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Research Article

Duality of (h, φ)-Multiobjective Programming Involving Generalized Invex Functions

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In the setting of Ben-Tal's generalized algebraic operations, this paper deals with Mond-Weir type dual theorems of multiobjective programming problems involving generalized invex functions. Two classes of functions, namely, (h, φ)-pseudoinvex and (h, φ)-quasi-invex, are defined for a vector function. By utilizing these two classes of functions, some dual theorems are established for conditionally proper efficient solution in (h, φ)-multiobjective programming problems.

1. Introduction

The theory and applications of multiobjective programming problems have been closely tied with convex analysis. Optimality conditions and duality theorems were established for the class of problems involving the optimizations of convex objective functions over convex feasible regions. Such assumptions were very convenient because of the known separation theorems and the guarantee that necessary conditions for optimality were sufficient under convexity. However, not all practical problems, when formulated as multiobjective programs, fulfill the requirements of convexity. Fortunately, such problems were often found to have some characteristics in common with convex problems, and these properties could be exploited to establish theoretical results or develop algorithms. Many notions of generalized convexity having some useful properties shared with convexity have been defined by a sizeable number of researchers. A meaningful generalization of convex functions is the introduction of invex functions, which was given by Hanson [1], for the scalar case. Nowadays, with and without differentiability, the invex functions are extended to vector functions in finite dimensions or infinite dimensions abstract spaces, and sufficient optimality criteria and duality results are obtained for multiobjective programming or vector optimization, respectively, see [1–15].

In 1976, Ben-Tal [8] introduced certain generalized operations of addition and multiplication. This kind of generalized algebraic means has many applications in pure and applied mathematical fields, see [6, 7, 10–16]. The biggest advantage under Ben-Tal's generalized means is that the function has some transformable properties. As pointed out in literature [12] that a function is not convex or differentiable, however it may be transformed into convex function or differentiable function in the setting of Ben-Tal's generalized algebraic operations. In this way, Ben-Tal's generalized means provided a manner in extension of convexity. Recently, more and more interest has been paid on dealing with optimality and duality of multiobjective program problems involving generalized convexity under Ben-Tal's generalized means circumstances, for instance, see [10–16].

The properness of the efficient solution of the multiobjective programming problem is of importance. In 1991, Singh and Hanson [9] introduced conditionally properly efficiency for multiobjective programming problems. This kind of proper efficiency has specific significance in the optimal problem with multicriteria. In present paper, we first extend the notions of the conditionally proper efficiency for multiobjective programming problems, pseudoinvexity and quasi-invexity for vector functions in the setting of Ben-Tal's generalized means. Then, for a class of constraint multiobjective program problem, we will establish several duality results by using the new defined proper efficient solutions and generalized invex functions. This paper is organized as follows. In Section 2, we present some preliminaries and related results which will be used in the rest of the paper. In Section 3, some duality theorems are derived.

2. Preliminaries

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space and $\mathbb{R}_{++}$ be the set of all positive real numbers. Throughout this paper, the following convention for vector in $\mathbb{R}^n$ will be used:

$$\begin{aligned} x > y \quad &\text{iff } x_i > y_i,\ i = 1, 2, \ldots, n, \\ x \geqq y \quad &\text{iff } x_i \geqq y_i,\ i = 1, 2, \ldots, n, \\ x \geqslant y \quad &\text{iff } x_i \geqq y_i,\ i = 1, 2, \ldots, n,\ \text{but } x \neq y. \end{aligned} \tag{2.1}$$

We first present the generalized algebraic operations given by Ben-Tal [8].

Definition 2.1 (see [6, 8]). Let $h : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous vector function. Suppose that the inverse function h^{-1} of h exists. Then the h-vector addition of $x, y \in \mathbb{R}^n$ defined by

$$x \oplus y = h^{-1}\left(h(x) + h(y)\right), \tag{2.2}$$

and the h-scalar multiplication of $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ is defined by

$$\alpha \otimes x = h^{-1}(\alpha h(x)). \tag{2.3}$$

Similarly, generalized algebraic operations for scalar-valued functions can be defined as follows.

Definition 2.2 (see [6, 8]). Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a continuous and scalar function. Suppose that the inverse function φ^{-1} of φ exists. Then the φ-addition of $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$, is given by

$$\alpha[+]\beta = \varphi^{-1}(\varphi(\alpha) + \varphi(\beta)), \tag{2.4}$$

and the φ-scalar multiplication of $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$ as

$$\beta[\cdot]\alpha = \varphi^{-1}(\beta\varphi(\alpha)). \tag{2.5}$$

Definition 2.3 (see [6, 8]). The (h,φ)-inner product of vector $x, y \in R^n$ is defined as

$$\left(x^T y\right)_{h,\varphi} = \varphi^{-1}\left(h(x)^T h(y)\right). \tag{2.6}$$

In this paper, we denote

$$\begin{gathered} \bigoplus_{i=1}^{m} x_i = x_1 \oplus x_2 \oplus \cdots \oplus x_m, \quad x_i \in R^n,\ i = 1,2,\ldots,m, \\ \left[\sum_{i=1}^{m}\right]\alpha_i = \alpha_1[+]\alpha_2[+]\cdots[+]\alpha_m, \quad \alpha_i \in \mathbb{R}, i = 1,2,\ldots,m, \\ \alpha[-]\beta = \alpha[+]((-1)[\cdot]\beta), \quad \alpha,\beta \in \mathbb{R}. \end{gathered} \tag{2.7}$$

For the differentiability of a real-valued function in the setting of generalized algebraic means, Avriel [6] introduced the following important concept.

Definition 2.4 (see [6]). Let f be a real-valued function defined on $\mathbb{R}^n$, denote $\widehat{f}(t) = \varphi(f(h^{-1}(t)))$, $t \in R^n$. For simplicity, write $\widehat{f}(t) = \varphi f h^{-1}(t)$. The function f is said to be (h,φ)-differentiable at $x \in R^n$, if $\widehat{f}(t)$ is differentiable at $t = h(x)$, and denoted by $\nabla^* f(x) = h^{-1}(\nabla \widehat{f}(t)|_{t=h(x)})$. In addition, It is said that f is (h,φ)-differentiable on $\mathrm{X} \subset \mathbb{R}^n$ if it is (h,φ)-differentiable at each $x \in \mathrm{X}$. A vector-valued function is called (h,φ)-differentiable on $\mathrm{X} \subset \mathbb{R}^n$ if each of its components is (h,φ)-differentiable at each $x \in \mathrm{X}$.

We collect some basic properties concerning Ben-Tal's generalized means from the literatures [12, 14], which will be used in the squeal.

Lemma 2.5 (see [12, 14]). *Suppose that f, f_i are real-valued functions defined on R^n, for $i = 1,2,\ldots,m$, and (h,φ)-differentiable at $\overline{x} \in R^n$. Then, the following statements hold:*

(1) $\nabla^*(\lambda[\cdot]f(\overline{x})) = \lambda \otimes \nabla^* f(\overline{x})$, *for* $\lambda \in \mathbb{R}$,

(2) $((\bigotimes_{i=1}^m \lambda_i \otimes \nabla^* f(\overline{x}))^T y)_{h,\varphi} = [\sum_{i=1}^m](\nabla^*(\lambda_i[\cdot]f(\overline{x}))^T y)_{h,\varphi}$, *for* $y \in R^n$, $\lambda_i \in \mathbb{R}$.

Lemma 2.6 (see [12, 14]). *Let $i = 1,2,\ldots,m$. The following statements hold:*

(1) $\lambda[\cdot](\mu[\cdot]\alpha) = \mu[\cdot](\lambda[\cdot]\alpha) = \lambda\mu[\cdot]\alpha$, *for* $\lambda,\mu,\alpha \in \mathbb{R}$;

(2) $\lambda[\cdot](\alpha[-]\beta) = \lambda[\cdot]\alpha[-]\lambda[\cdot]\beta$, *for* $\lambda,\alpha,\beta \in \mathbb{R}$;

(3) $[\sum_{i=1}^m](\alpha_i[-]\beta_i) = [\sum_{i=1}^m]\alpha_i[-][\sum_{i=1}^m]\beta_i$, *for* $\alpha_i,\beta_i \in \mathbb{R}$.

Lemma 2.7 (see [12, 14]). *Suppose that function φ, which appears in Ben-Tal generalized algebraic operations, is strictly monotone with $\varphi(0) = 0$. Then, the following statements hold:*

(1) *let $\lambda \geqq 0$, $\alpha, \beta \in \mathbb{R}$, and $\alpha \leqq \beta$. Then $\lambda[\cdot]\alpha \leqq \lambda[\cdot]\beta$;*

(2) *let $\lambda > 0$, $\alpha, \beta \in \mathbb{R}$, and $\alpha < \beta$. Then $\lambda[\cdot]\alpha < \lambda[\cdot]\beta$;*

(3) *let $\lambda < 0$, $\alpha, \beta \in \mathbb{R}$, and $\alpha \leqq \beta$. Then $\lambda[\cdot]\alpha \geqq \lambda[\cdot]\beta$;*

(4) *let $\alpha_i, \beta_i \in \mathbb{R}$, $i = 1, 2, \dots, m$. If $\alpha_i \leqq \beta_i$ for any $i \in M$, then*

$$\left[\sum_{i=1}^{m}\right]\alpha_i \leqq \left[\sum_{i=1}^{m}\right]\beta_i. \tag{2.8}$$

If $\alpha_i \leqq \beta_i$ for any $i = 1, 2, \dots, m$, and there exists at least an index k such that $x_k < y_k$, then

$$\left[\sum_{i=1}^{m}\right]\alpha_i < \left[\sum_{i=1}^{m}\right]\beta_i. \tag{2.9}$$

Lemma 2.8 (see [12, 14]). *Suppose that φ is a continuous one-to-one strictly monotone and onto function with $\varphi(0) = 0$. Let $\alpha, \beta \in \mathbb{R}$. Then,*

(1) *$\alpha < \beta$ if and only if $\alpha[-]\beta < 0$,*

(2) *$\alpha[+]\beta = 0$ if and only if $\alpha = (-1)[\cdot]\beta$.*

Throughout the rest of this paper, one further assumes that h is a continuous one-to-one and onto function with $h(0) = 0$. Similarly, suppose that φ is a continuous one-to-one strictly monotone and onto function with $\varphi(0) = 0$. Under the above assumptions, it is clear that $0[\cdot]\alpha = \alpha[\cdot]0 = 0$.

Let X be a nonempty subset of $\mathbb{R}^n$ and the functions $f = (f_1, \dots, f_p)^T : X \to \mathbb{R}^p$ and $g = (g_1, \dots, g_m)^T : X \to \mathbb{R}^m$ are (h, φ)-differentiable on the set X with respect to the same (h, φ). Consider the following (h, φ)-multiobjective programming problem:

$$\begin{aligned} \min \quad & f(x) = \left(f_1(x), f_2(x), \dots, f_p(x)\right)^T, \quad x \in X \subset \mathbb{R}^n \\ \text{s.t.} \quad & g(x) \leqq 0. \end{aligned} \tag{MOP$_{h,\varphi}$}$$

Definition 2.9. A point $\overline{x}$ is said to be an efficient solution for $(\text{MOP})_{h,\varphi}$ if $\overline{x} \in X$ and $f(x) \nleq f(\overline{x})$ for all $x \in X$.

Singh and Hanson [9] introduced the concept of conditionally properly efficient for multiobjective optimization. Now, we extend this notion under Ben-Tal's generalized algebraic operations as follows.

Definition 2.10. The point $\overline{x}$ is said to be (h, φ)-conditionally proper efficient solution for $(\text{MOP})_{h,\varphi}$ if $\overline{x}$ is an efficient solution and there exists a positive function $M(x)$ such that, for i, one has

$$\frac{f_i(\overline{x})[-]f_i(x)}{f_j(x)[-]f_j(\overline{x})} \leqq M(x), \tag{2.10}$$

for some j such that $f_j(x) > f_j(\overline{x})$, whenever $x \in X$ and

$$f_i(x) < f_i(\overline{x}). \tag{2.11}$$

Example 2.11. Consider the following multiobjective problem:

$$\begin{aligned} \min \quad & f(x) = \left(f_1(x), f_2(x)\right)^T = \left(\frac{x_1}{x_2}, \frac{x_2}{x_1}\right)^T \\ \text{s.t.} \quad & g(x) = \left(g_1(x), g_2(x)\right)^T = (x_1, x_2)^T \geqq 1 \\ & x = (x_1, x_2)^T \in \mathbb{R}^2. \end{aligned} \tag{MOP$)'_{h,\varphi}$}$$

Taking $h(x) = x$, $\varphi(t) = t^3$, it can be shown that every point of the feasible region is efficient. Let $x^* = (a, b)^T$ be an efficient solution. Choosing $M(x) \geqq (bx_2/ax_1)$, where $x = (x_1, x_2)^T$. For $i = 1$, we get

$$\frac{f_2(x^*)[-]f_2(x)}{f_1(x)[-]f_1(x^*)} = \frac{\sqrt[3]{(b/a)^3 - (x_2/x_1)^3}}{\sqrt[3]{(x_1/x_2)^3 - (a/b)^3}} = \frac{bx_2}{ax_1} \leqq M(x), \tag{2.12}$$

for $j = 2$ such that $f_2(x) = x_2/x_1 > b/a = f_2(x^*)$ whenever $x = (x_1, x_2)^T$ is feasible and

$$f_1(x) = \frac{x_1}{x_2} < \frac{a}{b} = f_1(x^*). \tag{2.13}$$

Thus, x^* is (h, φ)-conditionally proper efficient solution.

Xu and Liu [10] introduced (h, φ)-Kuhn-Tucker constraint qualification and used it to establish Kuhn-Tucker necessary condition for (h, φ)-multiobjective programming problems, for more details concerning (h, φ)-Kuhn-Tucker constraint qualification, please see [10]. We now state this result as the following (Lemma 2.12).

Lemma 2.12 (Kuhn-Tucker-type necessary condition). *Let f_i for $i = 1, 2, \ldots, p$, g_j for $j = 1, 2, \ldots, m$ be (h, φ)-differentiable on $\mathbb{R}^n$, $\overline{x}$ be an efficient solution of $(MOP)_{h,\varphi}$ and the (h, φ)-Kuhn-Tucker constraint qualification be satisfied at $\overline{x}$. Then there exist $\overline{\tau} = (\overline{\tau}_1, \overline{\tau}_2, \ldots, \overline{\tau}_p)^T > 0$ and $\overline{\lambda} = (\overline{\lambda}_1, \overline{\lambda}_2, \ldots, \overline{\lambda}_m)^T \geqq 0$ such that*

$$\begin{gathered} \left(\bigoplus_{i=1}^{m} \overline{\tau}_i \otimes \nabla^* f_i(\overline{x})\right) \oplus \left(\bigoplus_{j=1}^{m} \overline{\lambda}_j \otimes \nabla^* g_j(\overline{x})\right) = 0, \\ \overline{\lambda}_j [\cdot] g_j(\overline{x}) = 0, \quad j = 1, 2, \ldots, m. \end{gathered} \tag{2.14}$$

Jeyakumar and Mond [2] introduced the notion of V-invexity for a vector function $f = (f_1, f_2, \ldots, f_p)$ and discussed its applications to a class of constrained multi-objective optimization problems. One now gives the definitions of generalized V-invexity for a vector function in the setting of Ben-Tal's generalized algebraic operations as follows.

Definition 2.13. A vector function $f : X \subset \mathbb{R}^n \to \mathbb{R}^p$ is said to be (h,φ)-V-invex at $\overline{x} \in X$ if there exist functions $\eta : X \times X \to \mathbb{R}^n$ and $\alpha_i : X \times X \to \mathbb{R}_{++}$ such that for each $x \in X$ and for $i = 1, 2, \ldots, p$,

$$f_i(x)[-]f_i(\overline{x}) \geqq \alpha_i(x,\overline{x})[\cdot]\left(\nabla^* f_i(\overline{x})^T \eta(x,\overline{x})\right)_{h,\varphi}. \tag{2.15}$$

If we take h and φ as the identity functions, the above definitions reduce to the V-invex function given by Jeyakumar and Mond [2].

Example 2.14. The functions $f : \mathbb{R} \to \mathbb{R}^2$, $f(x) = (f_1(x), f_2(x))^T = (|x|, \sqrt{|x|})^T$. Let $h(x) = x$ and $\varphi(t) = t^3$. Then, f is (h,φ)-V-invex function at $\overline{x} = 0$ with respect to any $\eta(x,\overline{x})$ and $\alpha_i(x, \overline{x})$, $i = 1, 2$.

Definition 2.15. A vector function $f : X \subset \mathbb{R}^n \to R^p$ is said to be (h,φ)-V-pseudoinvex at $\overline{x} \in X$ if there exist functions $\eta : X \times X \to R^n$ and $\beta_i : X \times X \to \mathbb{R}_{++}$ such that for each $x \in X$ and for $i = 1, 2, \ldots, p$,

$$\left[\sum_{i=1}^{p}\right]\left(\nabla^* f_i(\overline{x})^T \eta(x,\overline{x})\right)_{h,\varphi} \geqq 0 \Longrightarrow \left[\sum_{i=1}^{p}\right]\beta_i(x,\overline{x})[\cdot]f_i(x) \geqq \left[\sum_{i=1}^{p}\right]\beta_i(x,\overline{x})[\cdot]f_i(\overline{x}). \tag{2.16}$$

If in the above definition $x \neq \overline{x}$ and (2.16) is satisfied as

$$\left[\sum_{i=1}^{p}\right]\left(\nabla^* f_i(\overline{x})^T \eta(x,\overline{x})\right)_{h,\varphi} \geqq 0 \Longrightarrow \left[\sum_{i=1}^{p}\right]\beta_i(x,\overline{x})[\cdot]f_i(x) > \left[\sum_{i=1}^{p}\right]\beta_i(x,\overline{x})[\cdot]f_i(\overline{x}), \tag{2.17}$$

then we say that f is strictly (h,φ)-V-pseudoinvex at $\overline{x} \in X$.

Example 2.16. The functions $f : (0,1] \to R^2$, $f(x) = (f_1(x), f_2(x)) = (\cos^2(x), -\sin^2(x))$. Let $h(t) = t$ and $\varphi(\alpha) = \arctan(\alpha)$. Then, f is (h,φ)-V-quasi-invex function at $\overline{x} = 1$ with respect to $\eta(x,\overline{x}) = 0$ and any $\beta_i(x,\overline{x}) > 0$ $(i = 1, 2)$. In fact, observing that $\varphi^{-1}(\alpha) = \tan(\alpha)$ and $h(0) = 0$, $\varphi(0) = \varphi^{-1}(0) = 0$. In this case, we have

$$0 = \left[\sum_{i=1}^{2}\right]\left(\nabla^* f_i(1)^T 0\right)_{h,\varphi} \geqq 0, \tag{2.18}$$

and for $x \in (0,1]$, it follows that

$$\begin{gathered}
f_1(x)[-]f_1(1) = \tan\left(\arctan\left(\cos^2(x)\right) - \arctan\left(\cos^2(1)\right)\right) = \frac{\cos^2(x) - \cos^2(1)}{1 + \cos^2(x)\cos^2(1)} \geqq 0, \\
f_2(x)[-]f_2(1) = \tan\left(\arctan\left(-\sin^2(x)\right) - \arctan\left(-\sin^2(1)\right)\right) = \frac{-\sin^2(x) - \left(-\sin^2(1)\right)}{1 + \sin^2(x)\sin^2(1)} \geqq 0.
\end{gathered} \tag{2.19}$$

Thus, we get from Lemmas 2.6 and 2.7 that

$$\left[\sum_{i=1}^{2}\right]\beta_i(x,\overline{x})[\cdot]f_i(x) \geqq \left[\sum_{i=1}^{2}\right]\beta_i(x,\overline{x})[\cdot]f_i(\overline{x}). \tag{2.20}$$

By Definition 2.15, we have shown that f is (h,φ)-V-pseudoinvex at $\overline{x} = 1$.

Definition 2.17. A vector function $f : X \subset \mathbb{R}^n \to R^p$ is said to be (h,φ)-V-quasi-invex at $\overline{x} \in X$ if there exist functions $\eta : X \times X \to R^n$ and $\delta_i : X \times X \to R^{++}$ such that for each $x \in X$ and for $i = 1, 2, \ldots, p$,

$$\left[\sum_{i=1}^{p}\right]\delta_i(x,\overline{x})[\cdot]f_i(x) \leqq \left[\sum_{i=1}^{p}\right]\delta_i(x,\overline{x})[\cdot]f_i(\overline{x}) \Longrightarrow \left[\sum_{i=1}^{p}\right]\left(\nabla^* f_i(\overline{x})^T\eta(x,\overline{x})\right)_{h,\varphi} \leqq 0. \tag{2.21}$$

Example 2.18. The function $f : \mathbb{R} \to \mathbb{R}$ is defined as $f(x) = x^3$. Taking $h(x) = x^3$ and $\varphi(t) = t$, then, f is (h,φ)-V-quasi-invex at $\overline{x} = 0$ with respect to $\eta(x,\overline{x}) = x \ominus \overline{x}$ and any $\delta(x,\overline{x}) > 0$.

3. Duality

In this section, we will establish the weak and strong duality theorems under the generalized (h,φ)-V-invexity assumptions for Mond and Weir type dual model in relation to $(\text{MOP})_{h,\varphi}$ Considering the following dual problem:

$$\max \quad f(u) = \left(f_1(u), f_2(u), \ldots, f_m(u)\right)^T \tag{$(\text{DMOP})_{h,\varphi}$}$$

$$\text{s.t.} \quad \left(\bigoplus_{i=1}^{p}\tau_i \otimes \nabla^* f_i(u)\right) \oplus \left(\bigoplus_{j=1}^{p}\lambda_j \otimes \nabla^* g_j(u)\right) = 0, \tag{3.1}$$

$$\lambda_j[\cdot]g_j(u) \geqq 0, \tag{3.2}$$

$$\tau > 0, \quad \tau = \left(\tau_1, \tau_2, \ldots, \tau_p\right)^T, \tag{3.3}$$

$$\lambda \geqq 0, \quad \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)^T \tag{3.4}$$

$$u \in X \subset \mathbb{R}^n. \tag{3.5}$$

Theorem 3.1 (weak duality). *Let x and (u,τ,λ) be any feasible solutions for $(MOP)_{h,\varphi}$ and $(DMOP)_{h,\varphi}$, respectively. Let either (a) or (b) below hold:*

(a) $(\tau_1[\cdot]f_1, \tau_2[\cdot]f_2, \ldots, \tau_p[\cdot]f_p)^T$ *is (h,φ)-V-pseudoinvex and* $(\lambda_1[\cdot]g_1, \lambda_2[\cdot]g_2, \ldots, \lambda_m g_m)^T$ *is (h,φ)-V-quasi-invex at u with respect to same η;*

(b) $(\tau_1[\cdot]f_1, \tau_2[\cdot]f_2, \ldots, \tau_p[\cdot]f_p)^T$ *is (h,φ)-V-quasi-invex and* $(\lambda_1[\cdot]g_1, \lambda_2[\cdot]g_2, \ldots, \lambda_m[\cdot]g_m)^T$ *is strictly (h,φ)-V-pseudoinvex at u with respect to same η. Then*

$$f(x)f(u). \tag{3.6}$$

Proof. Since (u,τ,λ) is a feasible solution for $(\text{DMOP})_{h,\varphi}$, by Lemma 2.5 and (3.1), for all $x' \in \mathbb{R}^n$ we obtain that

$$\left[\sum_{i=1}^{p}\right]\left(\nabla^*(\tau_i[\cdot]f_i(u))^T\eta(x',u)\right)_{h,\varphi}[+]\left[\sum_{j=1}^{m}\right]\left(\nabla^*(\lambda_j[\cdot]g_j(u))^T\eta(x',u)\right)_{h,\varphi} = 0. \tag{3.7}$$

(a) Let x be feasible for $(\text{MOP})_{h,\varphi}$ and $f(x) \leqslant f(u)$. Since $\tau > 0$ and $\beta_i(x,u) > 0$, for all $i = 1,\ldots,p$, it follows from Lemmas 2.6 and 2.7 that

$$\left[\sum_{i=1}^{p}\right]\beta_i(x,u)[\cdot]\tau_i[\cdot]f_i(x) < \left[\sum_{i=1}^{p}\right]\beta_i(x,u)[\cdot]\tau_i[\cdot]f_i(u), \tag{3.8}$$

and (h,φ)-V-pseudoinvexity at u of $(\tau_1[\cdot]f_1,\ldots,\tau_p[\cdot]f_p)^T$ implies

$$\left[\sum_{i=1}^{p}\right]\left(\nabla^*(\tau_i[\cdot]f_i(u))^T\eta(x,u)\right)_{h,\varphi} < 0. \tag{3.9}$$

Observing that x and (u,τ,λ) are feasible of $(\text{MOP})_{h,\varphi}$ and $(\text{DMOP})_{h,\varphi}$, respectively, we get from Lemma 2.7 that

$$\lambda_j[\cdot]g_j(u) \geqq 0 \geqq \lambda_j[\cdot]g_j(x), \quad \forall j = 1,2,\ldots m. \tag{3.10}$$

Again, since $\delta_j(x,u) > 0$, for all $j = 1,2,\ldots,m$, it follows from Lemma 2.7 that

$$\left[\sum_{j=1}^{m}\right]\delta_j(x,u)[\cdot]\lambda_j[\cdot]g_j(x) \leqq \left[\sum_{j=1}^{m}\right]\delta_j(x,u)[\cdot]\lambda_j[\cdot]g_j(u). \tag{3.11}$$

Now, (h,φ)-V-quasi-invexity at u of $(\lambda_1[\cdot]g_1,\ldots,\lambda_m[\cdot]g_m)^T$ implies that

$$\left[\sum_{j=1}^{m}\right]\left(\nabla^*(\lambda_j[\cdot]g_j(u))^T\eta(x,u)\right)_{h,\varphi} \leqq 0. \tag{3.12}$$

Together with (3.9) and (3.12), it yields from Lemma 2.7 that

$$\left[\sum_{i=1}^{p}\right]\left(\nabla^*(\tau_i[\cdot]f_i(u))^T\eta(x,u)\right)_{h,\varphi}[+]\left[\sum_{j=1}^{m}\right]\left(\nabla^*(\lambda_j[\cdot]g_j(u))^T\eta(x,u)\right)_{h,\varphi} < 0, \tag{3.13}$$

which contradicts to (3.7)

(b) Let x be feasible for $(\text{MOP})_{h,\varphi}$ and (u,τ,λ) feasible for $(\text{DMOP})_{h,\varphi}$. Suppose that $f(x) \leqslant f(u)$. Since $\delta_i(x,u) > 0$, for all $i = 1,\ldots,p$, and $\tau > 0$, we get from Lemmas 2.6 and 2.7 that

$$\left[\sum_{i=1}^{p}\right]\delta_i(x,u)[\cdot]\tau_i[\cdot]f_i(x) < \left[\sum_{i=1}^{p}\right]\delta_i(x,u)[\cdot]\tau_i[\cdot]f_i(u). \tag{3.14}$$

The (h,φ)-V-quasi-invexity at u of $(\tau_1[\cdot]f_1,\ldots,\tau_p[\cdot]f_p)^T$ implies that

$$\left[\sum_{i=1}^{p}\right]\left(\nabla^*(\tau_i[\cdot]f_i(u))^T\eta(x,u)\right)_{h,\varphi} \leqq 0. \tag{3.15}$$

By (3.7), we get from Lemmas 2.7 and 2.8 that

$$\left[\sum_{j=1}^{m}\right]\left(\nabla^*(\lambda_j[\cdot]g_j(u))^T\eta(x,u)\right)_{h,\varphi} \geqq 0. \tag{3.16}$$

and since $(\lambda_1[\cdot]g_1,\ldots,\lambda_m[\cdot]g_m)^T$ is strictly (h,φ)-V-pseudoinvex, we have

$$\left[\sum_{j=1}^{m}\right]\beta(x,u)_j[\cdot]\lambda_j[\cdot]g_j(x) > \left[\sum_{j=1}^{m}\right]\beta(x,u)_j[\cdot]\lambda_j[\cdot]g_j(u). \tag{3.17}$$

According to Lemma 2.7, this is a contradiction, since $\lambda_j[\cdot]g_j(x) \leqq 0$, $\lambda_j[\cdot]g_j(u) \geqq 0$ and $\beta(x,u)_j > 0$, for all $j = 1,2,\ldots,m$.

□

Theorem 3.2. *If $\overline{x}$ is feasible for $(MOP)_{h,\varphi}$ and $(\overline{u},\overline{\tau},\overline{\lambda})$ feasible for $(DMOP)_{h,\varphi}$ such that $f(\overline{x}) = f(\overline{u})$. Let neither (a′) or (b′) bellow hold:*

(a′) *$(\overline{\tau}_1[\cdot]f_1,\overline{\tau}_2[\cdot]f_2,\ldots,\overline{\tau}_p[\cdot]f_p)^T$ is (h,φ)-V-pseudoinvex and $(\overline{\lambda}_1[\cdot]g_1,\overline{\lambda}_2[\cdot]g_2,\ldots,\overline{\lambda}_m[\cdot]g_m)^T$ is (h,φ)-V-quasi-invex at $\overline{u}$ with respect to same η;*

(b′) *$(\overline{u}_1[\cdot]f_1,\overline{u}_2[\cdot]f_2,\ldots,\overline{u}_p[\cdot]f_p)^T$ is (h,φ)-V-quasi-invex and $(\overline{\lambda}_1[\cdot]g_1,\overline{\lambda}_2[\cdot]g_2,\ldots,\overline{\lambda}_m[\cdot]g_m)^T$ is strictly (h,φ)-V-pseudoinvex at $\overline{u}$ with respect to same η.*

Then $\overline{x}$ is (h,φ)-conditionally properly efficient for $(\text{MOP})_{h,\varphi}$ and $(\overline{u},\overline{\tau},\overline{\lambda})$ is (h,φ)-conditionally properly efficient solution for $(\text{DMOP})_{h,\varphi}$.

Proof. Suppose $\overline{x}$ is not an efficient solution for $(\text{MOP})_{h,\varphi}$, then there exists x feasible for $(\text{MOP})_{h,\varphi}$ such that

$$f(x) \leqslant f(\overline{x}). \tag{3.18}$$

Using the assumption $f(\overline{x}) = f(\overline{u})$, a contradiction to Theorem 3.1 is obtained. Hence, $\overline{x}$ is an efficient solution for $(\mathrm{MOP})_{h,\varphi}$. Similarly it can be ensured that $(\overline{u}, \overline{\tau}, \overline{\lambda}))$ is an efficient solution for $(\mathrm{DMOP})_{h,\varphi}$. □

Now suppose that $\overline{x}$ is not (h,φ)-conditionally properly efficient solution for $(\mathrm{MOP})_{h,\varphi}$. Therefore, for every positive function $M(x) > 0$, there exists $\widehat{x} \in X$ feasible for $(\mathrm{MOP})_{h,\varphi}$ and an index i such that

$$f_i(\overline{x})[-]f_i(\widehat{x}) > M(x)[\cdot]\big(f_j(\widehat{x})[-]f_j(\overline{x})\big), \tag{3.19}$$

for all j satisfying $f_j(\widehat{x}) > f_j(\overline{x})$, whenever $f_i(\widehat{x}) < f_i(\overline{x})$. This shows that $f_i(\overline{x})[-]f_i(\widehat{x})$ can be made arbitrarily large and hence for $\overline{\tau} > 0$ and $\beta_i(\widehat{x}, \overline{u}) > 0$, for all $i = 1, 2, \ldots, p$, the inequality

$$\left[\sum_{i=1}^{p}\right]\beta_i(\widehat{x},\overline{u})[\cdot]\overline{\tau}_i[\cdot]\big(f_i(\overline{x})[-]f_i(\widehat{x})\big) > 0. \tag{3.20}$$

is obtained. Consequently, we ge from Lemmas 2.7 and 2.8 that

$$\left[\sum_{i=1}^{p}\right]\beta_i(\widehat{x},\overline{u})[\cdot]\overline{\tau}_i[\cdot]f_i(\overline{x}) > \left[\sum_{i=1}^{p}\right]\beta_i(\widehat{x},\overline{u})[\cdot]\overline{\tau}_i[\cdot]f_i(\widehat{x}). \tag{3.21}$$

Now from feasibility conditions, we have

$$\overline{\lambda}_j[\cdot]g_j(\widehat{x}) \leqq \overline{\lambda}_j[\cdot]g_j(\overline{u}), \quad \forall j = 1, \ldots, m. \tag{3.22}$$

Since $\delta_j(\widehat{x}, \overline{u}) > 0$, for all $j = 1, \ldots, m$,

$$\left[\sum_{j=1}^{m}\right]\delta_j(\widehat{x},\overline{u})[\cdot]\overline{\lambda}_j[\cdot]g_j(\widehat{x}) \leqq \left[\sum_{j=1}^{m}\right]\delta_j(\widehat{x},\overline{u})[\cdot]\overline{\lambda}_j[\cdot]g_j(\overline{u}). \tag{3.23}$$

Suppose that the hypothesis (a′) holds at $\overline{u}$, we can get from (h,φ)-V-quasi-invexity at $\overline{u}$ of $(\overline{\lambda}_1[\cdot]g_1, \overline{\lambda}_2[\cdot]g_2, \ldots, \overline{\lambda}[\cdot]g_m)^T$ that

$$\left[\sum_{j=1}^{m}\right]\left(\nabla^*\left(\overline{\lambda}_j[\cdot]g_j(\overline{u})\right)^T\eta(\widehat{x},\overline{u})\right)_{h,\varphi} \leqq 0. \tag{3.24}$$

Therefore, from (3.1), we get from Lemmas 2.5, 2.7, and 2.8 that

$$\left[\sum_{i=1}^{p}\right]\left(\nabla^*(\overline{\tau}_i[\cdot]f_i(\overline{u}))^T\eta(\widehat{x},\overline{u})\right)_{h,\varphi} \geqq 0. \tag{3.25}$$

Since $(\overline{\tau}_1[\cdot]f_1, \overline{\tau}_2[\cdot]f_2, \ldots, \overline{\tau}_p[\cdot]f_p)^T$ is (h,φ)-V-pseudoinvex and $(\overline{\lambda}_1[\cdot]g_1, \overline{\lambda}_2[\cdot]g_2, \ldots, \overline{\lambda}_m[\cdot]g_m)^T$ is (h,φ)-V-quasi-invex at $\overline{u}$, we have

$$\left[\sum_{i=1}^{p}\right]\beta_i(\hat{x},\overline{u})[\cdot]\overline{\tau}_i[\cdot]f_i(\hat{x}) \geqq \left[\sum_{i=1}^{p}\right]\beta_i(\hat{x},\overline{u})[\cdot]\overline{\tau}_i[\cdot]f_i(\overline{u}). \tag{3.26}$$

On using the assumption $f(\overline{x}) = f(\overline{u})$ in the above equation, we get

$$\left[\sum_{i=1}^{p}\right]\beta_i(\hat{x},\overline{u})[\cdot]\overline{\tau}_i[\cdot]f_i(\hat{x}) \geqq \left[\sum_{i=1}^{p}\right]\beta_i(\hat{x},\overline{u})[\cdot]\overline{\tau}_i[\cdot]f_i(\overline{x}), \tag{3.27}$$

which is a contradiction to (3.21). Hence $\overline{x}$ is a (h,φ)-conditionally properly efficient solution for $(\text{MOP})_{h,\varphi}$.

We now suppose that $(\overline{u},\overline{\tau},\overline{\lambda})$ is not (h,φ)-conditionally properly efficient solution for $(\text{DMOP})_{h,\varphi}$. Therefore, for every positive function $M(x) > 0$, there exists a feasible $(\hat{u},\hat{\tau},\hat{\lambda})$ feasible for $(\text{DMOP})_{h,\varphi}$ and an index i such that

$$f_i(\hat{u})[-]f_i(\overline{u}) > M(x)[\cdot]\big(f_j(\overline{u})[-]f_j(\hat{u})\big), \tag{3.28}$$

for all j satisfying $f_j(\hat{u})[-]f_j(\overline{u}) > 0$ whenever $f_i(\hat{u})[-]f_i(\overline{u}) < 0$. This means $f_i(\hat{u})[-]f_i(\overline{u})$ can be made arbitrarily large and hence for $\overline{\tau} > 0$ and $\beta_i(\hat{x},\overline{u}) > 0$, for all $i = 1,2,\ldots,p$, the inequality

$$\left[\sum_{i=1}^{p}\right]\beta_i(\hat{x},\overline{u})[\cdot]\overline{\tau}_i[\cdot]f_i(\hat{u}) > \left[\sum_{i=1}^{p}\right]\beta_i(\hat{x},\overline{u})[\cdot]\overline{\tau}_i[\cdot]f_i(\overline{u}) \tag{3.29}$$

is obtained. Since $\overline{x}$ and $(\overline{u},\overline{\tau},\overline{\lambda})$ are feasible for $(\text{MOP})_{h,\varphi}$ and $(\text{DMOP})_{h,\varphi}$, respectively, it follows that as in first part:

$$\left[\sum_{i=1}^{p}\right]\beta_i(\hat{x},\overline{u})[\cdot]\overline{\tau}_i[\cdot]f_i(\hat{u}) \leqq \left[\sum_{i=1}^{p}\right]\beta_i(\hat{x},\overline{u})[\cdot]\overline{\tau}_i[\cdot]f_i(\overline{u}), \tag{3.30}$$

which contradicts (3.29). Hence, $(\overline{u},\overline{\tau},\overline{\lambda})$ is (h,φ)-conditionally properly efficient solution for $(\text{DMOP})_{h,\varphi}$.

Assuming that the hypothesis (b′) holds, we can finish the proof with the similar argument.

Theorem 3.3 (strong duality). *Let $\overline{x}$ be an efficient solution for $(MOP)_{h,\varphi}$. If the (h,φ)-Kuhn-Tucker constraint qualification is satisfied, then there are $\overline{\tau} > 0$, $\overline{\lambda} \geqq 0$ such that $(\overline{x},\overline{\tau},\overline{\lambda})$ is feasible for $(DMOP)_{h,\varphi}$ and the objective values of $(MOP)_{h,\varphi}$ and $(DMOP)_{h,\varphi}$ are equal at $\overline{x}$. Furthermore, if the hypothesis* (a′) or (b′) of Theorem 3.2 hold at $\overline{x}$, then $(\overline{x},\overline{\tau},\overline{\lambda})$ is (h,φ)-conditionally properly efficient for the problem $(\text{DMOP})_{h,\varphi}$.

Proof. Since $\overline{x}$ is an efficient solution for $(\text{MOP})_{h,\varphi}$ at which the (h,φ)-Kuhn-Tucker-type necessary conditions are satisfied, it follows from Lemma 2.12 that there exist $\overline{\tau} > 0$, $\overline{\lambda} \geqq 0$ such that $(\overline{x}, \overline{\tau}, \overline{\lambda})$ is feasible for $(\text{DMOP})_{h,\varphi}$. Evidently, the objective values of $(\text{MOP})_{h,\varphi}$ and $(\text{DMOP})_{h,\varphi}$ are equal at $\overline{x}$, since the objective functions for both problems are the same. The (h,φ)-conditionally proper efficiency of $(\overline{x}, \overline{\tau}, \overline{\lambda})$ for the problem $(\text{DMOP})_{h,\varphi}$ yields from Theorem 3.2. □

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