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Research Article

The Application of the Homotopy Perturbation Method and the Homotopy Analysis Method to the Generalized Zakharov Equations

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We introduce two powerful methods to solve the generalized Zakharov equations; one is the homotopy perturbation method and the other is the homotopy analysis method. The homotopy perturbation method is proposed for solving the generalized Zakharov equations. The initial approximations can be freely chosen with possible unknown constants which can be determined by imposing the boundary and initial conditions; the homotopy analysis method is applied to solve the generalized Zakharov equations. HAM is a strong and easy-to-use analytic tool for nonlinear problems. Computation of the absolute errors between the exact solutions of the GZE equations and the approximate solutions, comparison of the HPM results with those of Adomian's decomposition method and the HAM results, and computation the absolute errors between the exact solutions of the GZE equations with the HPM solutions and HAM solutions are presented.

1. Introduction

Nonlinearpartial differential equations are useful in describing the various phenomena in disciplines. Apart from a limited number of these problems, most of them do not have a precise analytical solution, so these nonlinear equations should be solved using approximate methods.

The application of the homotopy perturbation method (HPM) [1, 2] in nonlinear problems has been devoted by scientists and engineers, because this method is continuously.

Deform a simple problem which is easy to solve into the under study problem which is difficult to solve. The homotopy perturbation method was first proposed by He [3–6]. For solving differential and integral equations, linear and nonlinear has been the subject of extensive analytical and numerical studies. The method is a coupling of the traditional perturbation method and homotopy in topology. This method, which does not require a small parameter in an equation, has a significant advantage in that it provides an analytical approximate solution to a wide range of nonlinear problems in applied sciences. This HPM has already been applied successfully to solve the Laplace equation, nonlinear dispersive K(mp) equations, heat radiation equations, nonlinear integral equations, nonlinear heat conduction and convection equations, nonlinear oscillators, nonlinear Schrodinger equations, nonlinear wave equations, nonlinear chemistry problems, and other fields [7]. This HPM yields a very rapid convergence of the solution series in most cases, usually only a few iterations leading to very accurate solutions. Thus He's HPM is a universal one which can solve various kinds of nonlinear equations. The HPM yields a very rapid convergence of the solution series in the most cases. The method does not depend on a small parameter in the equation. Using homotopy technique in topology, a homotopy is constructed with an embedding parameter $p \in [0,1]$ which is considered as a "small parameter." No need to linearization or discretization; large computational work and round-off errors are avoided. It has been used to solve effectively, easily, and accurately a large class of nonlinear problems with approximations. These approximations converge rapidly to accurate solutions [7–10].

The HPM was successfully applied to nonlinear oscillators with discontinuities [4] and bifurcation of nonlinear problem [11]. In [6], a comparison of HPM and homotopy analysis method was made.

In [12] the homotopy perturbation method is applied to compute the Laplace transform and construct solitary wave solutions for a generalized Hirota-Satsuma-coupled KdV equation [13]. In [14] the HPM is employed to compute an approximation to the solution of the epidemic model. As well, in [15] is applied the homotopy perturbation method for solving the Lane-Emden-type singular IVPs problem, in [16] using the homotopy perturbation method to find exact solutions of nonlinear differential-difference equations.

In 1992, Liao employed the basic ideas of the homotopy in topology to propose a general analytic method for nonlinear problems, namely, homotopy analysis method (HAM) [17–20]. This method has been successfully applied to solve many types of nonlinear problems by others [21–25].

In this paper, we consider the generalized Zakharov equations (GZE) which are a set of coupled equations and can be written as [26–29]

$$\begin{gathered} i\frac{\partial E}{\partial t} + \frac{\partial^2 E}{\partial x^2} - 2\beta|E|^2E + 2nE = 0, \\ \frac{\partial^2 n}{\partial t^2} - \frac{\partial^2 n}{\partial x^2} + \frac{\partial^2 |E|^2}{\partial x^2} = 0, \end{gathered} \tag{1.1}$$

where β is an arbitrary constant and E is the envelope of the high-frequency electric field, and n is the plasma density measured from its equilibrium value. When $\beta = 0$, this system is reduced to the classical Zakharov equations of plasma physics. Because the GZE is much closer to the realistic model in plasma, it is meaningful for us to study the solitary wave solutions of the GZE. The motivation of this paper is to apply the Homotopy perturbation method and the homotopy analysis method to the problem mentioned above. When implementing the homotopy perturbation method (HPM) and the homotopy analysis method (HAM), we get the explicit solutions of the GZE equations without using any transformation method. Furthermore, we will show that considerably better approximations related to the accuracy level would be obtained. Comparing the HPM results for the study problem with the Adomian decomposition method (ADM) results takes six terms in evaluating the approximate solutions and HAM results which take eight terms in evaluating the approximate solutions of the generalized Zakharov equations.

2. Basic Idea of He's Homotopy Perturbation Method

The homotopy perturbation method is a combination of the classical perturbation technique and homotopy technique, which has eliminated the limitations of the traditional perturbation methods. This technique can have full advantage of the traditional perturbation techniques. To illustrate the basic idea of the homotopy perturbation method for solving nonlinear differential equations, we consider the following nonlinear differential equation:

$$A(u) - f(r) = 0, \qquad r \in \Omega, \tag{2.1}$$

subject to boundary condition

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \qquad r \in \Gamma, \tag{2.2}$$

where A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytic function, and Γ is the boundary of the domain Ω.

The operator A can, generally speaking, be divided into two parts: a linear part L and a nonlinear part N. Equation (2.1) therefore can be rewritten as follows:

$$L(u) + N(u) - f(r) = 0. \tag{2.3}$$

By the homotopy technique, we construct a homotopy $V(r,p) : \Omega \times [0,1] \rightarrow \Re$, which satisfies

$$H(V,p) = (1-p)[L(V) - L(u_0)] + p[A(V) - f(r)] = 0, \qquad p \in [0,1], r \in \Omega, \tag{2.4}$$

or

$$H(V,p) = L(V) - L(u_0) + pL(u_0) + p[N(V) - f(r)] = 0, \tag{2.5}$$

where $p \in [0,1]$ is an embedding parameter and u_0 is an initial approximation of (2.1) which satisfies the boundary conditions. It follows from (2.4) and (2.5) that we will have,

$$H(V,0) = L(V) - L(u_0), \qquad H(V,1) = A(V) - f(r). \tag{2.6}$$

Thus, the changing process of p from zero to unity is just that of $v(r,p)$ from $u_0(r)$ to $u(r)$. In topology, this is called deformation and $L(v) - L(u_0)$, $A(v) - f(r)$ are called homotopic.

According to the HPM, we can first use the embedding parameter p as a "small parameter," and assume that the solution of (2.4) and (2.5) can be written as a power series in p:

$$V = V_0 + pV_1 + p^2V_2 + \cdots . \tag{2.7}$$

Setting $p = 1$ results in the approximate solution of (2.1):

$$u = \lim_{p\to 1} V = V_0 + V_1 + V_2 + \cdots . \tag{2.8}$$

The series (2.8) is convergent for most cases; however, the convergent rate depends upon the nonlinear operator $A(V)$ (the following opinions are suggested by [2, 30]).

(1) The second derivative of $N(V)$ with respect to V must be small because the parameter may be relatively large; that is, $p \to 1$.

(2) The norm of $L^{-1}\partial N/\partial V$ must be smaller than one so that the series converges.

3. Basic Idea of Homotopy Analysis Method

In this paper, we apply the homotopy analysis method [17–20] to find the approximate solutions for the problem. Let us consider the following differential equation:

$$N[z(x,t)] = 0, \tag{3.1}$$

where N is a nonlinear operator, x and t denote independent variables, and $z(x,t)$ is an unknown function, respectively. For simplicity, we ignore all boundary or initial conditions, which can be treated in the similar way. By means of generalizing the traditional homotopy method, Liao [18] constructs the so-called zero-order deformation equation:

$$(1-p)L[\phi(x,t;p) - z_0(x,t)] = p\hbar N[\phi(x,t;p)], \tag{3.2}$$

where $p \in [0,1]$ is the embedding parameter, $\hbar \neq 0$ is a nonzero auxiliary parameter, L is an auxiliary linear operator, $z_0(x,t)$ is an initial guess of $z(x,t)$, and $\phi(x,t;p)$ is an unknown function, respectively. It is important that one has great freedom to choose auxiliary things in homotopy analysis method. When $p = 0$ and $p = 1$, it holds

$$\phi(x,t;\ 0) = z_0(x,t), \qquad \phi(x,t;\ 1) = z(x,t), \tag{3.3}$$

respectively. Thus as p increase from 0 to 1, the solution $\phi(x,t;p)$ varies from the initial guess $z_0(x,t)$ to the solution $z(x,t)$. Expanding $\phi(x,t;p)$ in the Taylor series with respect to p, one has

$$\phi(x,t;p) = z_0(x,t) + \sum_{m=1}^{\infty} z_m(x,t)p^m, \tag{3.4}$$

where

$$z_m(x,t) = \frac{1}{m!} \frac{\partial^m \phi(x,t;p)}{\partial p^m}\bigg|_{p=0}. \tag{3.5}$$

If the auxiliary linear operator, the initial guess, and the auxiliary parameter $\hbar$ are so properly chosen and the series (3.4) converges at $p = 1$, one has

$$z(x,t) = z_0(x,t) + \sum_{m=1}^{\infty} z_m(x,t), \tag{3.6}$$

which must be one of solutions of original nonlinear equation, as proved by Liao [18]. As $\hbar = -1$, (3.2) becomes

$$(1-p)L[\phi(x,t;p) - z_0(x,t)] + pN[\phi(x,t;p)] = 0, \tag{3.7}$$

which is used mostly in the homotopy perturbation method, whereas the solution obtained directly, without using the Taylor series [31]. The comparison between HAM and HPM can be found in [6].

According to (3.5), the governing equation can be deduced from the zero-order deformation equation (3.2). Define the vector

$$\vec{z_n} = \{z_0(x,t), z_1(x,t), \ldots, z_n(x,t)\}. \tag{3.8}$$

Differentiating (3.2) m times with respect to the embedding parameter p and setting $p = 0$ and finally dividing them by $m!$, we have the so-called mth-order deformation equation:

$$L[z_m(x,t) - \chi_m z_{m-1}(x,t)] = \hbar R_m(\vec{z}_{m-1}), \tag{3.9}$$

where

$$\begin{gathered} R_m(\vec{z}_{m-1}) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N[\phi(x,t;p)]}{\partial p^{m-1}}\bigg|_{p=0}, \\ \chi_m = \begin{Bmatrix} 0, & m \leq 1, \\ 1, & m > 1. \end{Bmatrix} \end{gathered} \tag{3.10}$$

It should be emphasized that $z_m(x,t)$ for $m \geq 1$ is governed by the linear equation (3.9) with the linear boundary conditions that come from original problem, which can be easily solved by symbolic computation software such as Maple.

First, we separate the complex high-frequency electric field E into real part and imaginary part; that is, $E = E_1 + iE_2$. Then we rewrite the system (1.1) in the following form:

$$\begin{aligned} \frac{\partial E_1}{\partial t} &= -\frac{\partial^2 E_2}{\partial x^2} + 2\beta\left(E_1^2 + E_2^2\right)E_2 - 2nE_2, \\ \frac{\partial E_2}{\partial t} &= \frac{\partial^2 E_1}{\partial x^2} - 2\beta\left(E_1^2 + E_2^2\right)E_1 + 2nE_1, \\ \frac{\partial^2 n}{\partial t^2} &= \frac{\partial^2 n}{\partial x^2} - \frac{\partial^2}{\partial x^2}\left(E_1^2 + E_2^2\right), \end{aligned} \tag{3.11}$$

with the following initial conditions [27]:

$$\begin{aligned} E_1(x,0) &= r\tanh(kx)\cos(k_1 x), \\ E_2(x,0) &= r\tanh(kx)\sin(k_1 x), \\ n(x,0) &= s + \frac{r^2}{-4k_1^2 + 1}\tanh^2(kx), \end{aligned} \tag{3.12}$$

where $r = \sqrt{k^2(4k_1^2 - 1)/(1 + (4k_1^2 - 1)\beta)}$, k, s, β, and k_1 are arbitrary constants.

4. Application the Homotopy Perturbation Method for the Generalized Zakharov Equations

To investigate the traveling wave solution of (3.11), we first construct a homotopy as follows:

$$\begin{aligned} (1-p)\left[\dot{v}_1 - \dot{E}_{1,0}\right] + p\left[\dot{v}_1 + v_2'' - 2\beta\left(v_1^2 + v_2^2\right)v_2 + 2v_2 v_3\right] &= 0, \\ (1-p)\left[\dot{v}_2 - \dot{E}_{2,0}\right] + p\left[\dot{v}_2 - v_1'' + 2\beta\left(v_1^2 + v_2^2\right)v_1 - 2v_1 v_3\right] &= 0, \\ (1-p)\left[\ddot{v}_3 - \ddot{n}_0\right] + p\left[\ddot{v}_3 - v_3'' + \left(v_1^2 + v_2^2\right)''\right] &= 0, \end{aligned} \tag{4.1}$$

where "$'$" denotes $\partial/\partial x$, and "$\cdot$" denotes $\partial/\partial t$, and the initial approximations are as follows,

$$\begin{aligned} v_{1,0}(x,t) &= E_{1,0}(x,t) = E_1(x,0) = r\tanh(kx)\cos(k_1 x), \\ v_{2,0}(x,t) &= E_{2,0}(x,t) = E_2(x,0) = r\tanh(kx)\sin(k_1 x), \\ v_{3,0}(x,t) &= n_0(x,t) = n(x,0) = s + \frac{r^2}{-4k_1^2 + 1}\tanh^2(kx), \end{aligned} \tag{4.2}$$

where $r = \sqrt{k^2(4k_1^2 - 1)/(1 + (4k_1^2 - 1)\beta)}$, k, s, β, and k_1 are arbitrary constants, and

$$\begin{aligned} v_1 &= v_{1,0} + pv_{1,1} + p^2 v_{1,2} + p^3 v_{1,3} + \cdots, \\ v_2 &= v_{2,0} + pv_{2,1} + p^2 v_{2,2} + p^3 v_{2,3} + \cdots, \\ v_3 &= v_{3,0} + pv_{3,1} + p^2 v_{3,2} + p^3 v_{3,3} + \cdots, \end{aligned} \tag{4.3}$$

where $v_{i,j}(x,t),\ i,j = 1,2,3,\ldots$, are functions yet to be determined. Substituting (4.3) into (3.11) and arranging the coefficients of "p" powers, we have

$$\begin{aligned}
&\left[\dot{v}_{1,1}+\dot{v}_{1,0}+v''_{2,0}-2\beta v_{1,0}^2 v_{2,0}-2\beta v_{2,0}^3+2v_{2,0}v_{3,0}\right]p \\
&\quad+\left[v''_{2,1}-4\beta v_{1,0}v_{1,1}v_{2,0}-2\beta v_{1,0}^2 v_{2,1}-6\beta v_{2,0}^2 v_{2,1}+2v_{2,1}v_{3,0}+2v_{2,0}v_{3,0}+\dot{v}_{1,2}\right]p^2 \\
&\quad+\left[-2\beta v_{1,1}^2 v_{2,0}-4\beta v_{1,0}v_{1,2}v_{2,0}-4\beta v_{1,0}v_{1,1}v_{2,1}-6\beta v_{2,0}v_{2,1}^2-2\beta v_{1,0}^2 v_{2,2}-6\beta v_{2,0}^2 v_{2,2}\right. \\
&\qquad\left.+2v_{2,2}v_{3,0}+2v_{2,1}v_{3,1}+2v_{2,0}v_{3,0}+\dot{v}_{1,3}+v''_{2,1}\right]p^3=0, \\
&\left[\dot{v}_{2,0}+\dot{v}_{2,1}-v''_{1.0}+2\beta v_{1,0}^3+2\beta v_{1,0}v_{2,0}^2-2v_{1,0}v_{3,0}\right]p \\
&\quad+\left[6\beta v_{1,0}^2 v_{1,1}+2\beta v_{1,1}v_{2,0}^2+4\beta v_{1,0}v_{2,0}v_{2,1}-2v_{1,1}v_{3,0}-2v_{1,0}v_{3,1}+\dot{v}_{2,2}-v''_{1,1}\right]p^2 \\
&\quad+\left[6\beta v_{1,0}v_{1,1}^2+6\beta v_{1,0}^2 v_{1,2}+2\beta v_{1,2}v_{2,0}^2+4\beta v_{1,1}v_{2,0}v_{2,1}+2\beta v_{1,0}v_{2,1}^2\right. \\
&\qquad\left.+4\beta v_{1,0}v_{2,0}v_{2,2}-2v_{1,2}v_{3,0}-2v_{1,1}v_{3,1}-2v_{1,0}\mathrm{v}_{3,2}+\dot{v}_{2,3}-v''_{1,2}\right]p^3=0, \\
&\left[\ddot{v}_{3,0}+\ddot{v}_{3,1}+2v'^2_{1,2}+2v'^2_{2,0}+2v_{1,0}v''_{1,0}+2v_{2,0}v''_{2,0}-v''_{3,0}\right]p \\
&\quad+\left[\ddot{v}_{3,2}+4v'_{1,0}v'_{1,1}+4v'_{2,0}v'_{2,1}+2v_{1,1}v''_{1,0}+2v_{1,0}v''_{1,1}+2v_{2,1}v''_{2,0}+2v_{2,0}v''_{2,1}-v''_{3,1}\right]p^2 \\
&\quad+\left[\ddot{v}_{3,3}+2v'^2_{1,1}+4v'_{1,0}v'_{1,2}+2v'^2_{2,1}+4v'_{2,0}v'_{2,2}+2v_{1,2}v''_{1,0}+2v_{1,1}v''_{1,1}+2v_{1,0}v''_{1,2}\right. \\
&\qquad\left.+2v_{2,2}v''_{2,0}+2v_{2,1}v''_{2,1}+2v_{2,0}v''_{2,2}-v''_{3,2}\right]p^3=0.
\end{aligned} \tag{4.4}$$

To obtain the unknown $v_{i,j}(x,t), i,j = 1,2,3$, we must construct and solve the following system which includes nine equations with nine unknowns, considering the initial conditions of $v_{i,j}(x,0) = 0, i,j = 1,2,3$,

$$\dot{v}_{1,1}+\dot{v}_{1,0}+v''_{2,0}-2\beta v_{1,0}^2 v_{2,0}-2\beta v_{2,0}^3+2v_{2,0}v_{3,0}=0, \tag{4.5}$$

$$v''_{2,1}-4\beta v_{1,0}v_{1,1}v_{2,0}-2\beta v_{1,0}^2 v_{2,1}-6\beta v_{2,0}^2 v_{2,1}+2v_{2,1}v_{3,0}+2v_{2,0}v_{3,0}+\dot{v}_{1,2}=0, \tag{4.6}$$

$$\begin{aligned}
&-2\beta v_{1,1}^2 v_{2,0}-4\beta v_{1,0}v_{1,2}v_{2,0}-4\beta v_{1,0}v_{1,1}v_{2,1}-6\beta v_{2,0}v_{2,1}^2-2\beta v_{1,0}^2 v_{2,2} \\
&\quad-6\beta v_{2,0}^2 v_{2,2}+2v_{2,2}v_{3,0}+2v_{2,1}v_{3,1}+2v_{2,0}v_{3,0}+\dot{v}_{1,3}+v''_{2,1}=0,
\end{aligned} \tag{4.7}$$

$$\dot{v}_{2,0}+\dot{v}_{2,1}-v''_{1.0}+2\beta v_{1,0}^3+2\beta v_{1,0}v_{2,0}^2-2v_{1,0}v_{3,0}=0, \tag{4.8}$$

$$6\beta v_{1,0}^2 v_{1,1}+2\beta v_{1,1}v_{2,0}^2+4\beta v_{1,0}v_{2,0}v_{2,1}-2v_{1,1}v_{3,0}-2v_{1,0}v_{3,1}+\dot{v}_{2,2}-v''_{1,1}=0, \tag{4.9}$$

$$6\beta v_{1,0}v_{1,1}^2 + 6\beta v_{1,0}^2 v_{1,2} + 2\beta v_{1,2}v_{2,0}^2 + 4\beta v_{1,1}v_{2,0}v_{2,1} + 2\beta v_{1,0}v_{2,1}^2 + 4\beta v_{1,0}v_{2,0}v_{2,2} \\ - 2v_{1,2}v_{3,0} - 2v_{1,1}v_{3,1} - 2v_{1,0}v_{2,3} + \dot{v}_{3,2} - v''_{1,2} = 0, \tag{4.10}$$

$$\ddot{v}_{3,0} + \ddot{v}_{3,1} + 2v'^2_{1,2} + 2v'^2_{2,0} + 2v_{1,0}v''_{1,0} + 2v_{2,0}v''_{2,0} - v''_{3,0} = 0, \tag{4.11}$$

$$\ddot{v}_{3,2} + 4v'_{1,0}v'_{1,1}4v'_{2,0}v'_{2,1} + 2v_{1,1}v''_{1,0} + 2v_{1,0}v''_{1,1} + 2v_{2,1}v''_{2,0} + 2v_{2,0}v''_{2,1} - v''_{3,1} = 0, \tag{4.12}$$

$$\ddot{v}_{3,3} + 2v'^2_{1,1} + 4v'_{1,0}v'_{1,2} + 2v'^2_{2,1} + 4v'_{2,0}v'_{2,2} + 2v_{1,2}v''_{1,0} + 2v_{1,1}v''_{1,1} \\ + 2v_{1,0}v''_{1,2} + 2v_{2,2}v''_{2,0} + 2v_{2,1}v''_{2,1} + 2v_{2,0}v''_{2,2} - v''_{3,2} = 0. \tag{4.13}$$

From (4.3), if the three approximations are sufficient, we will obtain

$$E_1(x,t) = \lim_{p\to 1} v_1(x,t) = \sum_{k=0}^{3} v_{1,k}(x,t), \\ E_2(x,t) = \lim_{p\to 1} v_2(x,t) = \sum_{k=0}^{3} v_{2,k}(x,t), \\ n(x,t) = \lim_{p\to 1} v_3(x,t) = \sum_{k=0}^{3} v_{3,k}(x,t). \tag{4.14}$$

To calculate the terms of the homotopy series (4.14) for $E_1(x,t), E_2(x,t)$, and $n(x,t)$, we substitute the initial conditions (4.2) into the system (4.4), and using Mathematica software, from (4.5), we obtain

$$v_{1,1}(x,t) = rt\Bigg[2k\operatorname{sech}^2(kx)(-k_1\cos(k_1x) + k\sin(k_1x)\tanh(kx)) + \sin(k_1x) \\ \times \tanh(kx)\left[k_1^2 - 2s + \frac{2r^2(1-\beta+4k_1^2\beta)\tanh^2(kx)}{(4k_1^2-1)}\right]\Bigg]; \tag{4.15}$$

substituting (4.15) into (4.8),we obtain

$$v_{2,1}(x,t) = rt\Bigg[-2k\operatorname{sech}^2(kx)(k_1\sin(k_1x) + k\cos(k_1x)\tanh(kx)) + \cos(k_1x) \\ \times \tanh(kx)\left[-k_1^2 + 2s + \frac{2r^2(-1+\beta-4k_1^2\beta)\tanh^2(kx)}{(4k_1^2-1)}\right]\Bigg]; \tag{4.16}$$

Table 1: The HPM results for $E(x,t)$ for the first three approximation in comparison with the analytical solution with initial conditions (3.11) when $k = 0.05,\ k_1 = 1,\ s = 0.33,\ \beta = 1$.

	t_i				
x_i	0.1	0.2	0.3	0.4	0.5
0.1	4.4605×10^{-14}	3.99649×10^{-12}	4.88487×10^{-11}	2.82493×10^{-10}	1.0943×10^{-9}
0.2	1.20054×10^{-17}	3.04251×10^{-12}	4.29073×10^{-11}	2.60288×10^{-10}	1.03179×10^{-9}
0.3	6.81758×10^{-14}	1.71124×10^{-12}	3.5057×10^{-11}	2.32051×10^{-10}	9.54569×10^{-10}
0.4	1.59939×10^{-13}	3.07249×10^{-15}	2.53001×10^{-11}	1.97794×10^{-10}	8.62668×10^{-10}
0.5	2.7525×10^{-13}	2.08147×10^{-12}	1.36395×10^{-11}	1.57525×10^{-10}	7.56107×10^{-10}

Table 2: The ADM results for $E(x,t)$ for the first six approximations in comparison with the analytical solution with initial conditions (3.11) when $k = 0.05,\ k_1 = 1,\ s = 0.33,\ \beta = 1$.

	t_i				
x_i	0.1	0.2	0.3	0.4	0.5
0.1	2.4373×10^{-17}	5.88838×10^{-16}	1.13172×10^{-14}	1.13331×10^{-13}	6.8677×10^{-13}
0.2	2.867×10^{-29}	3.45808×10^{-16}	8.9531×10^{-15}	9.68767×10^{-14}	6.093099×10^{-13}
0.3	2.219×10^{-17}	1.49642×10^{-16}	6.6379×10^{-15}	8.04947×10^{-14}	5.320314×10^{-13}
0.4	9.095×10^{-17}	8.5223×10^{-26}	4.3731×10^{-15}	6.4194×10^{-14}	4.549809×10^{-13}
0.5	2.063×10^{-16}	1.033×10^{-16}	2.1600×10^{-15}	4.79847×10^{-14}	3.78204×10^{-13}

substituting (4.15), (4.16) into (4.11), we obtain

$$v_{3,1}(x,t) = \frac{4k_1^2 k^2 r^2 t^2}{4k_1^2 - 1}\left[(-2 + \mathrm{Cosh}(2kx))\mathrm{Sec}\,h^4(kx)\right]. \tag{4.17}$$

in This Manner the Other Components $v_{1,2}(x,t), v_{2,2}(x,t),\ v_{3,2}(x,t),\ v_{1,3}(x,t),\ v_{2,3}(x,t)$, and $v_{3,3}(x,t)$ Can be obtains from (4.6), (4.9), (4.12), (4.7), (4.10), and (4.13), Respectively, and Substituting Into (4.14) to Obtain $E_1(x,t),\ E_2(x,t)$, and $n(x,t)$.

5. Application the Homotopy Analysis Method for the Generalized Zakharov Equations

In order to apply the homotopy analysis method, we choose the linear operator

$$\begin{gathered} L[\phi_i(x,t;p)] = \frac{\partial \phi_i}{\partial t}, \quad i = 1,2, \\ L[\phi_3(x,t;p)] = \frac{\partial^2 \phi_3}{\partial t^2}, \end{gathered} \tag{5.1}$$

Table 3: The HAM results for $E(x,t)$ for the 8th-order approximate in comparison with the analytical solution with initial conditions (3.11) when $\hbar=-1,\ k=0.05,\ k_1=1,\ s=0.33,\ \beta=1$.

	t_i				
x_i	0.1	0.2	0.3	0.4	0.5
0.1	5.1079×10^{-15}	1.008416×10^{-13}	5.6412×10^{-13}	1.779969×10^{-12}	4.42412×10^{-12}
0.2	1.517338×10^{-22}	6.7515×10^{-13}	4.35941×10^{-12}	1.527263×10^{-12}	3.9370×10^{-12}
0.3	5.167×10^{-15}	3.3395×10^{-14}	3.2705×10^{-13}	1.27024×10^{-12}	3.4467×10^{-12}
0.4	1.027×10^{-14}	6.0992×10^{-21}	2.1885×10^{-13}	1.02169×10^{-12}	$.9686\times10^{-12}$
0.5	1.525×10^{-14}	3.3136×10^{-14}	1.08417×10^{-13}	7.6228×10^{-13}	2.4632×10^{-12}

Table 4: The HPM results for $n(x,t)$ for the first three approximations in comparison with the analytical solution with initial conditions (3.11) when $k=0.05,\ k_1=1,\ s=0.33,\ \beta=1$.

	t_i				
x_i	0.1	0.2	0.3	0.4	0.5
0.1	4.95717×10^{-6}	3.71741×10^{-6}	2.47462×10^{-6}	1.22975×10^{-6}	1.62464×10^{-8}
0.2	2.19054×10^{-5}	1.95081×10^{-5}	1.70953×10^{-5}	1.4669×10^{-5}	1.22309×10^{-5}
0.3	4.9547×10^{-5}	4.61578×10^{-5}	4.27335×10^{-5}	3.92765×10^{-5}	3.57893×10^{-5}
0.4	8.58804×10^{-5}	8.17233×10^{-5}	77.507×10^{-5}	7.3241×10^{-5}	6.89074×10^{-5}
0.5	1.28493×10^{-4}	1.23818×10^{-4}	1.1906×10^{-4}	1.14219×10^{-4}	1.09299×10^{-4}

with the property $L[c]=0$ where c is constant; from (3.11), we define a system of nonlinear operators as

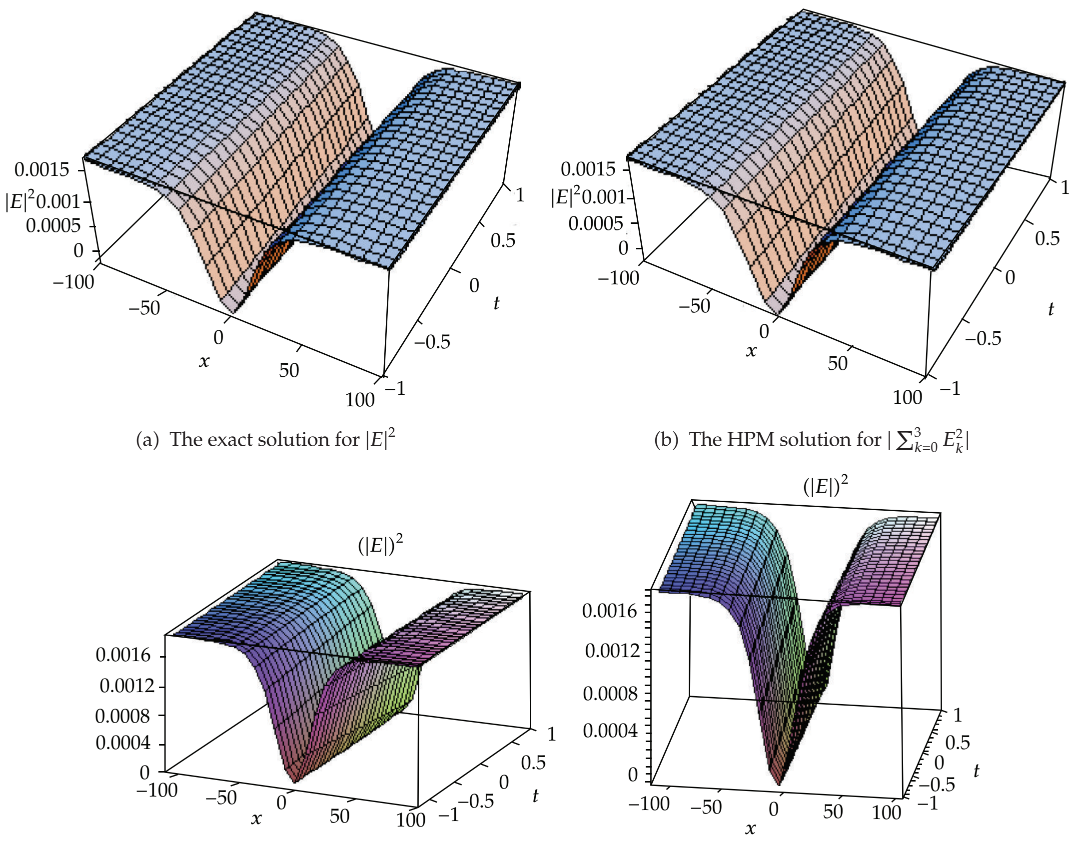
$$\begin{aligned}
N_1[\phi_1(x,t;p),\phi_2(x,t;p),\phi_3(x,t;p)] &= \frac{\partial\phi_1(x,t;p)}{\partial t}+\frac{\partial^2\phi_2(x,t;p)}{\partial x^2}\\
&\quad-2\beta\left(\phi_1^2(x,t;p)+\phi_2^2(x,t;p)\right)\phi_2(x,t;p)\\
&\quad+2\phi_2(x,t;p)\phi_3(x,t;p),\\
N_2[\phi_1(x,t;p),\phi_2(x,t;p),\phi_3(x,t;p)] &= \frac{\partial\phi_2(x,t;p)}{\partial t}-\frac{\partial^2\phi_1(x,t;p)}{\partial x^2}\\
&\quad+2\beta\left(\phi_1^2(x,t;p)+\phi_2^2(x,t;p)\right)\phi_1(x,t;p)\\
&\quad-2\phi_1(x,t;p)\phi_3(x,t;p),\\
N_3[\phi_1(x,t;p),\phi_2(x,t;p),\phi_3(x,t;p)] &= \frac{\partial^2\phi_3(x,t;p)}{\partial t^2}-\frac{\partial^2\phi_3(x,t;p)}{\partial x^2}\\
&\quad+\frac{\partial^2\left(\phi_1^2(x,t;p)+\phi_2^2(x,t;p)\right)}{\partial x^2}.
\end{aligned}\tag{5.2}$$

By using the above definition, we construct the zero-order deformation equations:

$$(1-p)L[\phi_i(x,t;p)-z_{i,0}(x,t)]=p\hbar N_i[\phi_1(x,t;p),\phi_2(x,t;p),\phi_3(x,t;p)],\quad i=1,2,3.\tag{5.3}$$

Table 5: The ADM results for $n(x,t)$ for the first six approximations in comparison with the analytical solution with initial conditions (3.11) when $k = 0.05,\ k_1 = 1,\ s = 0.33,\ \beta = 1$.

	t_i				
x_i	0.1	0.2	0.3	0.4	0.5
0.1	7.8×10^{-19}	1.42×10^{-17}	1.216×10^{-16}	8.0762×10^{-16}	4.14247×10^{-15}
0.2	9.4×10^{-19}	1.639×10^{-17}	1.3206×10^{-16}	8.4007×10^{-16}	4.22213×10^{-15}
0.3	1.16×10^{-18}	1.94×10^{-17}	1.466×10^{-16}	8.8452×10^{-16}	4.32766×10^{-15}
0.4	1.45×10^{-18}	2.322×10^{-17}	1.6455×10^{-16}	9.3918×10^{-16}	4.45507×10^{-15}
0.5	1.74×10^{-18}	2.758×10^{-17}	1.8546×10^{-16}	1.0022×10^{-15}	4.5994×10^{-15}



(a) The exact solution for $|E|^2$

(b) The HPM solution for $|\sum_{k=0}^{3} E_k^2|$

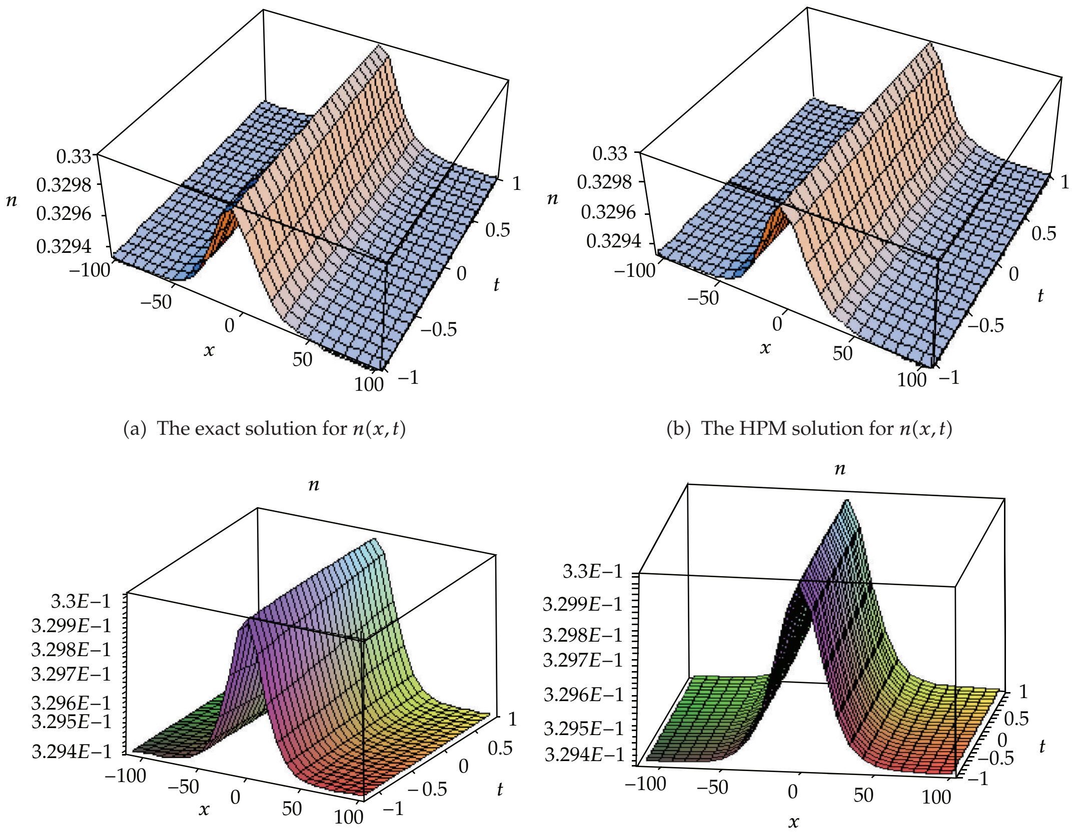
(c) The ADM solution for $|\sum_{k=0}^{6} E_i^2|$

(d) The HAM solution for $|E|^2$ for the 8th order

Figure 1: Comparison between the exact solution, the HPM solution, the ADM solution, and the HAM solution for $E(x,t)$ with initial conditions (3.11) when $k = 0.05,\ k_1 = 1,\ s = 0.33,\ \beta = 1,\ \hbar = -1$.

Table 6: The HAM results for $n(x,t)$ for the 8th-order approximate in comparison with the analytical solution with initial conditions (3.11) when $\hbar = -1$, $k = 0.05$, $k_1 = 1$, $s = 0.33$, $\beta = 1$.

	t_i				
x_i	0.1	0.2	0.3	0.4	0.5
0.1	6.26×10^{-8}	1.25×10^{-7}	1.875×10^{-7}	2.498×10^{-7}	3.123×10^{-7}
0.2	1.25×10^{-7}	2.499×10^{-7}	3.749×10^{-7}	4.996×10^{-7}	6.245×10^{-7}
0.3	1.874×10^{-7}	3.748×10^{-7}	5.62×10^{-7}	7.495×10^{-7}	9.364×10^{-7}
0.4	2.497×10^{-7}	4.996×10^{-7}	7.493×10^{-7}	9.989×10^{-7}	1.2482×10^{-6}
0.5	3.122×10^{-7}	6.244×10^{-7}	9.366×10^{-7}	1.2482×10^{-6}	1.5599×10^{-6}



(a) The exact solution for $n(x,t)$

(b) The HPM solution for $n(x,t)$

(c) The ADM solution for $n(x,t)$

(d) The HAM solution for $n(x,t)$ for the 8th order

Figure 2: Comparison between the exact solution, the HPM solution, the ADM solution, and the HAM solution for $n(x,t)$ with initial conditions (3.11) when $k = 0.05$, $k_1 = 1$, $s = 0.33$, $\beta = 1$, $\hbar = -1$.

When $p = 0$,

$$\begin{aligned}
\phi_1(x,t;\ 0) &= z_{1,0}(x,t) = E_1(x,0) = r\tanh(kx)\cos(k_1 x),\\
\phi_2(x,t;\ 0) &= z_{2,0}(x,t) = E_2(x,0) = r\tanh(kx)\sin(k_1 x),\\
\phi_3(x,t;\ 0) &= z_{3,0}(x,t) = n(x,0) = s + \frac{r^2}{-4k_1^2+1}\tanh^2(kx).
\end{aligned} \tag{5.4}$$

When $p = 1$,

$$\phi_1(x,t;\ 1) = E_1(x,t), \qquad \phi_2(x,t;\ 1) = E_2(x,t), \qquad \phi_3(x,t;\ 1) = n(x,t). \tag{5.5}$$

Therefore, as the embedding parameter p increases from 0 to 1, $\phi_i(x,t;p)$ varies from initial guess $z_{i,0}(x,t)$ to the solutions $E_1(x,t)$, $E_2(x,t)$, and $n(x,t)$, for $i = 1,2,3$, respectively.

Expanding $\phi_i(x,t;p)$ in the Taylor series with respect to p for $i = 1,2,3$, one has

$$\phi_i(x,t;p) = z_{i,0}(x,t) + \sum_{m=1}^{\infty} z_{i,m}(x,t)p^m, \tag{5.6}$$

where

$$z_{i,m}(x,t) = \frac{1}{m!}\left.\frac{\partial^m \phi_i(x,t;p)}{\partial p^m}\right|_{p=0}. \tag{5.7}$$

If the auxiliary linear operator, the initial guess, and the auxiliary parameters $\hbar_i$ are so properly chosen, the above series, converge at $p = 1$, has

$$\begin{aligned}
E_1(x,t) &= z_{1,0}(x,t) + \sum_{m=1}^{\infty} z_{1,m}(x,t),\\
E_2(x,t) &= z_{2,0}(x,t) + \sum_{m=1}^{\infty} z_{2,m}(x,t),\\
n(x,t) &= z_{3,0}(x,t) + \sum_{m=1}^{\infty} z_{3,m}(x,t),
\end{aligned} \tag{5.8}$$

which must be one of solutions of the original nonlinear equation, as proved by Liao [18]. Define the vectors

$$z_{\vec{i,n}} = \{z_{i,0}(x,t), z_{i,1}(x,t), \ldots, z_{i,n}(x,t)\}, \quad i = 1,2,3. \tag{5.9}$$

We have the mth-order deformatiom equations:

$$L\left[z_{i,m}(x,t) - \chi_m z_{i,m-1}(x,t)\right] = \hbar_i R_{i,m}\left(z_{\vec{1,m-1}}, z_{\vec{2,m-1}}, z_{\vec{3,m-1}}\right), \quad i = 1,2,3, \tag{5.10}$$

where

$$\begin{aligned}
R_{1,m}\left(\vec{z_{1,m-1}}, \vec{z_{2,m-1}}, \vec{z_{3,m-1}}\right) &= \frac{\partial z_{1,m-1}}{\partial t} + \frac{\partial^2 z_{2,m-1}}{\partial x^2} \\
&\quad - 2\beta\left[\sum_{j=0}^{m-1}\sum_{k=0}^{j} z_{1,k} z_{1,j-k} z_{2,m-1-j} + \sum_{j=0}^{m-1}\sum_{k=0}^{j} z_{2,k} z_{2,j-k} z_{2,m-1-j}\right] \\
&\quad + 2\sum_{j=0}^{m-1} z_{2,j} z_{3,m-1-j}, \\
R_{2,m}\left(\vec{z_{1,m-1}}, \vec{z_{2,m-1}}, \vec{z_{3,m-1}}\right) &= \frac{\partial z_{2,m-1}}{\partial t} - \frac{\partial^2 z_{1,m-1}}{\partial x^2} \\
&\quad + 2\beta\left[\sum_{j=0}^{m-1}\sum_{k=0}^{j} z_{1,k} z_{1,j-k} z_{1,m-1-j} + \sum_{j=0}^{m-1}\sum_{k=0}^{j} z_{1,k} z_{2,j-k} z_{2,m-1-j}\right] \\
&\quad - 2\sum_{j=0}^{m-1} z_{1,j} z_{3,m-1-j}, \\
R_{3,m}\left(\vec{z_{1,m-1}}, \vec{z_{2,m-1}}, \vec{z_{3,m-1}}\right) &= \frac{\partial^2 z_{3,m-1}}{\partial t^2} - \frac{\partial^2 z_{3,m-1}}{\partial x^2} \\
&\quad + \frac{\partial^2}{\partial x^2}\left[\sum_{j=0}^{m-1} z_{1,j} z_{1,m-1-j} + \sum_{j=0}^{m-1} z_{2,j} z_{2,m-1-j}\right],
\end{aligned} \tag{5.11}$$

where z_1, z_2, and z_3 are functions of x and t. Now, the solutions of the mth-order deformation equation (5.10) for $m \geq 1$ become

$$z_{i,m}(x,t) = \chi_m z_{i,m-1}(x,t) + \hbar_i L^{-1}\left[R_{i,m}\left(\vec{z_{1,m-1}}, \vec{z_{2,m-1}}, \vec{z_{3,m-1}}\right)\right], \quad i = 1, 2, \tag{5.12}$$

where $L^{-1} = \int_0^t (\cdot) dt$ and

$$z_{3,m}(x,t) = \chi_m z_{3,m-1}(x,t) + \hbar_3 L^{-1}\left[R_{3,m}\left(\vec{z_{1,m-1}}, \vec{z_{2,m-1}}, \vec{z_{3,m-1}}\right)\right], \quad i = 1, 2, \tag{5.13}$$

where $L^{-1} = \iint_0^t (\cdot) dt dt$. For simplicity, we suppose $\hbar_1 = \hbar_2 = \hbar_3 = \hbar$.

We consider the solutions of (3.11) with the initial conditions (5.4); we obtain

$$z_{1,1}(x,t) = rt\hbar\Big[2k\sec h^2(kx)(k_1\cos(k_1x) - k\sin(k_1x)\tanh(kx)) + \sin(k_1x) \times \tanh(kx)\Big[-k_1^2 + 2s + \frac{2r^2(-1+\beta-4k_1^2\beta)\tanh^2(kx)}{(4k_1^2-1)}\Big]\Big], \quad (5.14)$$

$$z_{2,1}(x,t) = rt\hbar\Big[2k\sec h^2(kx)(k_1\sin(k_1x) + k\cos(k_1x)\tanh(kx)) + \cos(k_1x) \times \tanh(kx)\Big[k_1^2 - 2s + \frac{2r^2(1-\beta+4k_1^2\beta)\tanh^2(kx)}{(4k_1^2-1)}\Big]\Big], \quad (5.15)$$

$$z_{3,1}(x,t) = -\frac{4k_1^2k^2r^2t^2\hbar}{4k_1^2-1}\Big[(-2+\cosh(2kx))\sec h^4(kx)\Big]. \quad (5.16)$$

Obviously, for $\hbar = -1$ the obtained solutions are the same homotopy perturbation method in (4.15)–(4.17); we continue to evaluate eight terms of HAM.

Using a Taylor series, then the closed-form solutions yield as follows:

$$E(x,t) = r\tanh(kx - wt)\exp[i(k_1x - \Omega t)],$$
$$n(x,t) = s + \frac{r^2}{-4k_1^2+1}\tanh^2(kx - wt), \quad (5.17)$$

where $w = 2k_1k$, $\Omega = -2s + k_1^2 + 2k^2$, $r = \sqrt{k^2(4k_1^2-1)/(1+(4k_1^2-1)\beta)}$, k, s, β, and k_1 are arbitrary constants.

6. Comparing the HPM Results with the HAM Results and the ADM Results and the Exact Solutions

To demonstrate the convergence of the HPM, the results of the numerical example are presented, and only few terms are required to obtain accurate solutions. Tables 1 and 4 show the absolute errors between the analytical solutions and the HPM solutions of the GZE with initial conditions (3.12) for $E(x,t), n(x,t)$ are very small with the present choice of t and x; Tables 2, 3, 5, and 6 help us to compare the HPM results with the ADM results, and the HAM results when $\hbar = -1$ through the absolute errors. Both the analytical solutions, the HPM result, the ADM result and the HAM result for $E(x,t)$ and $n(x,t)$ are plotted in Figures 1 and 2. The diagrams of the results obtained for $\hbar = -1.1, \hbar = -1$, and $\hbar = -0.9$ in comparison with the ADM solutions and the exact solutions for $E(x,t)$ and $n(x,t)$ are shown in Figures 3 and 4, respectively.

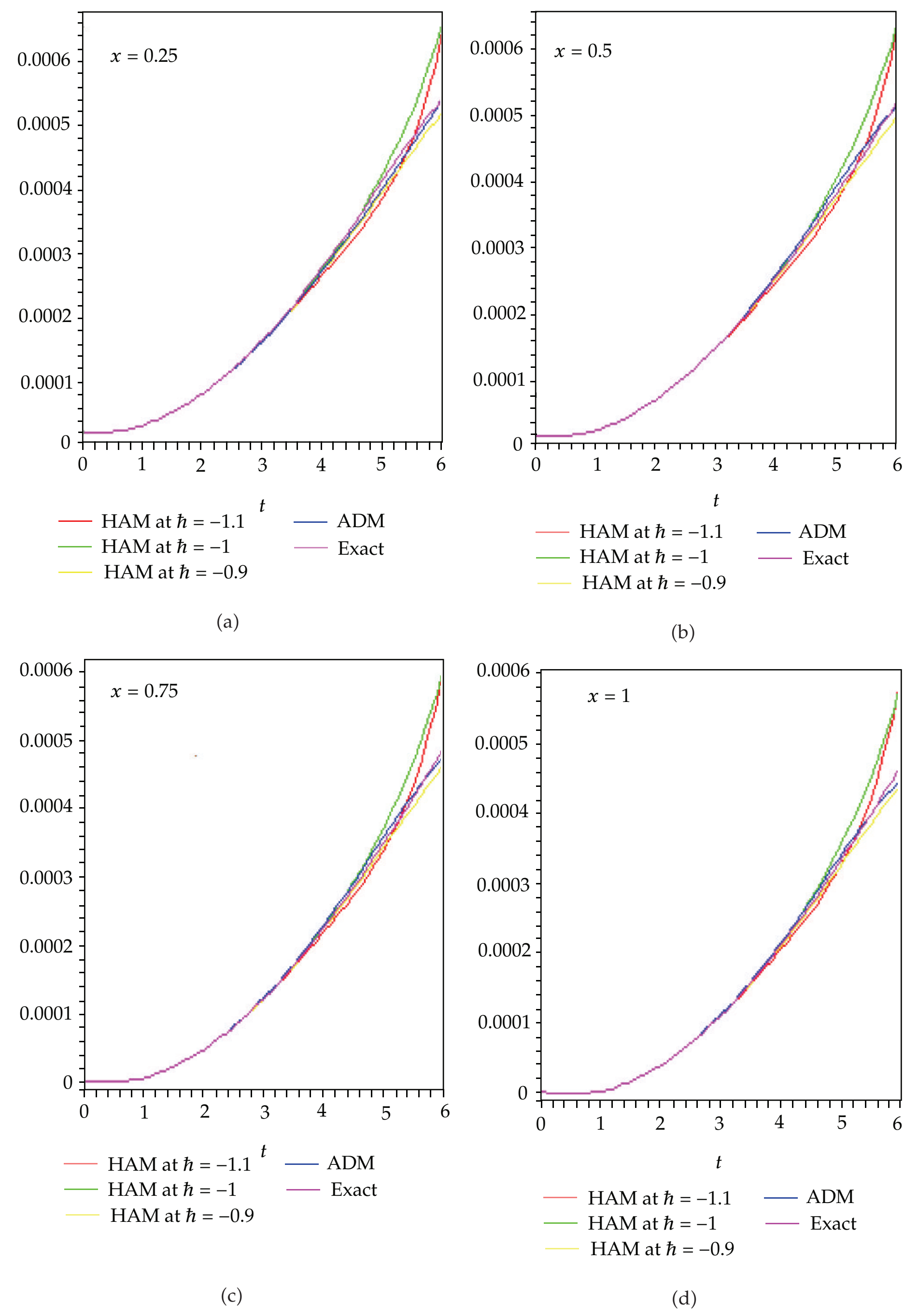


Figure 3: The results obtained by HAM for various $\hbar$ by the 8th-order approximate solutions for $E(x, t)$, in comparison with ADM solutions and the exact solutions with initial conditions (3.11) when $k = 0.05$, $k_1 = 1$, $s = 0.33$, $\beta = 1$, (a)$x = 0.25$, (b) $= 0.5$, (c) $= 0.75$, (d) $= 1$.

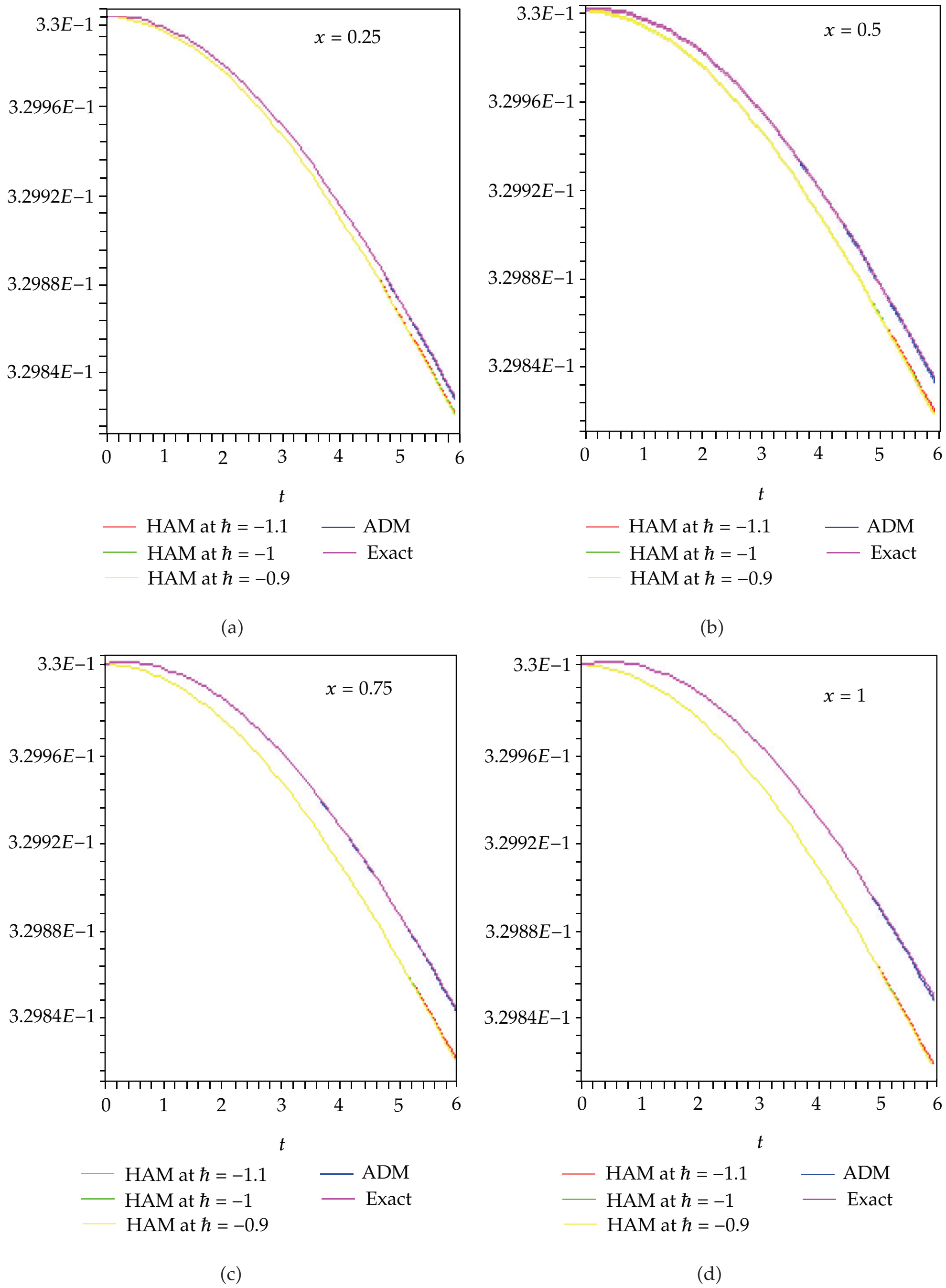


Figure 4: The results obtained by HAM for various $\hbar$ by 8th-order approximate solutions for $n(x,t)$, in comparison with ADM solutions and the exact solutions with initial conditions (3.11) when $k = 0.05$, $k_1 = 1$, $s = 0.33$, $\beta = 1$, (a)$x = 0.25$, (b) = 0.5, (c) = 0.75, (d) = 1.

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References

[1] D. D. Ganji and M. Rafei, "Solitary wave solutions for a generalized Hirota-Satsuma coupled KdV equation by homotopy perturbation method," *Physics Letters A*, vol. 356, no. 2, pp. 131–137, 2006.
[2] J.-H. He, "Homotopy perturbation technique," *Computer Methods in Applied Mechanics and Engineering*, vol. 178, no. 3-4, pp. 257–262, 1999.
[3] J.-H. He, "Asymptotology by homotopy perturbation method," *Applied Mathematics and Computation*, vol. 156, no. 3, pp. 591–596, 2004.
[4] J.-H. He, "The homotopy perturbation method nonlinear oscillators with discontinuities," *Applied Mathematics and Computation*, vol. 151, no. 1, pp. 287–292, 2004.
[5] J. H. He, "Application of homotopy perturbation method to nonlinear wave equations," *Chaos, Solitons & Fractals*, vol. 26, no. 3, pp. 695–700, 2005.
[6] J.-H. He, "Comparison of homotopy perturbation method and homotopy analysis method," *Applied Mathematics and Computation*, vol. 156, no. 2, pp. 527–539, 2004.
[7] D. D. Ganji, A. R. Sahouli, and M. Famouri, "A new modification of He's homotopy perturbation method for rapid convergence of nonlinear undamped oscillators," *Journal of Applied Mathematics and Computing*, vol. 30, no. 1-2, pp. 181–192, 2009.
[8] M. Rafei and D. D. Ganji, "Explicit solutions of helmholtz equation and fifth-order kdv equation using homotopy perturbation method," *International Journal of Nonlinear Sciences and Numerical Simulation*, vol. 7, no. 3, pp. 321–328, 2006.
[9] S. Abbasbandy, "Homotopy perturbation method for quadratic Riccati differential equation and comparison with Adomian's decomposition method," *Applied Mathematics and Computation*, vol. 172, no. 1, pp. 485–490, 2006.
[10] S. Abbasbandy, "Numerical solutions of the integral equations: homotopy perturbation method and Adomian's decomposition method," *Applied Mathematics and Computation*, vol. 173, no. 1, pp. 493–500, 2006.
[11] J. H. He, "Homotopy perturbation method for bifurcation of nonlinear problems," *International Journal of Nonlinear Sciences and Numerical Simulation*, vol. 6, no. 2, pp. 207–208, 2005.
[12] S. Abbasbandy, "Application of he's homotopy perturbation method for laplace transform," *Chaos, Solitons & Fractals*, vol. 30, no. 5, pp. 1206–1212, 2006.
[13] D. D. Ganji and A. Sadighi, "Application of he's homotopy-perturbation method to nonlinear coupled systems of reaction-diffusion equations," *International Journal of Nonlinear Sciences and Numerical Simulation*, vol. 7, no. 4, pp. 411–418, 2006.
[14] M. Rafei, D. D. Ganji, and H. Daniali, "Solution of the epidemic model by homotopy perturbation method," *Applied Mathematics and Computation*, vol. 187, no. 2, pp. 1056–1062, 2007.
[15] A. Yildirim and T. Öziş, "Solutions of singular ivps of lane-emden type by homotopy perturbation method," *Physics Letters, Section A*, vol. 369, no. 1-2, pp. 70–76, 2007.
[16] A. Yildirim, "Exact solutions of nonlinear differential-difference equations by he's homotopy perturbation method," *International Journal of Nonlinear Sciences and Numerical Simulation*, vol. 9, no. 2, pp. 111–114, 2008.
[17] S. J. Liao, *The proposed homotopy analysis technique fot the solution of nonlinear problems*, Ph.D. thesis, Shanghai Jiao Tong University, 1992.
[18] S. Liao, *Beyond Perturbation: Introduction to the Homotopy Analysis Method*, vol. 2 of *CRC Series: Modern Mechanics and Mathematics*, Chapman & Hall/CRC, Boca Raton, Fla, USA, 2004.
[19] S. Liao, "On the homotopy analysis method for nonlinear problems," *Applied Mathematics and Computation*, vol. 147, no. 2, pp. 499–513, 2004.
[20] S. Liao, "Comparison between the homotopy analysis method and homotopy perturbation method," *Applied Mathematics and Computation*, vol. 169, no. 2, pp. 1186–1194, 2005.
[21] S. Abbasbandy, "The application of homotopy analysis method to solve a generalized hirota-satsuma coupled kdv equation," *Physics Letters, Section A*, vol. 361, no. 6, pp. 478–483, 2007.

[22] S. Abbasbandy, "The application of homotopy analysis method to nonlinear equations arising in heat transfer," *Physics Letters A*, vol. 360, no. 1, pp. 109–113, 2006.
[23] H. A. Zedan, "Exact solutions for the generalized KdV equation by using Backlund transformations," *Journal of the Franklin Institute*, vol. 348, no. 8, pp. 1751–1768, 2011.
[24] H. A. Zedan, "Symmetry analysis of an integrable Ito coupled system," *Computers & Mathematics with Applications*, vol. 60, no. 12, pp. 3088–3097, 2010.
[25] M. M. Rashidi, G. Domairry, and S. Dinarvand, "Approximate solutions for the burger and regularized long wave equations by means of the homotopy analysis method," *Communications in Nonlinear Science and Numerical Simulation*, vol. 14, no. 3, pp. 708–717, 2009.
[26] B. Malomed, D. Anderson, M. Lisak, M. L. Quiroga-Teixeiro, and L. Stenflo, "Dynamics of solitary waves in the zakharov model equations," *Physical Review E*, vol. 55, no. 1, pp. 962–968, 1997.
[27] Y.-Y. Wang, C.-Q. Dai, L. Wu, and J.-F Zhang, "Exact and numerical solitary wave solutions of generalized Zakharov equation by the Adomian decomposition method," *Chaos, Solitons & Fractals*, vol. 32, no. 3, pp. 1208–1214, 2007.
[28] S. Abbasbandy, E. Babolian, and M. Ashtiani, "Numerical solution of the generalized Zakharov equation by homotopy analysis method," *Communications in Nonlinear Science and Numerical Simulation*, vol. 14, no. 12, pp. 4114–4121, 2009.
[29] M. Javidi and A. Golbabai, "Exact and numerical solitary wave solutions of generalized zakharov equation by the variational iteration method," *Chaos, Solitons & Fractals*, vol. 36, no. 2, pp. 309–313, 2008.
[30] L. Cveticanin, "Homotopy-perturbation method for pure nonlinear differential equation," *Chaos, Solitons & Fractals*, vol. 30, no. 5, pp. 1221–1230, 2006.
[31] J.-H. He, "Homotopy perturbation method for solving boundary value problems," *Physics Letters A*, vol. 350, no. 1-2, pp. 87–88, 2006.