylov subspace methods might be used. As known, Krylov subspace methods are considered as one kind of the important and efficient iterative techniques for solving the large sparse linear systems because these methods are cheap to be implemented and are able to fully exploit the sparsity of the coefficient matrix. It is well known that the convergence speed of Krylov subspace methods depends on the eigenvalue distribution of the coefficient matrix [6]. Since the coefficient matrix of (1.1) is often extremely ill-conditioned and highly indefinite, the convergence speed of Krylov subspace methods can be unacceptably slow. In this case, Krylov subspace methods are not competitive without a good preconditioner. That is, preconditioning technique is a key ingredient for the success of Krylov subspace methods in applications.

To efficiently and accurately solve (1.1), Semi-implicit method for pressure linked equations (SIMPLE) were presented in [7] by Patankar. Subsequently, combining the SIMPLE(R) algorithm and Krylov subspace method GCR [8], Vuik et al. [9] proposed the GCR-SIMPLE(R) algorithm for solving (1.1). In this algorithm, the SIMPLE iteration is used as a preconditioner in the GCR method. Numerical experiments show that the SIMPLE(R) preconditioning is effective and competitive.

It is well known that the spectral properties of the preconditioned matrix give important insight in the convergence behavior of the preconditioned Krylov subspace methods. In [10], the eigenvalue analysis was given for the SIMPLE preconditioned matrix with $B = C$ and $D = 0$, and two different formulations spectrum of the preconditioned matrix were derived. The relationship between the two different formulations has been built by using the theory of matrix singular value decomposition. If $B \neq C$ and $D \neq 0$, using matrix singular value decomposition to establish the relationship between the two different formulations is invalid. On this occasion, we present the relationship between the two different formulations by using the theory of matrix eigenvalue and overcome the shortcomings of [10]. Some corresponding results in [10] are extended to two-by-two generalized saddle point problems.

2. Spectral Analysis

For simplicity, $\sigma(\cdot)$ denotes the set of all eigenvalues of a matrix, and the diagonal entries of A are not equal to zero. If the SIMPLE algorithm is used as preconditioning, it is equivalent to choose the preconditioner $\mathcal{P}$ as

$$\mathcal{P} = \mathcal{M}\mathcal{B}^{-1}, \tag{2.1}$$

where

$$\mathcal{B} = \begin{bmatrix} I & -Q^{-1}B^T \\ 0 & I \end{bmatrix}, \qquad \mathcal{M} = \begin{bmatrix} A & 0 \\ C & R \end{bmatrix}, \qquad Q = \operatorname{diag}(A), \qquad R = -\left(D + CQ^{-1}B^T\right). \tag{2.2}$$

On the nonsingular of $\mathcal{A}$ and $\mathcal{P}$ we have the following proposition.

Proposition 2.1. *The matrices $\mathcal{A}$ and $\mathcal{P}$, respectively, in* (1.1) *and* (2.1) *are nonsingular if and only if the Schur complements $-(D + CA^{-1}B^T)$ and $-(D + CQ^{-1}B^T)$, respectively, are nonsingular.*

In this paper, we assume that $\mathcal{A}$ and $\mathcal{P}$ are nonsingular and that B and C are of full rank.

Proposition 2.2. *If the right preconditioner $\mathcal{P}$ is defined by* (2.1)*, then the preconditioned matrix is*

$$\widetilde{\mathcal{A}} = \mathcal{A}\mathcal{P}^{-1} = \begin{bmatrix} I - (I - AQ^{-1})B^T R^{-1} C A^{-1} & (I - AQ^{-1})B^T R^{-1} \\ 0 & I \end{bmatrix}. \tag{2.3}$$

Therefore, the spectrum of the SIMPLE preconditioned matrix $\tilde{\mathcal{A}}$ *is*

$$\sigma\left(\tilde{\mathcal{A}}\right) = \{1\} \cup \sigma\left(I - \left(I - AQ^{-1}\right)B^T R^{-1} C A^{-1}\right). \tag{2.4}$$

Proof. By simple computations, it is easy to verify that

$$\begin{aligned}
\mathcal{M}^{-1} &= \begin{bmatrix} A^{-1} & 0 \\ -R^{-1}CA^{-1} & R^{-1} \end{bmatrix} \\
\sigma\left(\tilde{\mathcal{A}}\right) &= \begin{bmatrix} A & B^T \\ C & -D \end{bmatrix} \begin{bmatrix} I & -Q^{-1}B^T \\ 0 & I \end{bmatrix} \begin{bmatrix} A^{-1} & 0 \\ -R^{-1}CA^{-1} & R^{-1} \end{bmatrix} \\
&= \begin{bmatrix} I - (I - AQ^{-1})B^T R^{-1} C A^{-1} & (I - AQ^{-1})B^T R^{-1} \\ 0 & I \end{bmatrix}.
\end{aligned} \tag{2.5}$$

Further, it is easy to find that the form of the spectrum of $\sigma(\tilde{\mathcal{A}})$ is described by (2.4). □

By the similarity invariance of the spectrum of the matrix, we have

$$\begin{aligned}
\sigma\left(I - \left(I - AQ^{-1}\right)B^T R^{-1} C A^{-1}\right) &= \sigma\left(I - \left(A^{-1} - Q^{-1}\right)B^T R^{-1} C\right) \\
&= \sigma\left(I - Q^{-1}(Q - A)A^{-1}B^T R^{-1} C\right) \\
&= \sigma\left(I - JA^{-1}B^T R^{-1} C\right),
\end{aligned} \tag{2.6}$$

where the matrix $J = Q^{-1}(Q - A)$ is the Jacobi iteration matrix of the matrix A. Further, we have the following proposition.

Proposition 2.3. *For the SIMPLE preconditioned matrix* $\tilde{\mathcal{A}}$,

(1) *1 is an eigenvalue with multiplicity at least of* m,

(2) *the remaining eigenvalues are* $1 - \mu_i$, $i = 1, 2, \ldots, n$, *where* μ_i *is the* i*th eigenvalue of*

$$ZEx = \mu x, \tag{2.7}$$

where

$$Z = JA^{-1} \in \mathbb{R}^{n\times n}, \qquad E = B^T R^{-1} C \in \mathbb{R}^{n\times n}. \tag{2.8}$$

In fact, we also have the following result.

Proposition 2.4. *For the SIMPLE preconditioned matrix* $\tilde{\mathcal{A}}$,

(1) *1 is an eigenvalue with (algebraic and geometric) multiplicity of* n,

(2) *the remaining eigenvalues are defined by the generalized eigenvalue problem*

$$Sx = \lambda Rx, \tag{2.9}$$

where $S = -(D + CA^{-1}B^T)$ *is the Schur complement of the matrix* $\mathcal{A}$.

Proof. Note that $\mathcal{A}\mathcal{P}^{-1}$ is the same spectrum as $\mathcal{P}^{-1}\mathcal{A}$. So, it is only needed to consider the following generalized eigenvalue problem

$$\mathcal{A}x = \lambda \mathcal{P}x, \tag{2.10}$$

where

$$\mathcal{A} = \begin{bmatrix} A & B^T \\ C & -D \end{bmatrix}, \qquad \mathcal{P} = \begin{bmatrix} A & AQ^{-1}B^T \\ C & -D \end{bmatrix}. \tag{2.11}$$

The generalized eigenvalue problem (2.10) can be written as

$$\begin{bmatrix} A & B^T \\ C & -D \end{bmatrix}\begin{bmatrix} u \\ p \end{bmatrix} = \lambda \begin{bmatrix} A & AQ^{-1}B^T \\ C & -D \end{bmatrix}\begin{bmatrix} u \\ p \end{bmatrix}, \tag{2.12}$$

that is,

$$Au + B^T p = \lambda\left(Au + AQ^{-1}B^T p\right), \tag{2.13}$$

$$Cu - Dp = \lambda(Cu - Dp). \tag{2.14}$$

From (2.13) and (2.14), it is easy to see that $\lambda = 1$ is an eigenvalue of (2.12). If the matrix $Q^{-1} - A^{-1}$ is nonsingular with $\lambda = 1$ and rank $(B^T) = m$, from (2.13) we have $p = 0$. Therefore, the eigenvectors corresponding to eigenvalue 1 are

$$v_i = \begin{bmatrix} u_i \\ 0 \end{bmatrix}, \quad u_i \in \mathbb{R}^n, \ i = 1, 2, \ldots, n, \tag{2.15}$$

where $\{u_i\}_i^n$ is a basis of $\mathbb{R}^n$.

For $\lambda \neq 1$, from (2.13) we obtain

$$u = \frac{1}{1-\lambda} A^{-1}\left(\lambda AQ^{-1}B^T p - B^T p\right). \tag{2.16}$$

Substituting it into (2.14) yields

$$Sp = \lambda Rp, \tag{2.17}$$

where $S = -(D + CA^{-1}B^T)$ is the Schur complement of the matrix $\mathcal{A}$. □

From Propositions 2.3 and 2.4, two different generalized eigenvalue problems (2.7) and (2.9) have been derived to describe the spectrum of $\widetilde{\mathcal{A}}$. Subsequently, we will investigate the relationship between both spectral formulations for the nonsymmetric case. Here we will make use of the theory of matrix eigenvalue to establish the relationship of the two different formulations spectrum of the SIMPLE preconditioned matrix. To this end, the following lemma is required.

Lemma 2.5 (See [11]). *Suppose that $M \in \mathbb{R}^{m\times n}$ and $N \in \mathbb{R}^{n\times m}$ with $m \le n$. Then NM has the same eigenvalues as MN, counting multiplicity, together with an additional $n - m$ eigenvalues equal to 0.*

By (2.7), it follows that

$$\begin{gathered} ZE = Z^{n\times n}\left(B^T\right)^{n\times m}\left(R^{-1}\right)^{m\times m}C^{m\times n} \in \mathbb{R}^{n\times n}, \\ \left(R^{-1}\right)^{m\times m}C^{m\times n}Z^{n\times n}\left(B^T\right)^{n\times m} \in \mathbb{R}^{m\times m}. \end{gathered} \tag{2.18}$$

From Lemma 2.5, we have

$$\sigma(ZE) = \{0\} \cup \sigma\left(R^{-1}CZB^T\right), \tag{2.19}$$

where the eigenvalue 0 is with multiplicity of $n - m$ and

$$\begin{aligned} \sigma\left(R^{-1}CZB^T\right) &= \sigma\left(R^{-1}CQ^{-1}(Q - A)A^{-1}B^T\right) \\ &= \sigma\left(R^{-1}C\left(A^{-1} - Q^{-1}\right)B^T\right) \\ &= \sigma\left(R^{-1}\left(CA^{-1}B^T - CQ^{-1}B^T\right)\right) \\ &= \sigma\left(R^{-1}\left[\left(CA^{-1}B^T + D\right) - \left(CQ^{-1}B^T + D\right)\right]\right) \\ &= \sigma\left(R^{-1}(R - S)\right) \\ &= \sigma\left(I - R^{-1}S\right) \\ &= \cup\{1 - \lambda_i\}, \quad i = 1, 2, \ldots, m. \end{aligned} \tag{2.20}$$

These relations lead to the following proposition.

Proposition 2.6. *For two generalized eigenvalue problems* (2.7) *and* (2.9)*, suppose that* $\mu_i \in \sigma(ZE)$*,* $i = 1, 2, \ldots, n$*, and* $\lambda_i \in \sigma(R^{-1}S)$*,* $i = 1, 2, \ldots, m$*, the relationship between two problems is that* $\mu = 0$ *is an eigenvalue of* (2.7) *with multiplicity of* $n - m$*, which can be denoted as* $\mu_{m+1} = \mu_{m+2} = \cdots = \mu_n = 0$*, and that* $\lambda_i = 1 - \mu_i$*,* $i = 1, 2, \ldots, m$*, holds for the remaining* m *eigenvalues.*

Some remarks on Proposition 2.6 are given as follows.

(i) In [10], the relationship between two different formulations spectrum of the preconditioned matrix with $B = C$ and $D = 0$ was built by using the theory of matrix singular value decomposition, but for the nonsymmetric case, the above strategy is invalid. Whereas, using the theory of matrix eigenvalue not only establishes the relationship between the two different formulations, but also overcomes the shortcomings of [10]. In this way, Propositions 2.2–2.6 can be regarded as the extension of Propositions 2–5 [10].

(ii) In [10], the diagonal entries of matrix A must be positive. But, in this paper, the diagonal entries of Q are only not equal to zero. Clearly, this assumption is weaker than that of [10]. If the diagonal entries of matrix A are complex and not equal to zero, then the diagonal entries of Q take the absolute diagonal entries of A. This idea is based on an absolute diagonal scaling technique, which is cheaply easy to implement, reducing computation times and amount of memory.

(iii) Recently, although Li et al. in [12] discussed the SIMPLE preconditioning for the generalized nonsymmetric saddle point problems and provided some results above the spectrum of the SIMPLE preconditioned matrix, some conditions of the supporting propositions may be defective. In fact, if A is nonsingular with rank (B^T) = rank $(C) = m$, then R and $\not{D}$ may be singular. For a counterexample, we take $C = [1\ 0]$, $A = Q = 2I$ and $B = [0\ 1]$, then $R = CQ^{-1}B^T = 0$. That is, this paper corrects some results in [12].

(iv) In fact, Q is not necessary the diagonal entries of A; in this case, the diagonal entries of A can be equal to zero. In actual implements, the choice of matrix Q is that the eigenvalue of the generalized eigenvalue problem (2.9) is close to one; Krylov subspace methods such as GMRES will converge quickly.

3. Conclusion

In this paper, the SIMPLE preconditioner for the nonsymmetric generalized saddle point problems is discussed. The relationship of the two different formulations spectrum of the SIMPLE preconditioned matrix has been built by using the theory of matrix eigenvalue.

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