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Research Article

Traveling Wave Solutions of the Nonlinear $(3+1)$-Dimensional Kadomtsev-Petviashvili Equation Using the Two Variables $(G'/G, 1/G)$-Expansion Method

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The two variables $(G'/G, 1/G)$-expansion method is proposed in this paper to construct new exact traveling wave solutions with parameters of the nonlinear $(3+1)$-dimensional Kadomtsev-Petviashvili equation. This method can be considered as an extension of the basic (G'/G)-expansion method obtained recently by Wang et al. When the parameters are replaced by special values, the well-known solitary wave solutions and the trigonometric periodic solutions of this equationwere rediscovered from the traveling waves.

1. Introduction

In the recent years, investigations of exact solutions to nonlinear PDEs play an important role in the study of nonlinear physical phenomena. Many powerful methods have been presented, such as the inverse scattering transform method [1], the Hirota method [2], the truncated Painlevé expansion method [3–6], the Backlund transform method [7, 8], the exp-function method [9–14], the tanh function method [15–18], the Jacobi elliptic function expansion method [19–21], the original (G'/G)-expansion method [22–33], the two variables $(G'/G, 1/G)$-expansion method [34, 35], and the first integral method [36]. The key idea of the original (G'/G)-expansion method is that the exact solutions of nonlinear PDEs can be expressed by a polynomial in one variable (G'/G) in which $G = G(\xi)$ satisfies the second ordinary differential equation $G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0$, where λ and μ are constants. In this paper, we will use the two variables $(G'/G, 1/G)$-expansion method, which can be considered as an extension of the original (G'/G)-expansion method. The key idea of the two variables $(G'/G, 1/G)$-expansion method is that the exact traveling wave solutions of nonlinear PDEs can be expressed by a polynomial in the two variables (G'/G) and $(1/G)$, in which $G = G(\xi)$ satisfies a second order linear ODE, namely, $G''(\xi) + \lambda G(\xi) = \mu$, where λ and μ are constants. The degree of this polynomial can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms in the given nonlinear PDEs, while the coefficients of this polynomial can be obtained by solving a set of algebraic equations resulted from the process of using the method. According to Aslan [29], the two variables $(G'/G, 1/G)$-expansion method becomes the basic (G'/G)-expansion method if $\mu = 0$ in (2.1) and $b_i = 0$ in (2.12). Recently,Li et al. [34] have applied the two variables $(G'/G, 1/G)$-expansion method and determined the exact solutions of the Zakharov equations, while Zayed andabdelaziz [35] have applied this method to determine the exact solutions of the nonlinear KdV-mKdV equation.

The objective of this paper is to apply the two variables $(G'/G, 1/G)$-expansion method to find the exact traveling wave solutions of the following nonlinear (3+1)-dimensional Kadomtsev-Petviashvili equation:

$$u_{xt} + 6(u_x)^2 + 6uu_{xx} - u_{xxxx} - u_{yy} - u_{zz} = 0. \tag{1.1}$$

This equation describes the dynamics of solitons and nonlinear wave in plasma and superfluids. Recently, Zayed [24] has found the exact solutions of (1.1) using the original (G'/G)-expansion method, while Aslan [14] has discussed (1.1) using the exp-function method. Comparison between our results and that obtained in [14, 24] will be discussed later. The rest of this paper is organized as follows. In Section 2, the description of the two variables $(G'/G, 1/G)$-expansion method is given. In Section 3, we apply this method to (1.1). In Section 4, conclusions are obtained.

2. Description of the Two Variables $(G'/G, 1/G)$-Expansion Method

Before we describe the main steps of this method, we need the following remarks (see [34, 35]).

Remark 2.1. If we consider the second order linear ODE

$$G''(\xi) + \lambda G(\xi) = \mu, \tag{2.1}$$

and set $\phi = G'/G$ and $\psi = 1/G$, then we get

$$\phi' = -\phi^2 + \mu\psi - \lambda, \qquad \psi' = -\phi\psi. \tag{2.2}$$

Remark 2.2. If $\lambda < 0$, then the general solutions of (2.1) is

$$G(\xi) = A_1 \sinh\left(\xi\sqrt{-\lambda}\right) + A_2 \cosh\left(\xi\sqrt{-\lambda}\right) + \frac{\mu}{\lambda}, \tag{2.3}$$

where A_1 and A_2 are arbitrary constants. Consequently, we have

$$\psi^2 = \frac{-\lambda}{\lambda^2\sigma + \mu^2}\left(\phi^2 - 2\mu\psi + \lambda\right), \tag{2.4}$$

where $\sigma = A_1^2 - A_2^2$.

Remark 2.3. If $\lambda > 0$, then the general solutions of (2.1) is

$$G(\xi) = A_1 \sin\left(\xi\sqrt{\lambda}\right) + A_2 \cos\left(\xi\sqrt{\lambda}\right) + \frac{\mu}{\lambda}, \tag{2.5}$$

and hence

$$\psi^2 = \frac{-\lambda}{\lambda^2\sigma - \mu^2}\left(\phi^2 - 2\mu\psi + \lambda\right). \tag{2.6}$$

where $\sigma = A_1^2 + A_2^2$.

Remark 2.4. If $\lambda = 0$, then the general solutions of (2.1) is

$$G(\xi) = \frac{\mu}{2}\xi^2 + A_1\xi + A_2, \tag{2.7}$$

and hence

$$\psi^2 = \frac{1}{A_1^2 - 2\mu A_2}\left(\phi^2 - 2\mu\psi\right), \tag{2.8}$$

Suppose we have the following NLPDEs in the form:

$$F(u, u_t, u_x, u_{xx}, u_{tt}, \ldots) = 0, \tag{2.9}$$

where F is a polynomial in u and its partial derivatives. In the following, we give the main steps of the two variables $(G'/G, 1/G)$-expansion method [34, 35].

Step 1. The traveling wave variable

$$u(x,t) = u(\xi), \quad \xi = x - Vt \tag{2.10}$$

reduces (2.9) to an ODE in the form

$$P(u, u', u'', \ldots) = 0, \tag{2.11}$$

where V is a constant and P is a polynomial in u and its total derivatives, while $\{\ \}' = d/d\xi$.

Step 2. Suppose that the solutions of (2.11) can be expressed by a polynomial in the two variables ϕ and ψ as follows:

$$u(\xi) = \sum_{i=0}^{i=N} a_i \phi^i + \sum_{i=1}^{i=N} b_i \phi^{i-1} \psi, \tag{2.12}$$

where a_i $(i = 0, 1, \ldots, N)$ and b_i $(i = 1, \ldots, N)$ are constants to be determined later.

Step 3. Determine the positive integer N in (2.12) by using the homogeneous balance between the highest order derivatives and the nonlinear terms in (2.11).

Step 4. Substitute (2.12) into (2.11) along with (2.2) and (2.4), the left-hand side of (2.11) can be converted into a polynomial in ϕ and ψ, in which the degree of ψ is not longer than 1. Equating each coefficients of this polynomial to zero yields a system of algebraic equations which can be solved by using the Maple or Mathematica to get the values of a_i, b_i, V, μ, A_1, A_2, and λ where $\lambda < 0$. Thus, we get the exact solutions in terms of the hyperbolic functions.

Step 5. Similar to Step 4, substitute (2.12) into (2.11) along with (2.2) and (2.6) for $\lambda > 0$ (or (2.2) and (2.8) for $\lambda = 0$), we obtain the exact solutions of (2.11) expressed by trigonometric functions (or by rational functions), respectively.

3. An Application

In this section, we apply the method described in Section 2, to find the exact traveling wave solutions of the nonlinear (3+1)-dimensional Kadomtsev-Petviashvili equation (1.1). To this end, we see that the traveling wave variables $\xi = x + y + z - Vt$ reduce (1.1) to the following ODE:

$$-(2 + V)u'' + 6\left(u'\right)^2 + 6uu'' - u'''' = 0. \tag{3.1}$$

Balancing u'''' with uu'' in (3.1) we get $N = 2$. Consequently, we get

$$u(\xi) = a_0 + a_1\phi(\xi) + a_2\phi^2(\xi) + b_1\psi(\xi) + b_2\phi(\xi)\psi(\xi), \tag{3.2}$$

where a_0, a_1, a_2, b_1, and b_2 are constants to be determined later. There are three cases to be discussed as follows.

Case 1. Hyperbolic function solutions ($\lambda < 0$).

If $\lambda < 0$, substituting (3.2) into (3.1) and using (2.2) and (2.4), the left-hand side of (3.1) becomes a polynomial in ϕ and ψ. Setting the coefficients of this polynomial to zero yields a system of algebraic equations in a_0, a_1, a_2, b_1, b_2, μ, σ, and λ which can be solved by using the Maple or Mathematica to find the following results:

$$a_0 = a_0, \qquad a_1 = 0, \qquad a_2 = 1, \qquad b_1 = -\mu, \qquad b_2 = \pm\sqrt{\frac{-(\lambda^2\sigma + \mu^2)}{\lambda}}, \qquad V = 6a_0 - 5\lambda - 2. \tag{3.3}$$

From (2.3), (3.2), and (3.3), we deduce the traveling wave solution of (1.1) as follows:

$$\begin{aligned} u(\xi) = a_0 &- \frac{\mu}{A_1 \sinh\left(\xi\sqrt{-\lambda}\right) + A_2 \cosh\left(\xi\sqrt{-\lambda}\right) + (\mu/\lambda)} \\ &- \frac{A_1 \cosh\left(\xi\sqrt{-\lambda}\right) + A_2 \sinh\left(\xi\sqrt{-\lambda}\right)}{\left(A_1 \sinh\left(\xi\sqrt{-\lambda}\right) + A_2 \cosh\left(\xi\sqrt{-\lambda}\right) + \mu/\lambda\right)^2} \\ &\times \left[A_1\lambda \cosh\left(\xi\sqrt{-\lambda}\right) + A_2\lambda \sinh\left(\xi\sqrt{-\lambda}\right) \mp \sqrt{\lambda^2\sigma + \mu^2}\right], \end{aligned} \tag{3.4}$$

where

$$\xi = x + y + z - (6a_0 - 5\lambda - 2)t. \tag{3.5}$$

In particular, by setting $A_1 = 0$, $A_2 > 0$ and $\mu = 0$ in (3.4), we have the solitary solution

$$u(\xi) = a_0 - \lambda \tanh\left(\xi\sqrt{-\lambda}\right)\left[\tanh\left(\xi\sqrt{-\lambda}\right) \mp i\,\mathrm{sech}\left(\xi\sqrt{-\lambda}\right)\right], \tag{3.6}$$

where $i = \sqrt{-1}$, while if $A_2 = 0$, $A_1 > 0$, and $\mu = 0$, then we have the solitary solution

$$u(\xi) = a_0 - \lambda \coth\left(\xi\sqrt{-\lambda}\right)\left[\coth\left(\xi\sqrt{-\lambda}\right) \mp \mathrm{csch}\left(\xi\sqrt{-\lambda}\right)\right]. \tag{3.7}$$

Case 2. Trigonometric function solutions ($\lambda > 0$).

If $\lambda > 0$, substituting (3.2) into (3.1) and using (2.2) and (2.6), we get a polynomial in ϕ and ψ. Vanishing each coefficient of this polynomial to get the algebraic equations which can be solved by using the Maple or Mathematica to find the following results:

$$a_0 = a_0, \qquad a_1 = 0, \qquad a_2 = 1, \qquad b_1 = -\mu, \qquad b_2 = \pm\sqrt{\frac{\lambda^2\sigma - \mu^2}{\lambda}}, \qquad V = 6a_0 - 5\lambda - 2. \tag{3.8}$$

From (2.5), (3.2), and (3.8), we deduce the traveling wave solution of (1.1) as follows:

$$\begin{aligned} u(\xi) = a_0 &- \frac{\mu}{A_1\sin\left(\xi\sqrt{\lambda}\right) + A_2\cos\left(\xi\sqrt{\lambda}\right) + (\mu/\lambda)} \\ &+ \frac{A_1\cos\left(\xi\sqrt{\lambda}\right) - A_2\sin\left(\xi\sqrt{\lambda}\right)}{\left(A_1\sin\left(\xi\sqrt{\lambda}\right) + A_2\cos\left(\xi\sqrt{\lambda}\right) + \mu/\lambda\right)^2} \\ &\times \left[A_1\lambda\cos\left(\xi\sqrt{\lambda}\right) - A_2\lambda\sin\left(\xi\sqrt{\lambda}\right) \pm \sqrt{\lambda^2\sigma - \mu^2}\right], \end{aligned} \tag{3.9}$$

where ξ has the same form (3.5).

In particular, by setting $A_1 = 0$, $A_2 > 0$, and $\mu = 0$ in (3.9), we have the periodic solution

$$u(\xi) = a_0 + \lambda\tan\left(\xi\sqrt{\lambda}\right)\left[\tan\left(\xi\sqrt{\lambda}\right) \mp \sec\left(\xi\sqrt{\lambda}\right)\right], \tag{3.10}$$

while if $A_2 = 0$, $A_1 > 0$, and $\mu = 0$, then we have the periodic solution

$$u(\xi) = a_0 + \lambda\cot\left(\xi\sqrt{\lambda}\right)\left[\cot\left(\xi\sqrt{\lambda}\right) \pm \csc\left(\xi\sqrt{\lambda}\right)\right]. \tag{3.11}$$

Case 3. Rational function solutions ($\lambda = 0$).

If $\lambda = 0$, substituting (3.2) into (3.1) and using (2.2) and (2.8), we get a polynomial in ϕ and ψ. Setting each coefficients of this polynomial to be zero to get the algebraic equations which can be solved by using the Maple or Mathematica to find the following results:

$$a_0 = a_0, \qquad a_1 = 0, \qquad a_2 = 1, \qquad b_1 = -\mu, \qquad b_2 = \pm\sqrt{A_1^2 - 2\mu A_2}, \qquad V = 6a_0 - 2. \tag{3.12}$$

From (2.7), (3.2), and (3.12), we deduce the traveling wave solution of (1.1) as follows:

$$u(\xi) = a_0 - \frac{\mu}{(\mu/2)\xi^2 + A_1\xi + A_2} + \frac{(\mu\xi + A_1)\left(\mu\xi + A_1 \pm \sqrt{A_1^2 - 2\mu A_2}\right)}{\left((\mu/2)\xi^2 + A_1\xi + A_2\right)^2}, \tag{3.13}$$

where

$$\xi = x + y + z - (6a_0 - 2)t. \tag{3.14}$$

Remark 3.1. All solutions of this paper have been checked with Maple by putting them back into the original equation (1.1).

4. Conclusions

The two variables $(G'/G, 1/G)$-expansion method has been used in this paper to discuss (1.1) and obtain the exact traveling wave solutions (3.4), (3.9), and (3.13) of Section 3. As the two parameters A_1 and A_2 take special values, we obtain the solitary wave solutions (3.6) and (3.7) and the trigonometric periodic solutions (3.10) and (3.11). On comparing these solutions with the result (11) of [14] obtained by Aslan using the exp-function method as well as the results (3.28)–(3.31) of [24] obtained by Zayed using the basic (G'/G)-expansion method, we conclude that all these solutions of (1.1) are different and satisfying that equation. The advantage of the two variables $(G'/G, 1/G)$-expansion method over the basic (G'/G)-expansion method is that the first method is an extension of the second one.

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