

Research Article

Approximation by Lupas-Type Operators and Szász-Mirakyan-Type Operators

Hee Sun Jung¹ and Ryozi Sakai²

¹ Department of Mathematics Education, Sungkyunkwan University, Seoul 110-745, Republic of Korea

² Department of Mathematics, Meijo University, Nagoya 468-8502, Japan

Correspondence should be addressed to Hee Sun Jung, hsun90@skku.edu

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Lupas-type operators and Szász-Mirakyan-type operators are the modifications of Bernstein polynomials to infinite intervals. In this paper, we investigate the convergence of Lupas-type operators and Szász-Mirakyan-type operators on $[0, \infty)$.

1. Introduction and Main Results

For $f \in C([0, 1])$, Bernstein operator $B_n f(x)$ is defined as follows: Let

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad 0 \leq k \leq n, \quad (1.1)$$

and then we define

$$(B_n f)(x) = \sum_{k=0}^n p_{n,k}(x) f\left(\frac{k}{n}\right). \quad (1.2)$$

Derriennic [1] gave a modified operator of $B_n f$ such as

$$(M_n^* f)(x) = (n+1) \sum_{k=0}^n p_{n,k}(x) \int_0^1 p_{n,k}(t) f(t) dt, \quad (1.3)$$

and obtained the result that for $f \in C^{(2)}([0, 1])$,

$$\lim_{n \rightarrow \infty} ((M_n^* f)(x) - f(x)) = (1 - 2x)f'(x) + x(1 - x)f''(x). \quad (1.4)$$

Lupas investigated a family of linear positive operators which mapped the class of all bounded and continuous functions on $[0, \infty)$ into $C[0, \infty)$ such that

$$(L_n f)(x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} f\left(\frac{k}{n}\right), \quad x \in [0, \infty). \quad (1.5)$$

Moreover, Sahai and Prasad [2] modified Lupas operators as follows: Let f be integrable on $[0, \infty)$ and let n be a positive integer. Then we define

$$(M_n[f])(x) = (n-1) \sum_{k=0}^{\infty} P_{n,k}(x) \int_0^{\infty} P_{n,k}(y) f(y) dy, \quad x \in [0, \infty), \quad (1.6)$$

where

$$P_{n,k}(x) = \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}}. \quad (1.7)$$

In this paper, we assume that n is a positive integer. Then they obtained the following;

Theorem 1.1 (see [2], Theorem 1). *If f is integrable on $[0, \infty)$ and admits its $(r+1)$ th and $(r+2)$ th derivatives, which are bounded at a point $x \in [0, \infty)$, and $f^{(r)}(x) = O(x^\alpha)$ (α is a positive integer ≥ 2) as $x \rightarrow \infty$, then*

$$\lim_{n \rightarrow \infty} n \left((M_n[f])^{(r)}(x) - f^{(r)}(x) \right) = (r+1)(1-2x)f^{(r+1)}(x) + x(1-x)f^{(r+2)}(x). \quad (1.8)$$

Theorem 1.1 holds only for bounded $x \leq K$, so it does not mean the norm convergence on $[0, \infty)$. In this paper, we improve Theorem 1.1 with respect to the norm convergence on $[0, \infty)$.

Let $0 < p \leq \infty$ and let w be a positive weight, that is, $w(x) \geq 0$ for $x \in \mathbb{R}$. For a function g on $[0, \infty)$, we define the norm by

$$\|g\|_{L_p([0, \infty))} := \begin{cases} \left(\int_{[0, \infty)} |g(t)|^p dt \right)^{1/p}, & 0 < p < \infty \\ \sup_{[0, \infty)} |g(t)|, & p = \infty. \end{cases} \quad (1.9)$$

For convenience, for nonnegative integers $n \geq 2$, r , and $n - r - 2 \geq 0$, we let

$$A_{n,r} := \frac{(n-1)!(n-2)!}{(n-r-2)!(n+r-1)!}. \quad (1.10)$$

Then we have the following results:

Theorem 1.2. Let $0 < p \leq \infty$. Let α and r be nonnegative integers and $n - r - 2 \geq 0$. Let $f \in C^{(r+1)}([0, \infty))$ satisfy

$$\left| f^{(r)}(x) \right| \leq O(1)(x+1)^\alpha, \quad \left| f^{(r+1)}(x) \right| \leq O(1)(x+1)^{\alpha+2}. \quad (1.11)$$

Then we have uniformly for f and n ,

$$\left| A_{n,r}(M_n[f])^{(r)}(x) - f^{(r)}(x) \right| = O\left(\frac{1}{n^{1/3}}\right)(x+1)^{\alpha+2}. \quad (1.12)$$

In particular, if $\|(x+1)^{\alpha+2}w(x)\|_{L_p([0,\infty))} < \infty$, then we have uniformly for n ,

$$\left\| \left(A_{n,r}(M_n[f])^{(r)}(x) - f^{(r)}(x) \right) w(x) \right\|_{L_p([0,\infty))} = O\left(\frac{1}{n^{1/3}}\right). \quad (1.13)$$

Remark 1.3. (a) We see that for nonnegative integers $n \geq 2$, r , and $n - r - 2 \geq 0$,

$$\frac{(n-1)!(n-2)!}{(n-r-2)!(n+r-1)!} \rightarrow 1 \quad \text{as } n \rightarrow \infty. \quad (1.14)$$

(b) The following weight is useful.

$$w_\lambda(x) = \frac{1}{1+x^\lambda} \begin{cases} \lambda > \frac{1}{p} + \alpha + 2, & 0 < p < \infty, \\ \lambda \geq \alpha + 2, & p = \infty. \end{cases} \quad (1.15)$$

Let

$$\psi(x) := \frac{1}{1+x}. \quad (1.16)$$

Theorem 1.4. Let r and β be nonnegative integers and $n - r - 2 \geq 0$. Let $f \in C^{(r+2)}([0, \infty))$ satisfy

$$\left\| f^{(r+1)}(x) \psi^{2\beta+1}(x) \right\|_{L_\infty([0,\infty))} < \infty, \quad \left\| f^{(r+2)}(x) \psi^{2\beta}(x) \right\|_{L_\infty([0,\infty))} < \infty. \quad (1.17)$$

Then we have uniformly for f and n ,

$$\begin{aligned} & \left| A_{n,r}(M_n[f])^{(r)}(x) - f^{(r)}(x) \right| \psi^{2\beta+2}(x) \\ & \leq O\left(\frac{1}{n}\right) \left(\left\| f^{(r+1)}(x) \psi^{2\beta+1}(x) \right\|_{L_\infty([0,\infty))} + \left\| f^{(r+2)}(x) \psi^{2\beta}(x) \right\|_{L_\infty([0,\infty))} \right). \end{aligned} \quad (1.18)$$

Let us define the weighted modulus of smoothness by

$$\omega_k(f; \eta; t) := \sup_{0 \leq h \leq t} \left\| \Delta_h^k f(\cdot) \eta(\cdot) \right\|_{L_\infty([0, \infty))}, \quad t \geq 0, \quad k = 1, 2, \quad (1.19)$$

where

$$\Delta_h^1 f(x) = f(x+h) - f(x), \quad \Delta_h^2 f(x) = f(x) - 2f(x+h) + f(x+2h). \quad (1.20)$$

Theorem 1.5. *Let β and r be nonnegative integers and $n - r - 2 \geq 0$. Let $f \in C^r([0, \infty))$. Then we have uniformly for f and n ,*

$$\begin{aligned} & \left\| (A_{n,r}(M_n[f])^{(r)}(x) - f^{(r)}(x)) \psi^{2\beta+2}(x) \right\|_{L_\infty([0, \infty))} \\ & \leq C \left(\frac{1}{\sqrt{n}} \omega_1 \left(f^{(r)}; \psi^{2\beta+1}; \frac{1}{\sqrt{n}} \right) + \omega_2 \left(f^{(r)}; \psi^{2\beta}; \frac{1}{\sqrt{n}} \right) \right). \end{aligned} \quad (1.21)$$

The Szász-Mirakyan operators are also generalizations of Bernstein polynomials on infinite intervals. They are defined by:

$$S_n(f)(x) = \sum_{k=0}^{\infty} S_{n,k}(x) f\left(\frac{k}{n}\right), \quad (1.22)$$

where

$$S_{n,k}(x) = \frac{e^{-nx} (nx)^k}{k!}. \quad (1.23)$$

In [3], the class of Szász-Mirakyan operators $S_{n,r,q}(f; x)$ was defined as follows:

$$S_{n,r,q}(f; x) := \frac{1}{A_r(nx)} \sum_{k=0}^{\infty} \frac{(nx)^{rk}}{(rk)!} f\left(\frac{rk}{n+q}\right), \quad x \in [0, \infty), \quad (1.24)$$

where $q > 0$ and

$$A_r(t) = \sum_{k=0}^{\infty} \frac{t^{rk}}{(rk)!}, \quad t \in [0, \infty). \quad (1.25)$$

Theorem 1.6 (see [3]). *Let $q > 0$ and $r \in \mathbb{N}$ be fixed numbers. Then there exists $M_{q,r} = \text{const.} > 0$ depending only on q and r such that, for every uniformly continuous and bounded function $f(x)e^{-qx}$ on $[0, \infty)$, the following inequalities hold;*

(a)

$$\begin{aligned} & \| (S_{n;q,r}(f;x) - f(x))\varphi(x)e^{-qx} \|_{L_\infty([0,\infty))} \\ & \leq M_{q,r} \frac{1}{n+q} \left(\|f'(x)e^{-qx}\|_{L_\infty([0,\infty))} + \|f''(x)e^{-qx}\|_{L_\infty([0,\infty))} \right); \end{aligned} \tag{1.26}$$

(b)

$$\begin{aligned} & \| (S_{n;q,r}(f;x) - f(x))\varphi(x)e^{-qx} \|_{L_\infty([0,\infty))} \\ & \leq M_{q,r} \frac{1}{n+q} (\delta_{n,q}\omega_1(f; e^{-qx}; \delta_{n,q}) + \omega_2(f; e^{-qx}; \delta_{n,q})), \end{aligned} \tag{1.27}$$

where $\delta_{n,q} := (n+q)^{-1/2}$.

(c) for every fixed $x \in [0, \infty)$, we have for every continuous f with $f^{(j)}(x)e^{-qx}$, $j = 0, 1, 2$, bounded on $[0, \infty)$,

$$\lim_{n \rightarrow \infty} n(S_{n;q,r}(f;x) - f(x)) = -qx f'(x) + \frac{x}{2} f''(x). \tag{1.28}$$

Now, we modify the Szász-Mirakyan operators as follows: let f be integrable on $[0, \infty)$, then we define

$$(Q_{n,\beta}[f])(x) = (n+\beta) \sum_{k=0}^{\infty} S_{n,k}(x) \int_0^{\infty} S_{n+\beta,k}(y) f(y) dy, \quad x \in [0, \infty), \tag{1.29}$$

where β is a nonnegative integer. Then we have the following results:

Theorem 1.7. Let α, β and r be nonnegative integers. Let $f \in C^{(r+1)}([0, \infty))$ satisfies

$$\left| f^{(r)}(x) \right| \leq O(1)e^{\beta x}(x+1)^\alpha, \quad \left| f^{(r+1)}(x) \right| \leq O(1)e^{\beta x}(x+1)^{\alpha+2}. \tag{1.30}$$

Then one has uniformly for f and n ,

$$\left| \left(\frac{n+\beta}{n} \right)^r (Q_{n,\beta}[f])^{(r)}(x) - f^{(r)}(x) \right| = O\left(\frac{1}{n^{1/3}} \right) e^{\beta x}(x+1)^{\alpha+2}. \tag{1.31}$$

In particular, let $0 < p \leq \infty$. If one supposes $\|e^{\beta x}(x+1)^{\alpha+2}\omega(x)\|_{L_p([0,\infty))} < \infty$, then one has uniformly for f and n ,

$$\left\| \left(\frac{n+\beta}{n} \right)^r (Q_{n,\beta}[f])^{(r)}(x) - f^{(r)}(x) \right\|_{L_p([0,\infty))} \omega(x) = O\left(\frac{1}{n^{1/3}} \right). \tag{1.32}$$

Remark 1.8. (a) We note that for nonnegative integers β and r ,

$$\left(\frac{n+\beta}{n}\right)^r \rightarrow 1 \quad \text{as } n \rightarrow \infty. \quad (1.33)$$

(b) The following weight is useful.

$$\omega_{\lambda,\beta}(x) = e^{-\beta x} \omega_\lambda(x), \quad (1.34)$$

where $\omega_\lambda(x)$ is defined in Remark 1.3.

Theorem 1.9. Let β, γ , and r be nonnegative integers. Let $f \in C^{(r+2)}([0, \infty))$ satisfies

$$\|f^{(r+1)}(x)e^{-\beta x}\psi^{2\gamma+1}(x)\|_{L_\infty([0,\infty))} < \infty, \quad \|f^{(r+2)}(x)e^{-\beta x}\psi^{2\gamma}(x)\|_{L_\infty([0,\infty))} < \infty. \quad (1.35)$$

Then one has uniformly for f and n ,

$$\begin{aligned} & \left| \left[\left(\frac{n+\beta}{n} \right)^r (Q_{n,\beta}[f])^{(r)}(x) - f^{(r)}(x) \right] e^{-\beta x} \psi^{2\gamma+2}(x) \right| \\ & \leq O\left(\frac{1}{n}\right) \left(\|f^{(r+1)}(x)e^{-\beta x}\psi^{2\gamma+1}(x)\|_{L_\infty([0,\infty))} + \|f^{(r+2)}(x)e^{-\beta x}\psi^{2\gamma}(x)\|_{L_\infty([0,\infty))} \right). \end{aligned} \quad (1.36)$$

Theorem 1.10. Let β, γ , and r be nonnegative integers. Then one has for $f \in C^r([0, \infty))$,

$$\begin{aligned} & \left\| \left[\left(\frac{n+\beta}{n} \right)^r (Q_{n,\beta}[f])^{(r)}(x) - f^{(r)}(x) \right] e^{-\beta x} \psi^{2\gamma+2}(x) \right\|_{L_\infty([0,\infty))} \\ & \leq C \left(\frac{1}{\sqrt{n}} \omega_1\left(f; e^{-\beta x} \psi^{2\gamma+1}(x); \frac{1}{\sqrt{n}}\right) + \omega_2\left(f; e^{-\beta x} \psi^{2\gamma}(x); \frac{1}{\sqrt{n}}\right) \right). \end{aligned} \quad (1.37)$$

2. Proofs of Results

First, we will prove results for Lupas-type operators such as Theorems 1.2, 1.4, and 1.5. To prove theorems, we need some lemmas.

Lemma 2.1. Let m and r be nonnegative integers and $n > m + r + 1$. Let

$$T_{n,m,r}(x) := (n-r-1) \sum_{k=0}^{\infty} P_{n+r,k}(x) \int_0^{\infty} P_{n-r,k+r}(y) (y-x)^m dy. \quad (2.1)$$

Then

$$(i) \quad T_{n,0,r}(x) = 1,$$

(ii)

$$T_{n,1,r}(x) = \frac{(r+1)(1+2x)}{(n-r+2)},$$

$$T_{n,2,r}(x) = \frac{2(n-1)x(1+x)}{(n-r-2)(n-r-3)} + \frac{(r+1)(r+2)(1+2x)^2}{(n-r-2)(n-r-3)}; \quad (2.2)$$

(iii) for $m \geq 1$,

$$(n-m-r-2)T_{n,m+1,r}(x)$$

$$= x(1+x)(T'_{n,m,r}(x) + 2mT_{n,m-1,r}(x)) + (m+r+1)(1+2x)T_{n,m,r}, \quad (2.3)$$

where $n > m+r+2$;(iv) for $m \geq 0$,

$$T_{n,m,r}(x) = O\left(\frac{1}{n^{\lfloor(m+1)/2\rfloor}}\right)q_{n,m,r}(x), \quad (2.4)$$

where $q_{n,m,r}(x)$ is a polynomial of degree $\leq m$ such that the coefficients are bounded independently of n and they are positive for $n > m+r+1$.

Proof. (i), (ii), and (iii) have been proved in [2, Lemma 1]. So we may show only the part of (2.4). For $m = 1, 2$, (2.4) holds. Let us assume (2.4) for $m(\geq 2)$. We note

$$T'_{n,m,r}(x) = O\left(\frac{1}{n^{\lfloor(m+1)/2\rfloor}}\right)q'_{m,r}(x), \quad q'_{m,r}(x) \in \mathcal{P}_{m-1}. \quad (2.5)$$

So, we have by the assumption of induction,

$$(n-m-r-2)T_{n,m+1,r}(x)$$

$$= x(1+x)(T'_{n,m,r}(x) + 2mT_{n,m-1,r}(x)) + (m+r+1)(1+2x)T_{n,m,r}$$

$$\leq x(1+x)\left(C\frac{1}{n^{\lfloor(m+1)/2\rfloor}}q'_{m,r}(x) + 2m\frac{1}{n^{\lfloor m/2\rfloor}}q_{m-1,r}(x)\right) \quad (2.6)$$

$$+ (m+r+1)(1+2x)\frac{1}{n^{\lfloor(m+1)/2\rfloor}}q_{m,r}(x).$$

Here, if m is even, then

$$\left[\frac{m+1}{2}\right] + 1 = \frac{m}{2} + 1 = \frac{m+2}{2} = \left[\frac{m+2}{2}\right], \quad \left[\frac{m}{2}\right] + 1 = \frac{m}{2} + 1 = \frac{m+2}{2} = \left[\frac{m+2}{2}\right], \quad (2.7)$$

and if m is odd, then

$$\left\lceil \frac{m+1}{2} \right\rceil + 1 = \frac{m+1}{2} + 1 = \left\lceil \frac{m+2}{2} \right\rceil, \quad \left\lfloor \frac{m}{2} \right\rfloor + 1 = \frac{m-1}{2} + 1 = \frac{m+1}{2} = \left\lfloor \frac{m+2}{2} \right\rfloor. \quad (2.8)$$

Hence, we have

$$T_{n,m+1,r}(x) = O\left(\frac{1}{n^{\lfloor (m+2)/2 \rfloor}}\right) q_{n,m+1,r}(x), \quad (2.9)$$

and here we see that $q_{n,m+1,r}(x)$ is a polynomial of degree $\leq m+1$ such that the coefficients of $q_{n,m+1,r}(x)$ are bounded independently of n . Moreover, we see from (2.6) that the coefficients of $q_{n,m+1,r}(x)$ are positive for $n > m+r+2$. \square

Lemma 2.2 (see [2, Lemma 2]). *Let r be a nonnegative integer and $n-r-2 \geq 0$. Then one has for $f \in C^r([0, \infty))$:*

$$(M_n[f])^{(r)}(x) = \frac{(n-r-1)!(n+r-1)!}{(n-1)!(n-2)!} \sum_{k=0}^{\infty} P_{n+r,k}(x) \int_0^{\infty} P_{n-r,k+r}(y) f^{(r)}(y) dy. \quad (2.10)$$

Let

$$\left(\widetilde{M}_n[f]\right)^{(r)}(x) = (n-r-1) \sum_{k=0}^{\infty} P_{n+r,k}(x) \int_0^{\infty} P_{n-r,k+r}(y) f^{(r)}(y) dy. \quad (2.11)$$

Then we have

$$\left(\widetilde{M}_n[f]\right)^{(r)}(x) = A_{n,r} (M_n[f])^{(r)}(x), \quad (2.12)$$

where $A_{n,r}$ is defined by (1.10).

Proof of Theorem 1.2. Let $|y-x| \leq 1$. By the second inequality in (1.11),

$$\begin{aligned} \left| f^{(r)}(y) - f^{(r)}(x) \right| &= |y-x| \left| f^{(r+1)}(\xi) \right| \\ &\leq O(1) |y-x| \left| \xi^{\alpha+2} \right| \leq O(1) |y-x| (x+1)^{\alpha+2}. \end{aligned} \quad (2.13)$$

Let $\varepsilon := n^{-\gamma}, 0 < \gamma < 1$,

$$\begin{aligned}
 & \left| \left(\left(\widetilde{M}_n[f] \right)^{(r)}(x) - f^{(r)}(x) \right) \right| \\
 &= \left| (n-r-1) \sum_{k=0}^{\infty} P_{n+r,k}(x) \right. \\
 & \quad \times \left(\int_{|y-x|<\varepsilon} P_{n-r,k+r}(y) \left| f^{(r)}(y) - f^{(r)}(x) \right| dy + \int_{|y-x|\geq\varepsilon} P_{n-r,k+r}(y) \left| f^{(r)}(y) - f^{(r)}(x) \right| dy \right) \left. \right| \\
 &=: A + B.
 \end{aligned} \tag{2.14}$$

First, we see by (2.13) and Lemma 2.1,

$$\begin{aligned}
 A &= O(1) \left| (n-r-1) \sum_{k=0}^{\infty} P_{n+r,k}(x) \int_{|y-x|<\varepsilon} P_{n-r,k+r}(y) |y-x|(x+1)^{\alpha+2} dy \right| \\
 &\leq O(1)\varepsilon |T_{n,0,r}(x)|(x+1)^{\alpha+2} = O(1)\varepsilon(x+1)^{\alpha+2}.
 \end{aligned} \tag{2.15}$$

Next, we estimate B . By the first inequality in (1.11),

$$\begin{aligned}
 B &\leq C \left| (n-r-1) \sum_{k=0}^{\infty} P_{n+r,k}(x) \int_{|y-x|>\varepsilon} P_{n-r,k+r}(y) \left(|f^{(r)}(y)| + |f^{(r)}(x)| \right) dy \right| \\
 &\leq C \left| (n-r-1) \sum_{k=0}^{\infty} P_{n+r,k}(x) \int_{|y-x|>\varepsilon} P_{n-r,k+r}(y) \left((y+1)^\alpha + (x+1)^\alpha \right) dy \right|.
 \end{aligned} \tag{2.16}$$

Here, using

$$(y+1)^\alpha = ((y-x) + x+1)^\alpha = \sum_{i=0}^{\alpha} \binom{\alpha}{i} (y-x)^i (x+1)^{\alpha-i} \tag{2.17}$$

and the notation:

$$\langle i \rangle = \begin{cases} 1, & (i : \text{odd}) \\ 0, & (i : \text{even}), \end{cases} \tag{2.18}$$

we have

$$B \leq C |(n-r-1)| \sum_{k=0}^{\infty} P_{n+r,k}(x) \int_{|y-x|>\varepsilon} P_{n-r,k+r}(y) \times \left(\sum_{i=1}^{\alpha} \binom{\alpha}{i} (y-x)^i (x+1)^{\alpha-i} + 2(x+1)^\alpha \right) dy$$

$$\begin{aligned}
&\leq C|(n-r-1)| \sum_{k=0}^{\infty} P_{n+r,k}(x) \int_{|y-x|>\varepsilon} P_{n-r,k+r}(y) \times \left(\sum_{i=1}^{\alpha} \binom{\alpha}{i} (y-x)^i \left| \frac{y-x}{\varepsilon} \right|^{\langle i \rangle} (x+1)^{\alpha-i} \right) dy \\
&\quad + C|(n-r-1)| \sum_{k=0}^{\infty} P_{n+r,k}(x) \int_{|y-x|>\varepsilon} P_{n-r,k+r}(y) \left(\frac{y-x}{\varepsilon} \right)^2 (x+1)^{\alpha} dy \\
&:= B_1 + B_2.
\end{aligned} \tag{2.19}$$

Then, we obtain

$$\begin{aligned}
B_1 &\leq C \left(\sum_{i=1}^{\alpha} \binom{\alpha}{i} |T_{n,i+\langle i \rangle,r}| \left(\frac{1}{\varepsilon} \right)^{\langle i \rangle} (x+1)^{\alpha-i} \right) \\
&\leq C \sum_{i=1}^{\alpha} \binom{\alpha}{i} O \left(\frac{n^{\gamma \langle i \rangle}}{n^{\lfloor (i+\langle i \rangle+1)/2 \rfloor}} |q_{n,i+\langle i \rangle,r}(x)| (x+1)^{\alpha-i} \right) \\
&\leq O \left(\frac{1}{n^{\lfloor (i+\langle i \rangle+1)/2 \rfloor - \gamma \langle i \rangle}} \right) (x+1)^{\alpha+\langle i \rangle} \leq O \left(\frac{1}{n^{1-\gamma}} \right) (x+1)^{\alpha+1}.
\end{aligned} \tag{2.20}$$

Here, we used the following that for $i \geq 1$,

$$\left\lfloor \frac{i+\langle i \rangle+1}{2} \right\rfloor - \gamma \langle i \rangle \geq 1 - \gamma, \tag{2.21}$$

because

$$\left\lfloor \frac{i+\langle i \rangle+1}{2} \right\rfloor - \gamma \langle i \rangle = \begin{cases} \frac{i+1}{2} - \gamma, & i : \text{odd}, \\ \frac{i}{2}, & i : \text{even}. \end{cases} \tag{2.22}$$

And we know that

$$\begin{aligned}
B_2 &\leq C |T_{n,2,r}(x)| \left(\frac{1}{\varepsilon} \right)^2 x^{\alpha} \leq O \left(\frac{1}{n^{\lfloor 3/2 \rfloor}} \right) |q_{n,2,r}(x)| \left(\frac{1}{\varepsilon} \right)^2 x^{\alpha} \\
&\leq O \left(\frac{n^{2\gamma}}{n^{\lfloor 3/2 \rfloor}} \right) |q_{n,2,r}(x)| x^{\alpha} \leq O \left(\frac{1}{n^{1-2\gamma}} \right) (x+1)^{\alpha+2}.
\end{aligned} \tag{2.23}$$

Thus, we obtain

$$B \leq O \left(\frac{1}{n^{1-2\gamma}} \right) (x+1)^{\alpha+2}. \tag{2.24}$$

Therefore, we have uniformly on n ,

$$\left| \left(\widetilde{M}_n[f] \right)^{(r)}(x) - f^{(r)}(x) \right| \leq O\left(\frac{1}{n^\gamma}\right)(x+1)^{\alpha+2} + O\left(\frac{1}{n^{1-2\gamma}}\right)(x+1)^{\alpha+2}. \quad (2.25)$$

Here, if we let $\gamma = 1/3$, then we have

$$\left| \left(\widetilde{M}_n[f] \right)^{(r)}(x) - f^{(r)}(x) \right| = O\left(\frac{1}{n^{1/3}}\right)(x+1)^{\alpha+2}, \quad (2.26)$$

that is, (1.12) is proved. So, we also have a norm convergence (1.13). \square

Proof of Theorem 1.4. We know that for $f \in C^{(r+2)}([0, \infty))$,

$$f^{(r)}(t) = f^{(r)}(x) + f^{(r+1)}(x)(t-x) + \int_x^t (t-u)f^{(r+2)}(u)du, \quad (2.27)$$

$$\left| \int_x^t (t-u)f^{(r+2)}(u)du \right| \leq C \left\| f^{(r+2)}(x)\varphi^{2\beta}(x) \right\|_{L_\infty([0, \infty))} \left((1+x)^{2\beta} + (1+t)^{2\beta} \right) (t-x)^2, \quad (2.28)$$

where $\varphi(t) = 1/(1+x)$. Then we obtain from (2.10) and (2.27),

$$\begin{aligned} & \left(\widetilde{M}_n[f] \right)^{(r)}(x) \\ &= f^{(r)}(x) + f^{(r+1)}(x)T_{n,1,r}(x) + (n-r-1) \sum_{k=0}^{\infty} P_{n+r,k}(x) \int_0^{\infty} P_{n-r,k+r}(y) \int_x^y (y-u)f^{(r+2)}(u)du dy \end{aligned} \quad (2.29)$$

and from (2.28),

$$\begin{aligned} & \left| (n-r-1) \sum_{k=0}^{\infty} P_{n+r,k}(x) \int_0^{\infty} P_{n-r,k+r}(y) \int_x^y (y-u)f^{(r+2)}(u)du dy \right| \\ & \leq \left\| f^{(r+2)}(x)\varphi^{2\beta}(x) \right\|_{L_\infty([0, \infty))} \\ & \quad \times \left((1+x)^{2\beta}|T_{n,2,r}(x)| + \left| (n-r-1) \sum_{k=0}^{\infty} P_{n+r,k}(x) \int_0^{\infty} P_{n-r,k+r}(y)(1+y)^{2\beta}(y-x)^2 dy \right| \right). \end{aligned} \quad (2.30)$$

Using $(1+y)^{2\beta} \leq C((y-x)^{2\beta} + (1+x)^{2\beta})$, we have

$$\begin{aligned} & \left| (n-r-1) \sum_{k=0}^{\infty} P_{n+r,k}(x) \int_0^{\infty} P_{n-r,k+r}(y)(1+y)^{2\beta}(y-x)^2 dy \right| \\ & \leq C \left(|T_{n,2\beta+2,r}(x)| + (1+x)^{2\beta}|T_{n,2,r}(x)| \right). \end{aligned} \quad (2.31)$$

Therefore, we have

$$\begin{aligned}
& \left| \left(\widetilde{M}_n[f] \right)^{(r)}(x) - f^{(r)}(x) \right| \psi^{2\beta+2}(x) \\
& \leq \left| f^{(r+1)}(x) \psi^{2\beta+1}(x) \right| |T_{n,1,r}(x)| \psi(x) \\
& \quad + C \left\| f^{(r+2)}(x) \psi^{2\beta}(x) \right\|_{L_\infty([0,\infty))} (1+x)^{2\beta} |T_{n,2,r}(x)| \psi^{2\beta+2}(x) \\
& \quad + C \left\| f^{(r+2)}(x) \psi^{2\beta}(x) \right\|_{L_\infty([0,\infty))} |T_{n,2\beta+2,r}(x)| \psi^{2\beta+2}(x) \\
& \leq O\left(\frac{1}{n}\right) \left| f^{(r+1)}(x) \psi^{2\beta+1}(x) \right| |g_{n,1,r}(x)| \psi(x) \\
& \quad + O\left(\frac{1}{n}\right) \left\| f^{(r+2)}(x) \psi^{2\beta}(x) \right\|_{L_\infty([0,\infty))} (1+x)^{2\beta} |g_{n,2,r}(x)| \psi^{2\beta+2}(x) \\
& \quad + O\left(\frac{1}{n^{\beta+1}}\right) \left\| f^{(r+2)}(x) \psi^{2\beta}(x) \right\|_{L_\infty([0,\infty))} |g_{n,2\beta+2,r}(x)| \psi^{2\beta+2}(x).
\end{aligned} \tag{2.32}$$

Since we know that for $x \in [0, \infty)$,

$$|g_{n,1,r}(x)| \psi(x) \leq C, \quad (1+x)^{2\beta} |g_{n,2,r}(x)| \psi^{2\beta+2}(x) \leq C, \quad |g_{n,2\beta+2,r}(x)| \psi^{2\beta+2}(x) \leq C, \tag{2.33}$$

we have

$$\begin{aligned}
& \left| \left(\widetilde{M}_n[f] \right)^{(r)}(x) - f^{(r)}(x) \right| \psi^{2\beta+2}(x) \\
& \leq O\left(\frac{1}{n}\right) \left(\left\| f^{(r+1)}(x) \psi^{2\beta+1}(x) \right\|_{L_\infty([0,\infty))} + \left\| f^{(r+2)}(x) \psi^{2\beta}(x) \right\|_{L_\infty([0,\infty))} \right).
\end{aligned} \tag{2.34}$$

□

Lemma 2.3. Let r and β be nonnegative integers and $n - r - 2 \geq 0$. Let $f \in C^{(r)}([0, \infty))$ satisfies

$$\left\| f^{(r)} \psi^{2\beta} \right\|_{L_\infty([0,\infty))} < \infty. \tag{2.35}$$

Then one has uniformly for n , f and $x \in [0, \infty)$,

$$\left| \left(\widetilde{M}_n[f] \right)^{(r)}(x) \right| \psi^{2\beta}(x) \leq C \left\| f^{(r)} \psi^{2\beta} \right\|_{L_\infty([0,\infty))}. \tag{2.36}$$

Proof. Using $(1 + y)^{2\beta} \leq C((y - x)^{2\beta} + (1 + x)^{2\beta})$, we have

$$\left| (n - r - 1) \sum_{k=0}^{\infty} P_{n+r,k}(x) \int_0^{\infty} P_{n-r,k+r}(y) (1 + y)^{2\beta} dy \right| \leq C \left(\psi^{-2\beta}(x) + |T_{n,2\beta,r}(x)| \right). \quad (2.37)$$

The assumption (2.35) means

$$|f^{(r)}(y)| \leq C(1 + y)^{2\beta}. \quad (2.38)$$

Then we can obtain by (2.10),

$$\begin{aligned} & \left| \left(\widetilde{M}_n[f] \right)^{(r)}(x) \right| \\ & \leq C \left| (n - r - 1) \sum_{k=0}^{\infty} P_{n+r,k}(x) \int_0^{\infty} P_{n-r,k+r}(y) (1 + y)^{2\beta} dy \right| \left\| f^{(r)} \psi^{2\beta} \right\|_{L_{\infty}([0, \infty))} \\ & \leq C \left(\psi^{-2\beta}(x) + |T_{n,2\beta,r}(x)| \right) \left\| f^{(r)} \psi^{2\beta} \right\|_{L_{\infty}([0, \infty))} \\ & \leq C \left(\psi^{-2\beta}(x) + O\left(\frac{1}{n^{\beta}}\right) |q_{n,2\beta,r}(x)| \right) \left\| f^{(r)} \psi^{2\beta} \right\|_{L_{\infty}([0, \infty))}. \end{aligned} \quad (2.39)$$

Consequently, since $|q_{n,2\beta,r}(x)|\psi^{2\beta}(x)$ is uniformly bounded on $[0, \infty)$, we have the result. \square

The Steklov function $[f]_h(x)$ for $f \in C([0, \infty))$ is defined as follows:

$$[f]_h(x) := \frac{4}{h^2} \int_0^{h/2} [2f(x + s + t) - f(x + 2(s + t))] ds dt, \quad x \geq 0, h > 0. \quad (2.40)$$

Then for the Steklov function $[f]_h(x)$ with respect to $f \in C([0, \infty))$, we have the following properties.

Lemma 2.4 (cf.[4]). *Let $f(x) \in C([0, \infty))$ and $\eta(x)$ be a positive and nonincreasing function on $[0, \infty)$. Then (i) $[f]_h(x) \in C^2([0, \infty))$;*

(ii)

$$\|([f]_h(x) - f(x))\eta(x)\|_{L_{\infty}([0, \infty))} \leq \omega_2\left(f; \eta; \frac{h}{2}\right); \quad (2.41)$$

(iii)

$$\|[f]_h'(x)\eta(x)\|_{L_{\infty}([0, \infty))} \leq \frac{4}{h}\omega_1\left(f; \eta; \frac{h}{2}\right) \frac{\eta(x)}{\eta(x + (h/2))} + \frac{1}{h}\omega_1(f; \eta; h) \frac{\eta(x)}{\eta(x + h)}; \quad (2.42)$$

(iv)

$$\| [f]''_h(x)\eta(x) \|_{L^\infty([0,\infty))} \leq \frac{4}{h^2} \left[2\omega_2\left(f; \eta; \frac{h}{2}\right) + \frac{1}{4}\omega_2(f; \eta; h) \right]. \quad (2.43)$$

Proof. (i) For $f \in C([0, \infty))$, we have the Steklov functions $[f]'_h(x)$ and $[f]''_h(x)$ as follows. We note

$$[f]_h(x) = \frac{4}{h^2} \int_0^{h/2} \left(\int_x^{x+h/2} 2f(u+t)du - \int_x^{x+h} \frac{1}{2}f(u+2t)du \right) dt, \quad x \geq 0, h > 0. \quad (2.44)$$

Then, we can see from (2.44),

$$\begin{aligned} [f]'_h(x) &= \frac{4}{h^2} \int_0^{h/2} \left[2\left(f\left(x + \frac{h}{2} + t\right) - f(x+t)\right) - \frac{1}{2}(f(x+h+2t) - f(x+2t)) \right] dt \\ &= \frac{4}{h^2} \int_0^{h/2} \left[2\Delta_{h/2}^1 f(x+t) - \frac{1}{2}\Delta_h^1 f(x+2t) \right] dt. \end{aligned} \quad (2.45)$$

Similarly to (2.44), we know

$$[f]'_h(x) = \frac{4}{h^2} \left[2 \int_x^{x+h/2} \left(f\left(u + \frac{h}{2}\right) - f(u) \right) du - \frac{1}{4} \int_x^{x+h} (f(u+h) - f(u)) du \right]. \quad (2.46)$$

Therefore, we have from (2.46),

$$\begin{aligned} [f]''_h(x) &= \frac{4}{h^2} \left[2\left(f(x+h) - 2f\left(x + \frac{h}{2}\right) + f(x)\right) - \frac{1}{4}(f(x+2h) - 2f(x+h) + f(x)) \right] \\ &= \frac{4}{h^2} \left[2\Delta_{h/2}^2 f(x) - \frac{1}{4}\Delta_h^2 f(x) \right]. \end{aligned} \quad (2.47)$$

Therefore, (i) is proved.

(ii) We easily see from (2.44) that

$$\begin{aligned} |(f(x) - [f]_h(x))\eta(x)| &= \left| \frac{4}{h^2} \iint_0^{h/2} \Delta_{s+t}^2 f(x)\eta(x) ds dt \right| \\ &\leq \omega_2\left(f; \eta; \frac{h}{2}\right). \end{aligned} \quad (2.48)$$

(iii) From (2.46), we have

$$\begin{aligned}
\left| [f]'_h(x)\eta(x) \right| &\leq \left| \frac{4}{h^2} \int_0^{h/2} 2 \left(\Delta_{h/2}^1 f(x+t)\eta(x+t) \right) \frac{\eta(x)}{\eta(x+t)} dt \right| \\
&\quad + \left| \frac{4}{h^2} \int_0^{h/2} \frac{1}{2} \left(\Delta_h^1 f(x+2t)\eta(x+2t) \right) \frac{\eta(x)}{\eta(x+2t)} dt \right| \\
&\leq \frac{4}{h} \omega_1 \left(f; \eta; \frac{h}{2} \right) \frac{\eta(x)}{\eta(x+h/2)} + \frac{1}{h} \omega_1(f; \eta; h) \frac{\eta(x)}{\eta(x+h)}.
\end{aligned} \tag{2.49}$$

(iv) From (2.47), we have

$$\left| [f]''_h(x)\eta(x) \right| \leq \frac{4}{h^2} \left[2\omega_2 \left(f; \eta; \frac{h}{2} \right) + \frac{1}{4}\omega_2(f; \eta; h) \right]. \tag{2.50}$$

□

Proof of Theorem 1.5. We know that for $f(x) \in C^r([0, \infty))$,

$$[f]_h^{(r)}(x) = [f^{(r)}]_h(x), \quad [f]_h^{(r+1)}(x) = [f^{(r)}]'_h(x), \quad [f]_h^{(r+2)}(x) = [f^{(r)}]''_h(x). \tag{2.51}$$

Then, we have

$$\begin{aligned}
&\left\| \left((\widetilde{M}_n [f])^{(r)}(x) - f^{(r)}(x) \right) \psi^{2\beta+2}(x) \right\|_{L_\infty([0, \infty))} \\
&\leq \left\| \left(\widetilde{M}_n [f - [f]_h] \right)^{(r)}(x) \psi^{2\beta+2}(x) \right\|_{L_\infty([0, \infty))} \\
&\quad + \left\| \left((\widetilde{M}_n [[f]_h])^{(r)}(x) - [f^{(r)}]_h(x) \right) \psi^{2\beta+2}(x) \right\|_{L_\infty([0, \infty))} \\
&\quad + \left\| [f^{(r)}]_h(x) - f^{(r)}(x) \right\|_{L_\infty([0, \infty))}.
\end{aligned} \tag{2.52}$$

From (2.51) and (2.41) of Lemma 2.4,

$$\begin{aligned}
\left\| \left(\widetilde{M}_n [f - [f]_h] \right)^{(r)}(x) \psi^{2\beta+2}(x) \right\|_{L_\infty([0, \infty))} &\leq \left\| [f^{(r)}(x) - [f]_h^{(r)}(x)] \psi^{2\beta+2}(x) \right\|_{L_\infty([0, \infty))} \\
&= \left\| [f^{(r)}(x) - [f^{(r)}]_h(x)] \psi^{2\beta+2}(x) \right\|_{L_\infty([0, \infty))} \\
&\leq \omega_2 \left(f^{(r)}; \psi^{2\beta+2}; h \right).
\end{aligned} \tag{2.53}$$

Here, we suppose $0 < h \leq 1$ and then we know that

$$\frac{\psi(x)}{\psi(x+h)} \leq 2, \quad \frac{\psi(x)}{\psi(x+h/2)} \leq 2. \quad (2.54)$$

From Theorem 1.4, (2.51), (2.42), and (2.43) of Lemma 2.4, we have

$$\begin{aligned} & \left\| \left((\widetilde{M}_n[[f]_h])^{(r)}(x) - [f]_h^{(r)}(x) \right) \psi^{2\beta+2}(x) \right\|_{L_\infty([0,\infty))} \\ & \leq O\left(\frac{1}{n}\right) \left(\left\| [f]_h^{(r+1)}(x) \psi^{2\beta+1}(x) \right\|_{L_\infty([0,\infty))} + \left\| [f]_h^{(r+2)}(x) \psi^{2\beta}(x) \right\|_{L_\infty([0,\infty))} \right) \\ & \leq O\left(\frac{1}{n}\right) \left(\frac{1}{h} \omega_1(f^{(r)}; \psi^{2\beta+1}; h) + \frac{1}{h^2} \omega_2(f^{(r)}; \psi^{2\beta}; h) \right). \end{aligned} \quad (2.55)$$

Therefore, we have

$$\begin{aligned} & \left\| \left((\widetilde{M}_n[f])^{(r)}(x) - f^{(r)}(x) \right) \psi^{2\beta+2}(x) \right\|_{L_\infty([0,\infty))} \\ & \leq O\left(\frac{1}{n}\right) \left(\frac{1}{h} \omega_1(f^{(r)}; \psi^{2\beta+1}; h) + \frac{1}{h^2} \omega_2(f^{(r)}; \psi^{2\beta}; h) \right) + \omega_2(f^{(r)}; \psi^{2\beta+2}; h). \end{aligned} \quad (2.56)$$

If we let $h = 1/\sqrt{n}$, then

$$\begin{aligned} & \left\| \left((\widetilde{M}_n[f])^{(r)}(x) - f^{(r)}(x) \right) \psi^{2\beta+2}(x) \right\|_{L_\infty([0,\infty))} \\ & \leq C \left(\frac{1}{\sqrt{n}} \omega_1\left(f^{(r)}; \psi^{2\beta+1}; \frac{1}{\sqrt{n}}\right) + \omega_2\left(f^{(r)}; \psi^{2\beta}; \frac{1}{\sqrt{n}}\right) \right), \end{aligned} \quad (2.57)$$

because $\omega_2(f^{(r)}; \psi^{2\beta+2}; 1/\sqrt{n}) \leq \omega_2(f^{(r)}; \psi^{2\beta}; 1/\sqrt{n})$. □

From now on, we will prove Theorems 1.7, 1.9, and 1.10, which are the results for the Szász-Mirakyan operators, analogously to the case of Lupas-type operators.

Lemma 2.5. *Let r be a nonnegative integer. Then one has for $f \in C^r([0, \infty))$,*

$$(Q_{n,\beta}[f])^{(r)}(x) = (n+\beta) \left(\frac{n}{n+\beta} \right)^r \sum_{k=0}^{\infty} S_{n,k}(x) \int_0^{\infty} S_{n+\beta,k+r}(y) f^{(r)}(y) dy, \quad x \in [0, \infty). \quad (2.58)$$

Proof. We know that

$$\begin{aligned} S_{n,k}^{(r)}(x) &= \sum_{i=0}^r \binom{r}{i} \frac{(e^{-ny})^{(r-i)} ((ny)^k)^{(i)}}{k!} = \sum_{i=0}^r \binom{r}{i} (-1)^r (-1)^i n^r S_{n,k-i}(x), \\ S_{n,k}^{(r)}(x) &= \sum_{i=0}^r \binom{r}{i} \frac{(e^{-ny})^{(i)} ((ny)^k)^{(r-i)}}{k!} = \sum_{i=0}^r \binom{r}{i} (-1)^i n^r S_{n,k-r+i}(x). \end{aligned} \quad (2.59)$$

Therefore, we have

$$\begin{aligned} (Q_{n,\beta}[f])^{(r)}(x) &= (n+\beta) \sum_{k=0}^{\infty} S_{n,k}^{(r)}(x) \int_0^{\infty} S_{n+\beta,k}(y) f(y) dy \\ &= (n+\beta) \sum_{i=0}^r \sum_{k=i}^{\infty} \binom{r}{i} (-1)^r (-1)^i n^r S_{n,k-i}(x) \int_0^{\infty} S_{n+\beta,k}(y) f(y) dy \\ &= (n+\beta) \left(\frac{n}{n+\beta}\right)^r \sum_{k=0}^{\infty} S_{n,k}(x) \int_0^{\infty} \sum_{i=0}^r \binom{r}{i} (-1)^r (-1)^i (n+\beta)^r S_{n+\beta,k+i}(y) f(y) dy \\ &= (n+\beta) \left(\frac{n}{n+\beta}\right)^r \sum_{k=0}^{\infty} S_{n,k}(x) \int_0^{\infty} (-1)^r S_{n+\beta,k+r}^{(r)}(y) f(y) dy \\ &= (n+\beta) \left(\frac{n}{n+\beta}\right)^r \sum_{k=0}^{\infty} S_{n,k}(x) \int_0^{\infty} S_{n+\beta,k+r}(y) f^{(r)}(y) dy. \end{aligned} \quad (2.60)$$

□

Lemma 2.6. Let a , b , and m be nonnegative integers.

$$R_{n,m,r}(a,b;x) := (n+b) \sum_{k=0}^{\infty} S_{n+a,k}(x) \int_0^{\infty} S_{n+b,k+r}(y) (y-x)^m dy. \quad (2.61)$$

Then one has

$$(i) R_{n,0,r}(a,b;x) = 1 \text{ and } R_{n,1,r}(a,b;x) = ((a-b)x + r + 1)/(n+b);$$

$$(ii) \text{ For } m \geq 1$$

$$\begin{aligned} (n+b)R_{n,m+1,r}(a,b;x) &= xR_{n,m,r}^{(1)}(a,b;x) + ((a-b)x + m + r + 1)R_{n,m,r}(a,b;x) + 2xmR_{n,m-1,r}(a,b;x); \end{aligned} \quad (2.62)$$

(iii)

$$R_{n,m,r}(a,b;x) = O\left(\frac{1}{n^{\lfloor(m+1)/2\rfloor}}\right)g_{n,m,r}(a,b;x), \quad (2.63)$$

where $g_{n,m,r}(a,b;x)$ is a polynomial of degree $\leq m$ such that the coefficients of $g_{n,m,r}(a,b;x)$ are bounded independently of n .

Proof. Let $R_{n,m,r}(x) := R_{n,m,r}(a,b;x)$. Then (i)

$$\begin{aligned} R_{n,0,r}(x) &= (n+b) \sum_{k=0}^{\infty} S_{n+a,k}(x) \int_0^{\infty} S_{n+b,k+r}(y) dy = 1, \\ R_{n,1,r}(x) &= (n+b) \sum_{k=0}^{\infty} S_{n+a,k}(x) \int_0^{\infty} S_{n+b,k+r}(y)(y-x) dy \\ &= \sum_{k=0}^{\infty} S_{n+a,k}(x) \frac{k+r+1}{n+b} - x \\ &= \sum_{k=0}^{\infty} S_{n+a,k}(x) \frac{k}{n+b} + \frac{r+1}{n+b} - x \\ &= \frac{(n+a)x}{n+b} + \frac{r+1}{n+b} - x = \frac{(a-b)x+r+1}{n+b}. \end{aligned} \quad (2.64)$$

(ii) Using $xS_{n,k}^{(1)}(x) = (k-nx)S_{n,k}(x)$, we obtain

$$\begin{aligned} &x\left(R_{n,m,r}^{(1)}(x) + mR_{n,m-1,r}(x)\right) \\ &= (n+b) \sum_{k=0}^{\infty} xS_{n+a,k}^{(1)}(x) \int_0^{\infty} S_{n+b,k+r}(y)(y-x)^m dy \\ &= (n+b) \sum_{k=0}^{\infty} S_{n,k}(x) \int_0^{\infty} (k-(n+a)x)S_{n+b,k+r}(y)(y-x)^m dy. \end{aligned} \quad (2.65)$$

Here, we see

$$\begin{aligned} &(k-(n+a)x)S_{n+b,k+r}(y) \\ &= ((k+r-(n+b)y) - (r+(a-b)x) + (n+b)(y-x))S_{n+b,k+r}(y) \\ &= yS_{n+b,k+r}^{(1)}(y) - (r+(a-b)x)S_{n+b,k+r}(y) + (n+b)S_{n+b,k+r}(y)(y-x). \end{aligned} \quad (2.66)$$

Then substituting (2.66) for (2.65), we consider the following;

$$\begin{aligned}
 & x \left(R_{n,m,r}^{(1)}(x) + mR_{n,m-1,r}(x) \right) \\
 &= (n+b) \sum_{k=0}^{\infty} S_{n,k}(x) \int_0^{\infty} (yS_{n+b,k+r}^{(1)}(y) - (r+(a-b)x)S_{n+b,k+r}(y) \\
 &\quad + (n+b)S_{n+b,k+r}(y)(y-x))(y-x)^m dy \\
 &:= \int_1 + \int_2 + \int_3.
 \end{aligned} \tag{2.67}$$

Then, we have

$$\begin{aligned}
 \int_1 &= (n+b) \sum_{k=0}^{\infty} S_{n,k}(x) \int_0^{\infty} yS_{n+b,k+r}^{(1)}(y)(y-x)^m dy \\
 &= (n+b) \sum_{k=0}^{\infty} S_{n+a,k}(x) \int_0^{\infty} S_{n+b,k+r}^{(1)}(y)(y-x)^{m+1} dy \\
 &\quad + x(n+b) \sum_{k=0}^{\infty} S_{n+a,k}(x) \int_0^{\infty} S_{n+b,k+r}^{(1)}(y)(y-x)^m dy \\
 &= -(m+1)R_{n,m,r}(x) - xmR_{n,m-1,r}(x).
 \end{aligned} \tag{2.68}$$

Here the last equation follows by parts of integration. Furthermore, we have

$$\int_1 + \int_2 = -(r+(a-b)x)R_{n,m,r}(x) + (n+b)R_{n,m+1,r}(x). \tag{2.69}$$

Therefore, we have

$$(n+b)R_{n,m+1,r}(x) = xR_{n,m,r}^{(1)}(x) + ((a-b)x + m+r+1)R_{n,m,r}(x) + 2xmR_{n,m-1,r}(x). \tag{2.70}$$

(iii) It is proved by the same method as the proof of Lemma 2.1 (iv). \square

Proof of Theorem 1.7. Let $|y-x| \leq 1$. By the second inequality in (1.30),

$$\begin{aligned}
 \left| f^{(r)}(y) - f^{(r)}(x) \right| &= |y-x| \left| f^{(r+1)}(\xi) \right| \\
 &\leq C|y-x| \left| e^{\beta\xi} \xi^{\alpha+2} \right| \leq C|y-x| e^{\beta x} (x+1)^{\alpha+2}.
 \end{aligned} \tag{2.71}$$

Let $\varepsilon = n^{-\gamma}$, $0 < \gamma < 1$,

$$\begin{aligned} & \left| \left(\frac{n+\beta}{n} \right)^r (Q_{n,\beta}[f])^{(r)}(x) - f^{(r)}(x) \right| \\ &= \left| (n+\beta) \sum_{k=0}^{\infty} S_{n,k}(x) \left(\int_{|y-x|<\varepsilon} S_{n+\beta,k+r}(y) |f^{(r)}(y) - f^{(r)}(x)| dy \right. \right. \\ & \quad \left. \left. + \int_{|y-x|\geq\varepsilon} S_{n+\beta,k+r}(y) |f^{(r)}(y) - f^{(r)}(x)| dy \right) \right| \\ &=: A + B. \end{aligned} \tag{2.72}$$

First, we see that by (2.71) and Lemma 2.6(i),

$$\begin{aligned} A &\leq C(n+\beta) \sum_{k=0}^{\infty} S_{n,k}(x) \int_{|y-x|<\varepsilon} S_{n+\beta,k+r}(y) |y-x| dy e^{\beta x} (x+1)^{\alpha+2} \\ &\leq C\varepsilon e^{\beta x} (x+1)^{\alpha+2} \leq O\left(\frac{1}{n^\gamma}\right) e^{\beta x} (x+1)^{\alpha+2}. \end{aligned} \tag{2.73}$$

Next, to estimate B , we split it into two parts:

$$\begin{aligned} B &= \left| (n+\beta) \sum_{k=0}^{\infty} S_{n,k}(x) \int_{|y-x|\geq\varepsilon} S_{n+\beta,k+r}(y) |f^{(r)}(y) - f^{(r)}(x)| dy \right| \\ &\leq (n+\beta) \sum_{k=0}^{\infty} S_{n,k}(x) \int_{|y-x|\geq\varepsilon} S_{n+\beta,k+r}(y) (|f^{(r)}(y)| + |f^{(r)}(x)|) dy \\ &\leq (n+\beta) \sum_{k=0}^{\infty} S_{n,k}(x) \int_{|y-x|\geq\varepsilon} S_{n+\beta,k+r}(y) (e^{\beta y} (y+1)^\alpha + e^{\beta x} (x+1)^\alpha) dy \\ &=: B_1 + B_2. \end{aligned} \tag{2.74}$$

First, we estimate

$$B_1 = (n+\beta) \sum_{k=0}^{\infty} S_{n,k}(x) \int_{|y-x|>\varepsilon} S_{n+\beta,k+r}(y) e^{\beta y} (y+1)^\alpha dy. \tag{2.75}$$

Then, using the following facts:

$$S_{n+\beta,k+r}(y) e^{\beta y} = S_{n,k+r}(y) \left(\frac{n+\beta}{n} \right)^{k+r}, \tag{2.76}$$

$$\left(\frac{n+\beta}{n}\right)^{k+r} S_{n,k}(x) = \left(\frac{n+\beta}{n}\right)^r S_{n+\beta,k}(x) e^{\beta x}, \quad (2.77)$$

$$(y+1)^\alpha = ((y-x)+x+1)^\alpha = \sum_{i=0}^{\alpha} \binom{\alpha}{i} (y-x)^i (x+1)^{\alpha-i}, \quad (2.78)$$

we have

$$\begin{aligned} B_1 &\leq C e^{\beta x} (n+\beta) \left(\frac{n+\beta}{n}\right)^r \sum_{k=0}^{\infty} S_{n+\beta,k}(x) \int_{|y-x|>\varepsilon} S_{n,k+r}(y) (y+1)^\alpha dy \\ &= C e^{\beta x} n \left(\frac{n+\beta}{n}\right)^{r+1} \sum_{k=0}^{\infty} S_{n+\beta,k}(x) \\ &\quad \times \int_{|y-x|>\varepsilon} S_{n,k+r}(y) \sum_{i=1}^{\alpha} \binom{\alpha}{i} (y-x)^i (x+1)^{\alpha-i} dy \\ &\quad + C e^{\beta x} n \left(\frac{n+\beta}{n}\right)^r \sum_{k=0}^{\infty} S_{n+\beta,k}(x) \int_{|y-x|>\varepsilon} S_{n,k+r}(y) (x+1)^\alpha dy \\ &=: C e^{\beta x} (B_{11} + B_{12}). \end{aligned} \quad (2.79)$$

Then, using (2.18) and Lemma 2.6, we have

$$\begin{aligned} B_{11} &= n \sum_{k=0}^{\infty} S_{n+\beta,k}(x) \int_{|y-x|>\varepsilon} S_{n,k+r}(y) \sum_{i=1}^{\alpha} \binom{\alpha}{i} (y-x)^i (x+1)^{\alpha-i} dy \\ &\leq n \sum_{k=0}^{\infty} S_{n+\beta,k}(x) \int_{|y-x|>\varepsilon} S_{n,k+r}(y) \sum_{i=1}^{\alpha} \binom{\alpha}{i} (y-x)^i \left|\frac{y-x}{\varepsilon}\right|^{\langle i \rangle} (x+1)^{\alpha-i} dy \\ &\leq n \sum_{k=0}^{\infty} S_{n+\beta,k}(x) \int_{|y-x|>\varepsilon} S_{n,k+r}(y) \sum_{i=1}^{\alpha} \binom{\alpha}{i} (y-x)^{i+\langle i \rangle} \left(\frac{1}{\varepsilon}\right)^{\langle i \rangle} (x+1)^{\alpha-i} dy \\ &\leq \sum_{i=1}^{\alpha} \binom{\alpha}{i} R_{n,i+\langle i \rangle r}(\beta, 0; x) \left(\frac{1}{\varepsilon}\right)^{\langle i \rangle} (x+1)^{\alpha-i} \\ &= \sum_{i=1}^{\alpha} \binom{\alpha}{i} O\left(\frac{\langle i \rangle}{n^{\lfloor (i+\langle i \rangle + 1)/2 \rfloor}}\right) q_{n,i+\langle i \rangle r}(\beta, 0; x) (x+1)^{\alpha-i}. \end{aligned} \quad (2.80)$$

Then by (2.21) we have

$$B_{11} \leq O\left(\frac{1}{n^{1-\gamma}}\right) (x+1)^{\alpha+1}. \quad (2.81)$$

For B_{12} , we have

$$\begin{aligned}
 B_{12} &= n \sum_{k=0}^{\infty} S_{n+\beta,k}(x) \int_{|y-x|>\varepsilon} S_{n,k+r}(y) (x+1)^\alpha dy \\
 &\leq n \sum_{k=0}^{\infty} S_{n+\beta,k}(x) \int_{|y-x|>\varepsilon} S_{n,k+r}(y) \left(\frac{y-x}{\varepsilon}\right)^2 dy (x+1)^\alpha \\
 &\leq R_{n,2,r}(\beta, 0; x) \left(\frac{1}{\varepsilon}\right)^2 (x+1)^\alpha = O\left(\frac{n^{2\gamma}}{n^{[3/2]}}\right) q_{n,2,r}(\beta, 0; x) (x+1)^\alpha \\
 &= O\left(\frac{1}{n^{1-2\gamma}}\right) (x+1)^{\alpha+2}.
 \end{aligned} \tag{2.82}$$

From (2.81), (2.82) and (2.79), we have

$$B_1 \leq O\left(\frac{1}{n^{1-\gamma}}\right) e^{\beta x} (x+1)^{\alpha+2}. \tag{2.83}$$

We estimate

$$B_2 = (n+\beta) \sum_{k=0}^{\infty} S_{n,k}(x) \int_{|y-x|>\varepsilon} S_{n+\beta,k+r}(y) e^{\beta x} (x+1)^\alpha dy. \tag{2.84}$$

Then we can estimate B_2 by the same method as B_{12} ,

$$\begin{aligned}
 B_2 &= (n+\beta) \sum_{k=0}^{\infty} S_{n,k}(x) \int_{|y-x|>\varepsilon} S_{n+\beta,k+r}(y) e^{\beta x} (x+1)^\alpha dy \\
 &\leq (n+\beta) \sum_{k=0}^{\infty} S_{n,k}(x) \int_{|y-x|>\varepsilon} S_{n+\beta,k+r}(y) \left(\frac{y-x}{\varepsilon}\right)^2 dy e^{\beta x} (x+1)^\alpha \\
 &\leq R_{n,2,r}(0, \beta; x) \left(\frac{1}{\varepsilon}\right)^2 e^{\beta x} (x+1)^\alpha,
 \end{aligned} \tag{2.85}$$

so we have

$$B_2 \leq O\left(\frac{n^{2\gamma}}{n^{[3/2]}}\right) q_{n,2,r}(0, \beta; x) e^{\beta x} (x+1)^\alpha \leq O\left(\frac{1}{n^{1-2\gamma}}\right) e^{\beta x} (x+1)^{\alpha+2}. \tag{2.86}$$

Consequently, we obtain from (2.83) and (2.86),

$$B \leq \left(O\left(\frac{1}{n^{1-\gamma}}\right) + O\left(\frac{1}{n^{1-2\gamma}}\right)\right) e^{\beta x} (x+1)^{\alpha+2} \leq O\left(\frac{1}{n^{1-2\gamma}}\right) e^{\beta x} (x+1)^{\alpha+2}. \tag{2.87}$$

Therefore, from (2.73) and (2.87), we have uniformly on n ,

$$\left| \left(\frac{n+\beta}{n} \right)^r (Q_{n,\beta}[f])^{(r)}(x) - f^{(r)}(x) \right| \leq O\left(\frac{1}{n^\gamma}\right) e^{\beta x} x^{\alpha+2} + O\left(\frac{1}{n^{1-2\gamma}}\right) e^{\beta x} x^{\alpha+2}. \quad (2.88)$$

Here, if we let $\gamma = 1/3$, then we have

$$\left| \left(\frac{n+\beta}{n} \right)^r (Q_{n,\beta}[f])^{(r)}(x) - f^{(r)}(x) \right| = O\left(\frac{1}{n^{1/3}}\right) e^{\beta x} x^{\alpha+2}, \quad (2.89)$$

that is, (1.31) is proved. So, we also have a norm convergence (1.32). \square

Lemma 2.7. *Let m and b be nonnegative integers. Let*

$$U_{n,m,r}(b; x) := (n+b) \sum_{k=0}^{\infty} S_{n,k}(x) \int_0^{\infty} S_{n+b,k+r}(y) (y-x)^m e^{by} dy. \quad (2.90)$$

Then one has

$$U_{n,m,r}(b; x) = \left(\frac{n+b}{n} \right)^{r+1} e^{bx} R_{n,m,r}(b, 0; x). \quad (2.91)$$

Proof. From (2.76) and (2.77) we have

$$S_{n+b,k+r}(x) e^{bx} = S_{n,k+r}(x) \left(\frac{n+b}{n} \right)^{k+r}, \quad (2.92)$$

$$\left(\frac{n+b}{n} \right)^{k+r} S_{n,k}(x) = \left(\frac{n+b}{n} \right)^r S_{n+b,k}(x) e^{bx}. \quad (2.93)$$

We have from (2.92), (2.93), and noting (2.61),

$$\begin{aligned} U_{n,m,r}(b; x) &= (n+b) \sum_{k=0}^{\infty} S_{n,k}(x) \int_0^{\infty} S_{n+b,k+r}(y) (y-x)^m e^{by} dy \\ &= (n+b) \sum_{k=0}^{\infty} \left(\frac{n+b}{n} \right)^{k+r} S_{n,k}(x) \int_0^{\infty} S_{n,k+r}(y) (y-x)^m dy \\ &= (n+b) \left(\frac{n+b}{n} \right)^r e^{bx} \sum_{k=0}^{\infty} S_{n+b,k}(x) \int_0^{\infty} S_{n,k+r}(y) (y-x)^m dy \\ &= \left(\frac{n+b}{n} \right)^{r+1} e^{bx} R_{n,m,r}(b, 0; x). \end{aligned} \quad (2.94)$$

\square

Proof of Theorem 1.9. We prove this theorem, similarly to the proof of Theorem 1.4. Using (2.93) and (2.27), we have for $f \in C^{(r+2)}([0, \infty))$,

$$\begin{aligned}
& \left(\frac{n+\beta}{n}\right)^r (Q_{n,\beta}[f])^{(r)}(x) \\
&= (n+\beta) \sum_{k=0}^{\infty} S_{n,k}(x) \int_0^{\infty} S_{n+\beta,k+r}(y) f^{(r)}(y) dy \\
&= f^{(r)}(x) + f^{(r+1)}(x) (n+\beta) \sum_{k=0}^{\infty} S_{n,k}(x) \int_0^{\infty} S_{n+\beta,k+r}(y) (y-x) dy \\
&\quad + (n+\beta) \sum_{k=0}^{\infty} S_{n,k}(x) \int_0^{\infty} S_{n+\beta,k+r}(y) \left(\int_x^y (y-u) f^{(r+2)}(u) du\right) dy \\
&= f^{(r)}(x) + f^{(r+1)}(x) R_{n,1,r}(0, \beta; x) \\
&\quad + (n+\beta) \sum_{k=0}^{\infty} S_{n,k}(x) \int_0^{\infty} S_{n+\beta,k+r}(y) \left(\int_x^y (y-u) f^{(r+2)}(u) du\right) dy.
\end{aligned} \tag{2.95}$$

We estimate the last term. We note the given condition:

$$\left\| f^{(r+2)}(x) e^{-\beta x} \psi^{2\gamma}(x) \right\|_{L_{\infty}([0, \infty))} < \infty. \tag{2.96}$$

Using the inequality $(1+t)^{2\gamma} \leq C((1+x)^{2\gamma} + (t-x)^{2\gamma})$ and Lemma 2.7, we have

$$\begin{aligned}
\sum &:= (n+\beta) \sum_{k=0}^{\infty} S_{n,k}(x) \int_0^{\infty} S_{n+\beta,k+r}(y) \left(e^{\beta y} \psi^{-2\gamma}(y) + e^{\beta x} \psi^{-2\gamma}(x) \right) (y-x)^2 dy \\
&\leq C(n+\beta) \sum_{k=0}^{\infty} S_{n,k}(x) \int_0^{\infty} S_{n+\beta,k+r}(y) \\
&\quad \times \left(e^{\beta y} \psi^{-2\gamma}(x) + e^{\beta y} \left((y-x)^{2\gamma} + e^{\beta x} \psi^{-2\gamma}(x) \right) (y-x)^2 dy \right) \\
&\leq C \left(\psi^{-2\gamma}(x) U_{n,2,r}(\beta; x) + U_{n,2\gamma+2,r}(\beta; x) + e^{\beta x} \psi^{-2\gamma}(x) R_{n,2,r}(0, \beta; x) \right) \\
&\leq C \left(e^{\beta x} \psi^{-2\gamma}(x) R_{n,2,r}(\beta, 0; x) \left(\frac{n+\beta}{n}\right)^{r+1} \right. \\
&\quad \left. + e^{\beta x} R_{n,2\gamma+2,r}(\beta, 0; x) \left(\frac{n+\beta}{n}\right)^{r+1} + e^{\beta x} \psi^{-2\gamma}(x) R_{n,2,r}(0, \beta; x) \right).
\end{aligned} \tag{2.97}$$

Then, we can estimate as follows:

$$\left| (n+\beta) \sum_{k=0}^{\infty} S_{n,k}(x) \int_0^{\infty} S_{n+\beta,k+r}(y) \left(\int_x^y (y-u) f^{(r+2)}(u) du\right) dy \right|$$

$$\begin{aligned}
&\leq \left\| f^{(r+2)}(x)e^{-\beta x}\psi^{2\gamma}(x) \right\|_{L_\infty([0,\infty))} \sum \\
&\leq C \left\| f^{(r+2)}(x)e^{-\beta x}\psi^{2\gamma}(x) \right\|_{L_\infty([0,\infty))} e^{\beta x} \\
&\quad \times \left(\psi^{-2\gamma}(x)R_{n,2,r}(\beta, 0; x) \left(\frac{n+\beta}{n}\right)^{r+1} + R_{n,2\gamma+2,r}(\beta, 0; x) \left(\frac{n+\beta}{n}\right)^{r+1} \right. \\
&\quad \left. + \psi^{-2\gamma}(x)R_{n,2,r}(0, \beta; x) \right).
\end{aligned} \tag{2.98}$$

Then, we have by (iv) of Lemma 2.6,

$$\begin{aligned}
&\left| \left(\frac{n+\beta}{n}\right)^r (Q_{n,\beta}[f])^{(r)}(x) - f^{(r)}(x) \right| e^{-\beta x}\psi^{2\gamma+2}(x) \\
&\leq \left\| f^{(r+2)}(x)e^{-\beta x}\psi^{2\gamma}(x) \right\|_{L_\infty([0,\infty))} \\
&\quad \times \left(O\left(\frac{1}{n}\right)\psi^2(x)g_{n,2,r}(\beta, 0; x) + O\left(\frac{1}{n^\gamma}\right)g_{n,2\gamma+2,r}(\beta, 0; x)\psi^{2\gamma+2}(x) \right. \\
&\quad \left. + O\left(\frac{1}{n}\right)\psi^2(x)g_{n,2,r}(0, \beta; x) \right) \\
&\quad + O\left(\frac{1}{n}\right) \left| f^{(r+1)}(x)e^{-\beta x}\psi^{2\gamma+1}(x)g_{n,1,r}(0, \beta; x)\psi(x) \right|.
\end{aligned} \tag{2.99}$$

Consequently, we have

$$\begin{aligned}
&\left| \left[\left(\frac{n+\beta}{n}\right)^r (Q_{n,\beta}[f])^{(r)}(x) - f^{(r)}(x) \right] e^{-\beta x}\psi^{2\gamma+2}(x) \right| \\
&\leq O\left(\frac{1}{n}\right) \left(\left\| f^{(r+1)}(x)e^{-\beta x}\psi^{2\gamma+1}(x) \right\|_{L_\infty([0,\infty))} + \left\| f^{(r+2)}(x)e^{-\beta x}\psi^{2\gamma}(x) \right\|_{L_\infty([0,\infty))} \right),
\end{aligned} \tag{2.100}$$

since we know that $|g_{n,2,r}(\beta, 0; x)\psi^2(x)|$, $|g_{n,2,r}(0, \beta; x)\psi^2(x)|$, $|g_{n,2\gamma+2,r}(\beta, 0; x)\psi^{2\gamma+2}(x)|$, and $|g_{n,1,r}(0, \beta; x)\psi(x)|$ are uniformly bounded on $[0, \infty)$. \square

Theorem 2.8. *Let β and γ be nonnegative integers and $r > 0$ be a positive integer. Then one has uniformly for n , f and $x \in [0, \infty)$,*

$$\left| \left[(Q_{n,\beta}[f])^{(r)}(x) \right] e^{-\beta x}\psi^{2\gamma}(x) \right| \leq C \frac{n+\beta}{n} \left\| f^{(r)}(x)e^{-\beta x}\psi^{2\gamma}(x) \right\|_{L_\infty([0,\infty))}. \tag{2.101}$$

Proof. Using $(1+y)^{2\gamma} \leq C((1+x)^{2\gamma} + (y-x)^{2\gamma})$, we have

$$\begin{aligned} & \left| (n+\beta) \sum_{k=0}^{\infty} S_{n,k}(x) \int_0^{\infty} S_{n+\beta,k+r}(y) e^{\beta y} (1+y)^{2\gamma} dy \right| \\ & \leq \left| (n+\beta) \sum_{k=0}^{\infty} S_{n,k}(x) \int_0^{\infty} S_{n+\beta,k+r}(y) e^{\beta y} \left((1+x)^{2\gamma} + (y-x)^{2\gamma} \right) dy \right| \\ & \leq C \left(U_{n,0,r}(\beta; x) \psi^{-2\gamma}(x) + U_{n,2\gamma,r}(\beta; x) \right). \end{aligned} \quad (2.102)$$

By Lemma 2.7 and (i) of Lemma 2.6, we know

$$\begin{aligned} U_{n,0,r}(\beta; x) &= \left(\frac{n+\beta}{n} \right)^{r+1} e^{\beta x} R_{n,0,r}(\beta, 0; x) = \left(\frac{n+\beta}{n} \right)^{r+1} e^{\beta x}, \\ U_{n,2\gamma,r}(\beta; x) &= \left(\frac{n+\beta}{n} \right)^{r+1} e^{\beta x} R_{n,2\gamma,r}(\beta, 2\gamma; x) \\ &= O\left(\frac{1}{n^\gamma} \right) \left(\frac{n+\beta}{n} \right)^{r+1} e^{\beta x} g_{n,2\gamma,r}(\beta, 2\gamma; x). \end{aligned} \quad (2.103)$$

Therefore, we have

$$\begin{aligned} & \left| (n+\beta) \sum_{k=0}^{\infty} S_{n,k}(x) \int_0^{\infty} S_{n+\beta,k+r}(y) e^{\beta y} (1+y)^{2\gamma} dy \right| \\ & \leq C \left(\frac{n+\beta}{n} \right)^{r+1} e^{\beta x} \left(\psi^{-2\gamma}(x) + O\left(\frac{1}{n^\gamma} \right) g_{n,2\gamma,r}(\beta, 2\gamma; x) \right). \end{aligned} \quad (2.104)$$

Since $|g_{n,2\gamma,r}(\beta, 2\gamma; x) \psi^{2\gamma}(x)|$ is uniformly bounded on $[0, \infty)$, we have

$$\begin{aligned} & \left| \left(\frac{n+\beta}{n} \right)^r (Q_{n,\beta}[f])^{(r)}(x) e^{-\beta x} \psi^{2\gamma}(x) \right| \\ & \leq \left| (n+\beta) \sum_{k=0}^{\infty} S_{n,k}(x) \int_0^{\infty} S_{n+\beta,k+r}(y) e^{\beta y} (1+y)^{2\gamma} dy \right| e^{-\beta x} \psi^{2\gamma}(x) \\ & \quad \times \left\| f^{(r)}(x) e^{-\beta x} \psi^{2\gamma}(x) \right\|_{L_\infty([0, \infty))} \\ & \leq C \left(\frac{n+\beta}{n} \right)^{r+1} \left(1 + O\left(\frac{1}{n^\gamma} \right) \right) \left\| f^{(r)}(x) e^{-\beta x} \psi^{2\gamma}(x) \right\|_{L_\infty([0, \infty))}. \end{aligned} \quad (2.105)$$

Therefore, we have the result. \square

Proof of Theorem 1.10. We will prove this theorem by the same method as the proof of Theorem 1.5. First, we split it as follows:

$$\begin{aligned}
& \left\| \left[\left(\frac{n+\beta}{n} \right)^r (Q_{n,\beta}[f])^{(r)}(x) - f^{(r)}(x) \right] e^{-\beta x} \psi^{2\gamma+2}(x) \right\|_{L_\infty([0,\infty))} \\
& \leq \left\| \left(\frac{n+\beta}{n} \right)^r (Q_{n,\beta}[f - [f]_h])^{(r)}(x) e^{-\beta x} \psi^{2\gamma+2}(x) \right\|_{L_\infty([0,\infty))} \\
& \quad + \left\| \left[\left(\frac{n+\beta}{n} \right)^r (Q_{n,\beta}[[f]_h])^{(r)}(x) - [f]_h^{(r)}(x) \right] e^{-\beta x} \psi^{2\gamma+2}(x) \right\|_{L_\infty([0,\infty))} \\
& \quad + \left\| [f]_h^{(r)}(x) - f^{(r)}(x) \right\|_{L_\infty([0,\infty))} e^{-\beta x} \psi^{2\gamma+2}(x).
\end{aligned} \tag{2.106}$$

Then for the first term, we have, using Theorem 2.8 and (2.41),

$$\begin{aligned}
& \left\| (Q_{n,\beta}[f - [f]_h])^{(r)}(x) e^{-\beta x} \psi^{2\gamma+2}(x) \right\|_{L_\infty([0,\infty))} \\
& \leq C \frac{n+\beta}{n} \left\| [f^{(r)}(x) - [f^{(r)}]_h(x)] e^{-\beta x} \psi^{2\gamma+2}(x) \right\|_{L_\infty([0,\infty))} \leq C \omega_2(f^{(r)}; e^{-\beta x} \psi^{2\gamma+2}(x); h).
\end{aligned} \tag{2.107}$$

For the second term, we have from Theorem 1.9,

$$\begin{aligned}
& \left\| \left[\left(\frac{n+\beta}{n} \right)^r (Q_{n,\beta}[f_h])^{(r)}(x) - f_h^{(r)}(x) \right] e^{-\beta x} \psi^{2\gamma+2}(x) \right\|_{L_\infty([0,\infty))} \\
& \leq O\left(\frac{1}{n}\right) \left(\left\| f_h^{(r+1)}(x) e^{-\beta x} \psi^{2\gamma+1}(x) \right\|_{L_\infty([0,\infty))} + \left\| f_h^{(r+2)}(x) e^{-\beta x} \psi^{2\gamma}(x) \right\|_{L_\infty([0,\infty))} \right).
\end{aligned} \tag{2.108}$$

Here, we suppose $0 < h \leq 1$ and then we know that $(e^{-\beta x} \psi(x)) / (e^{-\beta(x+h)} \psi(x+h))$ and $e^{-\beta x} / e^{-\beta(x+h)}$ are uniformly bounded on $[0, \infty)$. Therefore, we have from (2.42) and (2.43) of Lemma 2.4,

$$\begin{aligned}
& \left\| f_h^{(r+1)}(x) e^{-\beta x} \psi^{2\gamma+1}(x) \right\|_{L_\infty([0,\infty))} \leq C \frac{1}{h} \omega_1(f; e^{-\beta x} \psi^{2\gamma+1}(x); h), \\
& \left\| f_h^{(r+2)}(x) e^{-\beta x} \psi^{2\gamma}(x) \right\|_{L_\infty([0,\infty))} \leq C \frac{1}{h^2} \omega_2(f; e^{-\beta x} \psi^{2\gamma}(x); h).
\end{aligned} \tag{2.109}$$

Therefore, we have

$$\begin{aligned}
& \left\| \left[\left(\frac{n+\beta}{n} \right)^r (Q_{n,\beta}[f_h])^{(r)}(x) - f_h^{(r)}(x) \right] e^{-\beta x} \psi^2(x) \right\|_{L_\infty([0,\infty))} \\
& \leq O\left(\frac{1}{n}\right) \left(\frac{1}{h} \omega_1(f; e^{-\beta x} \psi^{2+1}(x); h) + \frac{1}{h^2} \omega_2(f; e^{-\beta x} \psi^2(x); h) \right).
\end{aligned} \tag{2.110}$$

Consequently, we have

$$\begin{aligned} & \left\| \left[\left(\frac{n+\beta}{n} \right)^r (Q_{n,\beta}[f])^{(r)}(x) - f^{(r)}(x) \right] e^{-\beta x} \psi^{2\gamma+2}(x) \right\| \\ & \leq O\left(\frac{1}{n}\right) \left(\frac{1}{h} \omega_1(f; e^{-\beta x} \psi^{2\gamma+1}(x); h) + \frac{1}{h^2} \omega_2(f; e^{-\beta x} \psi^{2\gamma}(x); h) \right) \\ & \quad + C \omega_2(f^{(r)}; e^{-\beta x} \psi^{2\gamma+2}(x)(x); h). \end{aligned} \quad (2.111)$$

If we let $h = 1/\sqrt{n}$, then

$$\begin{aligned} & \left\| \left[\left(\frac{n+\beta}{n} \right)^r (Q_{n,\beta}[f])^{(r)}(x) - f^{(r)}(x) \right] e^{-\beta x} \psi^{2\gamma+2}(x) \right\| \\ & \leq C \left(\frac{1}{\sqrt{n}} \omega_1\left(f; e^{-\beta x} \psi^{2\gamma+1}(x); \frac{1}{\sqrt{n}}\right) + \omega_2\left(f; e^{-\beta x} \psi^{2\gamma}(x); \frac{1}{\sqrt{n}}\right) \right), \end{aligned} \quad (2.112)$$

since $\omega_2(f^{(r)}; e^{-\beta x} \psi^{2\gamma+2}(x); 1/\sqrt{n}) \leq \omega_2(f; e^{-\beta x} \psi^{2\gamma}(x); 1/\sqrt{n})$. \square

3. Conclusion

In this paper, Lupas-type operators and Szász-Mirakyan-type operators are treated and the various weighted norm convergence on $[0, \infty)$ of these operators are investigated. Moreover, this paper proves theorems on degree of approximation of $f \in C^r([0, \infty))$ by these operators using the modulus of smoothness of f .

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