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Research Article

Noether Symmetries of the Area-Minimizing Lagrangian

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It is shown that the Lie algebra of Noether symmetries for the Lagrangian minimizing an $(n-1)$-area enclosing a constant n-volume in a Euclidean space is $so(n)\oplus_s\mathbb{R}^n$ and in a space of constant curvature the Lie algebra is $so(n)$. Furthermore, if the space has one section of constant curvature of dimension n_1, another of n_2, and so on to n_k and one of zero curvature of dimension m, with $n \geq \sum_{j=1}^{k} n_j + m$ (as some of the sections may have no symmetry), then the Lie algebra of Noether symmetries is $\oplus_{j=1}^{k} so(n_j+1) \oplus (so(m)\oplus_s\mathbb{R}^m)$.

1. Introduction

Symmetries of the Lagrangian and the corresponding Euler-Lagrange (EL) equations play an important role in the study of the differential equations (DEs). These symmetries can be used to reduce the order of the DEs or the number of variables in the case of partial differential equations (PDEs) and for the linearization of nonlinear DEs [1–3]. Symmetries that keep the action integral invariant (called Noether symmetries) are more important than the symmetries of the corresponding EL equation (called Lie point symmetries) as these symmetries give double reduction of the DEs and provide conserved quantities [4].

As the differential equations "live" on manifolds, it is natural to search for the connection between symmetries of differential equations and geometry. The first such attempt looked for the connection through the system of geodesic equations [5, 6]. Some connections between Noether symmetries and isometries have been found in the context of general relativity [7–9]. There are some errors in [7, 8] and it is incomplete (regarding all Noether symmetries under discussion, the corrected list was given in [10, 11]). Recently, the relation of both the Lie and Noether symmetries of the geodesic for a general Riemannian manifold has been given [12]. The geodesic equations are the EL equations for the arc-length-minimizing action. Their symmetries and the corresponding geodesic equations are known for maximally and nonmaximally symmetric spaces. A connection was obtained between isometries (the symmetries of the geometry) and Lie symmetries of the geodesic equations of the underlying space, which leads to the geometric linearization for ordinary differential equations (ODEs). An additional benefit of this approach is that one can obtain the solution of the linearized equations by the transformation to the metric tensor coordinates given by the geodesic equations from Cartesian coordinates [13–15]. In searching for an extension of the geometric methods to partial differential equations (PDEs), we tried to obtain a relation between isometries and Noether symmetries for the area-minimizing Lagrangian. In this paper, it is shown that the Lie algebra of the symmetries for the area-minimizing Lagrangian in an n-dimensional Euclidean space is $so(n)\oplus_s\mathbb{R}^n$ and in a space of constant curvature is $so(n)$. It is also shown that if the space has one section of constant curvature of dimension n_1, another of n_2, and so forth to n_k, and one of zero curvature of dimension m and $n \geq \sum_{j=1}^{k} n_j + m$, so that some of the sections have no symmetry, then the Lie algebra of Noether symmetries is $A = \oplus_{j=1}^{k} so(n_j + 1) \oplus (so(m)\oplus_s\mathbb{R}^m)$.

The plan of the paper is as follows. A brief review of the mathematical formalism is given in the next section. In the subsequent sections we present the symmetries of the area-minimizing Lagrangian for maximally and nonmaximally symmetric spaces. In the sixth section we present the symmetries for the less symmetric spaces. A summary and brief discussion are given in the last section.

2. Preliminaries

Let $\mathbf{x} = (x^i)$ and $\mathbf{u} = (u^\alpha)$ be n-independent and m-dependent variables, respectively. The derivatives of $\mathbf{u}$ with respect to $\mathbf{x}$ are denoted by $u_i^\alpha = D_i(u^\alpha)$, $u_{ij}^\alpha = D_i D_j(u^\alpha), \ldots$, where

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \cdots \tag{2.1}$$

is the total derivative operator and we have used the Einstein summation convention. The set of all the first derivatives u_i^α is denoted by $\mathbf{u}_{(1)}$ and the sets of all the higher-order derivatives are denoted by $\mathbf{u}_{(2)}, \mathbf{u}_{(3)}, \ldots, \mathbf{u}_{(i)}$. Let $L(\mathbf{x}, \mathbf{u}, \mathbf{u}_{(1)}, \ldots, \mathbf{u}_{(s)})$ be a Lagrangian of order s, corresponding to when the Euler Lagrange equations are

$$\frac{\delta L}{\delta u^\alpha} = 0, \quad \alpha = 1, \ldots, m, \tag{2.2}$$

where

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{k=1}^{\infty} (-1)^k D_{i_1} \ldots D_{i_k} \frac{\partial}{\partial u_{i_1 \ldots ik}^\alpha}. \tag{2.3}$$

A vector field

$$\mathbf{X} = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} \tag{2.4}$$

is a Noether symmetry of the Lagrangian [3] which is also a symmetry of (A.1), if there exists a vector-valued gauge function $\mathbf{A} = (A^1, A^2, \ldots, A^n)$, $A^i \in \mathcal{A}$, where $\mathcal{A}$ is the space of differential functions, such that

$$\mathbf{X}(L) + LD_i\left(\xi^i\right) = D_i\left(A^i\right). \tag{2.5}$$

Isometries are the directions, $\mathbf{k} = k^a(\partial/\partial x^a)$, along which the Lie derivative of the metric tensor $\mathbf{g}$ is zero, that is,

$$\mathcal{L}_k \mathbf{g} = 0. \tag{2.6}$$

This equation can be written in component form as

$$g_{ab,c}k^c + g_{bc}k^c_{,a} + g_{ac}k^c_{,b} = 0, \tag{2.7}$$

where "," denotes derivative with respect to x^a $(a = 1, 2, \ldots, n)$. This equation forms a set of $n(n+1)/2$ linear first-order partial differential equations for n functions of n variables in general, called the Killing equations.

In the n-dimensional space we minimize the $(n-1)$-area $A(S)$ of a hypersurface S given by $x^n = x^n(x^\alpha)$, $\alpha = 1, 2, \ldots, n-1$, keeping the n-volume $V(S)$ fixed. We define $y_\alpha = \partial x^n / \partial x^\alpha$. The area-minimizing action is [16]

$$I = A(S) + \lambda V(S) = \int_s n^p d^{n-1} s_p + \lambda \int_{\mathbf{v}} d^n V, \quad p = 1, 2, \ldots, n-1, \tag{2.8}$$

and the resulting EL-equation is

$$D_\alpha \left(\ln \sqrt{|g_{n-1}|} \right)_{,}^{\alpha} + \lambda \left(\ln \sqrt{|g_n|} \right)_{,n} = 0, \tag{2.9}$$

where g_{n-1} is the determinant of the $(n-1)$-metric of the hypersurface, g_n is the determinant of the n-metric of the volume, and "$_{,}{}^{\alpha}$" represents $\partial/\partial y_\alpha$.

3. Symmetries for Flat Spaces

3.1. Two-Area Minimization

The flat space metric in spherical coordinates is

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2. \tag{3.1}$$

Let the enclosing surface be $r = r(\theta, \phi)$. The 2-area is then given by

$$A(S) = \int \left(r^4 \sin^2\theta + r^2 \sin^2\theta\, r_{,\theta}^2 + r^2 r_{,\phi}^2\right)^{1/2} d\theta d\phi, \tag{3.2}$$

and the variational principle for the action (2.8) becomes

$$\delta \int \left[\Sigma + \lambda \frac{r^3 \sin\theta}{3}\right] d\theta d\phi = 0, \tag{3.3}$$

where

$$\Sigma = \left(r^4 \sin^2\theta + r^2 \sin^2\theta\, r_{,\theta}^2 + r^2 r_{,\phi}^2\right)^{1/2}. \tag{3.4}$$

Thus the Lagrangian is

$$L = \left(r^4 \sin^2\theta + r^2 \sin^2\theta\, r_{,\theta}^2 + r^2 r_{,\phi}^2\right)^{1/2} + \lambda \frac{r^3 \sin\theta}{3}. \tag{3.5}$$

The Noether symmetry condition (2.5) results in the following system of linear PDEs:

$$\begin{gathered}
r\xi^{\theta} \cos\theta + 2\eta \sin\theta + r \sin\theta\left(\xi^{\theta}_{,\theta} + \xi^{\phi}_{,\phi}\right) = 0, \qquad \eta_{,\theta} + r^2 \xi^{\theta}_{,r} = 0, \\
r\xi^{\theta} \cos\theta + \eta \sin\theta + r\left(\eta_{,r} + \xi^{\phi}_{,\phi}\right) \sin\theta = 0, \qquad \eta_{,\phi} + r^2 \sin^2\theta\, \xi^{\phi}_{,r} = 0, \\
A^{\theta}_{,\theta} + A^{\phi}_{,\phi} - \lambda\left[\frac{\xi^{\theta} r^3}{3} \cos\theta + \eta r \sin\theta + \frac{r^3}{3} \sin\theta\left(\xi^{\theta}_{,\theta} + \xi^{\phi}_{,\phi}\right)\right] = 0, \\
\eta + r\left(\eta_{,r} + \xi^{\theta}_{,\theta}\right) = 0, \qquad \sin^2\theta\, \xi^{\phi}_{,\theta} + \xi^{\theta}_{,\phi} = 0, \\
A^{\theta}_{,r} - \frac{\lambda r^3}{3} \xi^{\theta}_{,r} = 0, \qquad A^{\phi}_{,r} - \frac{\lambda r^3}{3} \xi^{\phi}_{,r} = 0.
\end{gathered} \tag{3.6}$$

This system gives us the following six symmetries:

$$\begin{gathered}
\mathbf{X}_1 = \sin\phi \frac{\partial}{\partial\theta} + \cos\phi \cot\theta \frac{\partial}{\partial\phi}, \qquad \mathbf{X}_2 = \cos\phi \frac{\partial}{\partial\theta} - \sin\phi \cot\theta \frac{\partial}{\partial\phi}, \\
\mathbf{X}_3 = \frac{\partial}{\partial\phi}, \qquad \mathbf{X}_4 = \frac{\cos\theta \sin\phi}{r} \frac{\partial}{\partial\theta} + \frac{\cos\phi}{r \sin\theta} \frac{\partial}{\partial\phi} + \sin\theta \sin\phi \frac{\partial}{\partial r}, \\
\mathbf{X}_5 = \frac{\cos\theta \cos\phi}{r} \frac{\partial}{\partial\theta} - \frac{\sin\phi}{r \sin\theta} \frac{\partial}{\partial\phi} + \sin\theta \cos\phi \frac{\partial}{\partial r}, \qquad \mathbf{X}_6 = \frac{\sin\theta}{r} \frac{\partial}{\partial\theta} - \cos\theta \frac{\partial}{\partial r}.
\end{gathered} \tag{3.7}$$

The corresponding Lie algebra of the Noether symmetries is $so(3)\oplus_s\mathbb{R}^3$, where $\oplus_s$ is the semidirect sum, $so(3) = \langle \mathbf{X_1}, \mathbf{X_2}, \mathbf{X_3}\rangle$, $\mathbb{R}^3 = \langle \mathbf{X_4}, \mathbf{X_5}, \mathbf{X_6}\rangle$, and

$$A_1 = \frac{\lambda r^2}{6}[-\cos\theta\sin\theta\sin\phi, -\cos\phi], \qquad A_2 = \frac{\lambda r^2}{6}[-\sin\theta\cos\theta\cos\phi, \sin\phi],$$
$$A_3 = -\frac{\lambda r^2}{6}\left[\sin^2\theta, 0\right], \tag{3.8}$$

are the vector gauge functions corresponding to the translations ($\mathbb{R}^3$).

3.2. Three-Area Minimization

Following the same procedure for a 3-area enclosing a constant 4-volume in hyperspherical coordinates, the Lagrangian is

$$L = \left(r^6\sin\chi^4\sin^2\theta + r^4 r_{,\chi}^2\sin\chi^4\sin\theta^2 + r^4 r_{,\theta}^2\sin\chi^2\sin\theta^2 + r^4 r_{,\phi}^2\sin\chi^2\right)^{1/2} + \lambda\frac{r^4\sin\chi^2\sin\theta}{4}. \tag{3.9}$$

The Lie algebra of the Noether symmetries of this Lagrangian is $so(4)\oplus_s\mathbb{R}^4$ and

$$A_1 = \frac{\lambda r^3}{12}\left[\sin^2\chi\cos\chi\sin^2\theta\cos\phi, \sin\chi\sin\theta\cos\theta\cos\phi, -\sin\chi\sin\phi\right],$$
$$A_2 = -\frac{\lambda r^3}{12}\left[\sin^2\chi\cos\chi\sin^2\theta\sin\phi, \sin\chi\sin\theta\cos\theta\sin\phi, \sin\chi\cos\phi\right], \tag{3.10}$$
$$A_3 = \frac{\lambda r^3}{12}\left[-\sin^2\chi\cos\chi\sin\theta\cos\theta, \sin\chi\sin^2\theta, 0\right], \qquad A_4 = \frac{\lambda r^3}{12}\left[-\sin^3\chi\sin\theta, 0, 0\right],$$

are the vector gauge functions corresponding to the translations ($\mathbb{R}^4$).

3.3. Four-Area Minimization

Extending to the 4-area enclosing a constant 5-volume the Lagrangian is

$$L = \left(r^8\sin^6\psi\sin^4\chi\sin^2\theta + r^6 r_{,\psi}^2\sin^6\psi\sin^4\chi\sin^2\theta + r^6 r_{,\chi}^2\sin^4\psi\sin^4\chi\sin^2\theta\right.$$
$$\left.+r^6 r_{,\theta}^2\sin\psi^4\sin^2\chi\sin^2\theta + r^6 r_{,\phi}\sin^4\psi\sin^2\chi\right)^{1/2} + \lambda\frac{1}{5}r^5\sin^3\psi\sin^2\chi\sin\theta. \tag{3.11}$$

The Lie algebra of the Noether symmetries for this Lagrangian is $so(5)\oplus_s\mathbb{R}^5$ and

$$\begin{gathered}
A_1=\frac{\lambda r^4}{20}\Big[\sin^3\psi\cos\psi\sin^3\chi\sin^2\theta\cos\phi,\cos\chi\cos\phi\sin^2\chi\sin^2\psi\sin^2\theta,\\
\sin^2\psi\sin\chi\sin\theta\cos\theta\cos\phi,-\sin\chi\sin\phi\sin^2\psi\Big],\\
A_2=-\frac{\lambda r^4}{20}\Big[\sin^3\chi\sin^3\psi\cos\psi\sin\phi\sin^2\theta,\sin\phi\cos\chi\sin^2\psi\sin^2\chi\sin^2\theta,\\
\sin^2\psi\sin\chi\sin\theta\cos\theta\sin\phi,\sin\chi\cos\phi\sin^2\psi\Big],\\
A_3=\frac{\lambda r^4}{20}\Big[-\sin^3\chi\sin^3\psi\cos\theta\sin\theta\cos\psi,-\cos\theta\sin\theta\cos\chi\sin^2\chi\sin^2\psi,\sin^2\psi\sin\chi\sin^2\theta,0\Big],\\
A_4=\frac{\lambda r^4}{20}\Big[-\sin^3\psi\cos\psi\cos\chi\sin^2\chi\sin\theta,\sin^3\chi\sin\theta\sin^2\psi,0,0\Big],\\
A_5=\frac{\lambda r^4}{20}\Big[-\sin^4\psi\sin^2\chi\sin\theta,0,0,0\Big],
\end{gathered}\tag{3.12}$$

are the vector gauge functions corresponding to the translations ($\mathbb{R}^5$).

We can now prove the results generalized to $(m-1)$-area minimization for a constant m-volume in a flat space by using a method of reduction and induction as done earlier for the connection between geometry and Lie symmetries [6].

Theorem 3.1. *The Lagrangian for minimizing the $(m-1)$-area enclosing a constant m-volume in a Euclidian space has a Lie algebra of Noether symmetries identical with the Lie algebra of isometries of the Euclidean space, $so(m)\oplus_s\mathbb{R}^{(m)}$, with the vector gauge functions corresponding to the translations.*

Proof. The Lie algebra of Noether symmetries of the Lagrangian that minimizes the 2-area enclosing a 3-volume in Euclidean space, 3-area enclosing a 4-volume, and the 4-area enclosing a 5-volume in Euclidean space are $so(3)\oplus_s\mathbb{R}^3$, $so(4)\oplus_s\mathbb{R}^4$, and $so(5)\oplus_s\mathbb{R}^5$, respectively. Now, suppose that the Lie algebra of Noether symmetries of the Lagrangian that minimizes an $(n-1)$-area enclosing a constant n-volume in Euclidean space is $so(n)\oplus_s\mathbb{R}^n$. The Lagrangian for minimizing the n-area enclosing a constant $(n+1)$-volume in Euclidean space contains a subset of Noether symmetries identical to the isometries of S^n, that is, $so(n+1)$. In the Euclidean space, S^n minimizes the n-area enclosing a constant $(n+1)$-volume. For the full set of Lie algebra, first reduce the n-area to an $(n-1)$-area and $(n+1)$-volume to an n-volume. The Lagrangian minimizing the n-area enclosing a constant $(n+1)$-volume reduces to the Lagrangian which minimizes the $(n-1)$-area enclosing a constant n-volume in the Euclidean space. The corresponding Lie algebra is $so(n)\oplus_s\mathbb{R}^n$ (the Lagrangian which minimizes the 4-area enclosing 5-volume can be transformed to the Lagrangian which minimizes 3-area enclosing 4-volume). Now, working in reverse, from the Lagrangian minimizing an $(n-1)$-area enclosing a constant n-volume to the Lagrangian minimizing the n-area enclosing a constant $(n+1)$-volume, it takes n more generators of rotation and one generator of translation from the previous one, that is, $(n+1)$ more generators. Thus the Lie algebra of the Noether symmetries of the Lagrangian which minimizes $(m-1)$-area enclosing a constant m-volume in the Euclidean space is identical to the Lie algebra of isometries of the Euclidean space, that is, $so(m)\oplus_s\mathbb{R}^{(m)}$. $\square$

4. Symmetries for Curved Spaces

4.1. Two-Area Minimization

The metric for a three-dimensional curved space is

$$ds^2 = d\chi^2 + \sinh^2\chi d\theta^2 + \sinh^2\chi \sin^2\theta d\phi^2. \tag{4.1}$$

Using the variational principle (2.8) for minimizing two area, we obtain the Lagrangian

$$L = \Sigma_1 + \lambda\frac{1}{2}(\sinh\chi\cosh\chi - \chi)\sin\theta, \tag{4.2}$$

where

$$\Sigma_1 = \left(\sinh^4\chi\sin^2\theta + \chi_{,\theta}^2\sinh^2\chi\sin^2\theta + \chi_{,\phi}^2\sinh^2\chi\right)^{1/2}. \tag{4.3}$$

The Lie algebra of the Noether symmetries of the Lagrangian is $so(3)$. Notice that there is no translational symmetry arising here.

4.2. Three-Area Minimization

The metric for four-dimensional curved space is

$$ds^2 = d\psi^2 + \sinh^2\psi d\chi^2 + \sinh^2\psi\sin^2\chi d\theta^2 + \sinh^2\psi\sin^2\chi\sin^2\theta d\phi^2. \tag{4.4}$$

The Lagrangian for minimizing three-area is

$$L = \Sigma_2 + \lambda\frac{1}{3}\left(\sinh^2\psi\cosh\psi - 2\cosh\psi\right)\sin^2\chi\sin\theta, \tag{4.5}$$

where

$$\Sigma_2 = \left(\sinh^6\psi\sin^4\chi\sin^2\theta + \sinh^4\psi\psi_{,\chi}^2\sin^4\chi\sin^2\theta + \sinh^4\psi\psi_{,\theta}^2\sin^2\chi\sin^2\theta + \sinh^4\psi\psi_{,\phi}^2\sin^2\chi\right)^{1/2}. \tag{4.6}$$

The Lie algebra of the Noether symmetries of the Lagrangian is $so(4)$, there is no translation in this case.

For spaces of constant nonzero curvature we present the following theorem.

Theorem 4.1. *The Lie algebra of Noether symmetries for the Lagrangian for minimizing the $(m-1)$-area keeping a constant m-volume in a space of nonzero constant curvature is $so(m)$.*

Proof. The proof of this theorem can be provided by arguments similar to those in Theorem 3.1. □

These two theorems provide the Noether symmetries of the area-minimizing Lagrangian for maximally symmetric spaces (constant curvature and zero curvature). Notice that when we go to spaces of constant curvature from spaces of zero curvature we lose m symmetries of the area-minimizing Lagrangian. In the case of zero curvature m symmetries (translational symmetries) come out only with particular vector gauge functions, while the remaining symmetries (rotational symmetries) have a zero gauge function. In the case of nonzero curvature there is no translational symmetry and we have only rotational symmetries corresponding to a zero gauge function.

5. Symmetries for Spaces Having Flat Section

5.1. Three-Area Minimization in One-Dimensional Flat and Three-Dimensional Curved Space

The metric for a four-dimensional space having one-dimensional flat section is

$$ds^2 = d\psi^2 + d\chi^2 + \sinh^2\chi d\theta^2 + \sinh^2\chi \sin^2\theta d\phi^2. \tag{5.1}$$

Following the same procedure, the three-area-minimizing Lagrangian is

$$L = \Sigma_3 + \lambda\, \psi \sinh^2\chi \sin\theta, \tag{5.2}$$

where

$$\Sigma_3 = \left(\sinh^4\chi \sin^2\theta + \psi_{,\chi}^2 \sinh^4\chi \sin^2\theta + \psi_{,\theta}^2 \sinh^2\chi \sin^2\theta + \psi_{,\phi}^2 \sinh^2\chi\right)^{1/2}. \tag{5.3}$$

The Lie algebra of the Noether symmetries of this Lagrangian is $so(4) \oplus \mathbb{R}^1$ and $\mathbb{R}^1$ corresponds to the vector gauge function, $A = \lambda(0, 0, \phi \sinh^2\chi \sin\theta)$.

5.2. Four-Area Minimization in Two-Dimensional Flat and Three-Dimensional Curved Space

The metric for a five-dimensional flat space having two-dimensional flat section is

$$ds^2 = dr^2 + r^2 d\chi^2 + d\psi^2 + \sinh^2\psi d\theta^2 + \sinh^2\psi \sin^2\theta d\phi^2. \tag{5.4}$$

Thus the Lagrangian for four-area minimization is

$$L = \left(r^2\sinh^4\psi\sin^2\theta + r_{,\chi}^2\sinh^4\psi\sin^2\theta + r^2r_{,\psi}^2\sinh^4\psi\sin^2\theta + r^2r_{,\theta}^2\sinh^2\psi\sin^2\theta + r^2r_{,\phi}^2\sinh^2\psi\right)^{1/2} + \lambda\frac{1}{2}r^2\sinh^2\psi\sin\theta. \tag{5.5}$$

The Lie algebra of Noether symmetries for this Lagrangian is $so(4)\oplus(so(2)\oplus_s\mathbb{R}^2)$, where $\mathbb{R}^2$ corresponds to the vector gauge function

$$A_1 = \frac{\lambda r}{2}\left[-\sin\theta\cos\chi\sinh^2\psi, 0, 0, 0\right], \quad A_2 = \frac{\lambda r}{2}\left[-\sin\theta\sin\chi\sinh^2\psi, 0, 0, 0\right]. \tag{5.6}$$

6. Symmetries for the Less Symmetric Spaces

The metric for a spheroid which is a less symmetric surface of positive curvature is

$$ds^2 = \left(a^2\cos^2\theta + b^2\sin^2\theta\right)d\theta^2 + a^2\sin^2\theta d\phi^2, \tag{6.1}$$

and the Lagrangian is

$$L = \frac{\left(a^2 - a^2\theta_{,\phi}^2 + b^2\theta_{,\phi}^2\right)\sin\theta\cos\theta}{\left(a^2\cos^2\theta\theta_{,\phi}^2 + b^2\sin^2\theta\theta_{,\phi}^2 + a^2\sin^2\theta\right)^{1/2}} + \lambda a\sin\theta\left(a^2\cos^2\theta + b^2\sin^2\theta\right)^{1/2}. \tag{6.2}$$

This Lagrangian has only one symmetry, that is, $\partial/\partial\phi$.

The metric for the ellipsoid is

$$ds^2 = \left(a^2\cos^2\theta\cos^2\phi + b^2\cos^2\theta\sin^2\phi + c^2\sin^2\theta\right)d\theta^2 + 2\left(b^2 - a^2\right)\sin\theta\cos\theta\sin\phi\cos\phi d\theta d\phi + \left(a^2\sin^2\theta\sin^2\phi + b^2\sin^2\theta\cos^2\phi\right)d\phi^2, \tag{6.3}$$

and the Lagrangian is

$$L = \frac{1}{\Sigma_4}\left[\sin\theta\cos\theta\left(c^2 - a^2\cos^2\phi - b^2\sin^2\phi\right)\theta_{,\phi}^2 + \sin\phi\cos\phi\left(\cos^2\theta - \sin^2\theta\right)\left(b^2 - a^2\right)\theta_{,\phi} + \sin\theta\cos\theta\left(b^2\cos^2\phi + a^2\sin^2\phi\right)\right] + \lambda\left(a^2b^2\sin^2\theta\cos^2\theta + a^2c^2\sin^4\theta\sin^2\phi + b^2c^2\sin^4\theta\cos^2\phi\right)^{1/2}, \tag{6.4}$$

where

$$\Sigma_4 = \left(a^2\left(\cos\theta\cos\phi\theta_{,\phi} - \sin\theta\sin\phi\right)^2 + b^2\left(\cos\theta\sin\phi\theta_{,\phi} + \sin\theta\cos\phi\right)^2 + c^2\theta_{,\phi}^2\sin^2\theta\right)^{1/2}. \tag{6.5}$$

This Lagrangian admits no symmetry.

We can now generalize the results for spaces having sections of different constant curvatures using the geometric method as done in [13].

Theorem 6.1. *The Lie algebra of the Noether symmetries for the Lagrangian which minimizes an $(n-1)$-area enclosing a constant n-volume, in a space which has one section of constant curvature of dimension n_1, another of n_2, and so forth up to n_k and a flat section of dimension m and $n \geq \sum_{j=1}^{k} n_j + m$ (as some of the sections may have no symmetry) is $\oplus_{j=1}^{k} so(n_j+1) \oplus (so(m)\oplus_s \mathbb{R}^m)$.*

Proof. First consider a manifold N of dimension n containing a maximal m-dimensional flat section M such that $N = M \oplus M^{\perp}$. Now, the orthogonal subspace $M^{\perp}$ has no flat section but can be further broken into sections of constant curvature of dimension $n_1, n_2, \ldots,$ up to n_k and possibly a remnant section with no symmetry. For this manifold we have an n-volume in the space having an m-dimensional flat section and one section of constant curvature of dimension n_1, another of n_2, and so forth up to n_k. We minimize the $(n-1)$-area in a subspace having an $(m-1)$-dimensional flat section and the sections of constant curvature remaining unchanged. The Lie algebra of Noether symmetries of the $(m-1)$-area-minimizing Lagrangian in flat space is $so(m) \oplus_s \mathbb{R}^m$. In the manifold $M^{\perp}$, each section retains its Lie algebra of Noether symmetries. Thus, the full Lie algebra of Noether symmetries, in this case, is the direct sum of all these Lie algebras, that is, $A = \oplus_{j=1}^{k} so(n_j+1) \oplus (so(m) \oplus_s \mathbb{R}^m)$.

If instead there is no reduction of dimension of the flat section, but one constant curvature section reduces by one-dimension, say $n_j \to n_j - 1$ the Lie algebra of the other sections remains unchanged while that of the reduced section now becomes $so(n_j)$. Now consider the case that there is only a one-dimensional flat section, that is, $m = 1$, and $(n-1)$-dimensional section of constant curvature. We have an n-volume in a space having a one-dimensional flat section and an $(n-1)$-dimensional section of constant curvature. We minimize the $(n-1)$-area in a subspace of constant curvature keeping a constant n-volume. Then, the Lie algebra of Noether symmetries is $so(n) \oplus (so(1)\oplus_s \mathbb{R}^1)$, that is, $so(n) \oplus \mathbb{R}^1$ (as $so(1)$ is the identity).

By increasing the dimension of the flat section by one, as in Theorem 3.1, the algebra for it becomes $so(m+1)\oplus_s \mathbb{R}^{m+1}$. Similarly, by increasing the dimension of one constant curvature section by 1, $n_j \to n_{j+1}$, the Lie algebra of that section becomes $so(n_j+2)$ while the other sections retain their Lie algebras.

Thus, reduction and induction show that the formula continues to hold. This completes the proof. □

7. Summary and Discussion

In this paper we have dealt with the Noether symmetries of the $(n-1)$-area-minimizing Lagrangian keeping a constant n-volume for the maximally and nonmaximally symmetric spaces. For spaces of maximal symmetry, the Lie algebra of the Noether symmetries is $so(n)\oplus_s \mathbb{R}^n$ in an n-dimensional flat space and $so(n)$ in an n-dimensional space of constant curvature. For an n-dimensional space of constant curvature, the area-minimizing Lagrangian has $(n(n-1))/2$ rotational symmetries with a zero gauge function and for the zero curvature there are n translational symmetries with specific vector gauge functions, along with $(n(n-1))/2$ rotational symmetries with zero gauge function.

The second theorem provides the Noether symmetries for the area-minimizing Lagrangian in nonmaximally symmetric spaces. In this case, we have a space consisting of sections of different constant curvatures, one section of zero curvature, and possibly a section with no symmetry. The Lie algebra of Noether symmetries is then the direct sum of the Lie algebras of Noether symmetries of each section. If the space has a flat section of only one dimension the Lie algebra becomes $\oplus_{j=1}^{k} so(n_j+1)\oplus\mathbb{R}^1$.

Appendix

A. For 2-Area

The EL equation corresponding to the Lagrangian (3.5) is

$$\begin{aligned}
&\left(2r^5+3r^3r_\theta^2-r^4r_{\theta\theta}\right)\sin^3\theta-r^2r_\theta\cos\theta\left(r^2+{r_\theta}^2\right)\sin^2\theta\\
&-r^2\left(r_{\theta\theta}{r_\phi}^2-2r_{\theta\phi}r_\theta r_\phi-3r{r_\phi}^2+r_{\phi\phi}{r_\theta}^2+r^2r_{\phi\phi}\right)\sin\theta\\
&-2r^2r_\theta\cos\theta{r_\phi}^2+\lambda\left(r^4\sin^2\theta+r^2\sin\theta^2{r_\theta}^2+r^2{r_\phi}^2\right)^{3/2}=0,
\end{aligned}\tag{A.1}$$

and the conserved quantities for the Noether symmetries $\mathbf{X_1}$, $\mathbf{X_2}$, $\mathbf{X_3}$, $\mathbf{X_4}$, $\mathbf{X_5}$, $\mathbf{X_6}$ are

$$\begin{aligned}
\mathbf{I_1}=\frac{r^2}{\Sigma}\Bigg[&\frac{1}{3}r\lambda\sin\phi\sin\theta\Sigma+\left(r^2\sin\phi-r_\theta r_\phi\cot\theta\cos\theta\right)\sin^2\theta+r_\phi^2\sin\phi,\\
&\frac{1}{3}r\lambda\cos\theta\cos\phi\Sigma+\cos\phi\left(r^2+r_\theta^2\right)\cos\theta\sin\theta-r_\theta r_\phi\sin\phi\Bigg],\\
\mathbf{I_2}=\frac{r^2}{\Sigma}\Bigg[&\frac{1}{3}r\lambda\cos\phi\sin\theta\Sigma+\left(r^2\cos\phi+r_\theta r_\phi\cot\theta\sin\phi\right)\sin^2\theta+r_\phi^2\cos\phi,\\
&-\frac{1}{3}r\lambda\cos\theta\sin\phi\Sigma-\sin\phi\left(r^2+r_\theta^2\right)\cos\theta\sin\theta-r_\theta r_\phi\cos\phi\Bigg],\\
\mathbf{I_3}=\frac{r^2}{\Sigma}\Bigg[&-r_\theta r_\phi\sin^2\theta,\frac{1}{3}r\lambda\sin\theta\Sigma+\left(r^2+r_\theta^2\right)\sin^2\theta\Bigg],\\
\mathbf{I_4}=\frac{r}{\Sigma}\Bigg[&\frac{1}{2}\lambda r\cos\theta\sin\theta\sin\phi\Sigma+rr_\theta\sin^3\theta\sin\phi\cos\theta\sin\phi\left(2r_\theta^2+r^2\right)\sin^2\theta-r_\theta r_\phi\sin\theta\cos\phi\\
&+r_\phi^2\cos\theta\sin\phi,\frac{1}{2}r\lambda\cos\phi\Sigma+\left(\left(r^2+r_\theta^2\right)\cos\phi+r_\theta r_\phi\sin\phi\right)\sin\theta-r_\theta r_\phi\cos\theta\sin\phi\Bigg],
\end{aligned}$$

$$\mathbf{I}_5 = \frac{r}{\Sigma}\Big[r^2\cos\theta\cos\phi\sin^2\theta + r_\phi^2\cos\theta\cos\phi + \frac{1}{2}r\lambda\cos\theta\sin\theta\cos\phi\Sigma + rr_\theta\sin^3\theta\cos\phi$$
$$+r_\theta r_\phi\sin\theta\sin\phi, -\frac{1}{2}r\lambda\sin\phi\Sigma - \left(\left(r^2+r_\theta^2\right)\sin\phi - rr_\phi\cos\phi\right)\sin\theta - r_\theta r_\phi\cos\theta\cos\phi\Big],$$
$$\mathbf{I}_6 = \frac{r}{\Sigma}\left[r^2\sin^3\theta + r_\phi^2\sin\theta + \frac{1}{2}r\lambda\sin^2\theta\Sigma - rr_\theta\sin^2\theta\cos\theta, -rr_\phi\cos\theta - r_\theta r_\phi\sin\theta\right], \tag{A.2}$$

where

$$\Sigma = \left(r^4\sin^2\theta + r^2r_\theta^2\sin^2\theta + r^2r_\phi^2\right)^{1/2}. \tag{A.3}$$

B. For 3-Area

The metric for a 4-dimensional flat space in hyperspherical coordinates is

$$ds^2 = dr^2 + r^2d\chi^2 + r^2\sin^2\chi d\theta^2 + r^2\sin^2\chi\sin^2\theta d\phi^2. \tag{B.1}$$

Let the enclosing surface be $r = r(\chi,\theta,\phi)$. The 3-area is

$$A(S) = \int\left(r^6\sin^4\chi\sin^2\theta + r^4r_{,\chi}^2\sin^4\chi\sin^2\theta + r^4r_{,\theta}^2\sin^2\chi\sin^2\theta + r^4r_{,\phi}^2\sin^2\chi\right)^{1/2}d\chi d\theta d\phi. \tag{B.2}$$

Then the variational principle (2.8) becomes

$$\delta\int\left[\Sigma + \lambda\frac{1}{4}r^4\sin^2\chi\sin\theta\right]d\chi d\theta d\phi = 0, \tag{B.3}$$

where

$$\Sigma = \left(r^6\sin^4\chi\sin^2\theta + r^4r_{,\chi}^2\sin^4\chi\sin^2\theta + r^4r_{,\theta}^2\sin^2\chi\sin^2\theta + r^4r_{,\phi}^2\sin^2\chi\right)^{1/2}. \tag{B.4}$$

Thus, the Lagrangian is

$$L = \Sigma + \lambda\frac{1}{4}r^4\sin^2\chi\,\sin\theta. \tag{B.5}$$

C. For 4-Area

The metric for a 5-dimensional flat space in hyperspherical coordinates is

$$ds^2 = dr^2 + r^2d\psi^2 + r^2\sin^2\psi d\chi^2 + r^2\sin^2\psi\sin^2\chi d\theta^2 + r^2\sin^2\psi\sin^2\chi\sin^2\theta d\phi^2. \tag{C.1}$$

Let the enclosing surface be $r = r(\psi, \chi, \theta, \phi)$. The 4-area is

$$A(S) = \int \Big(r^8 \sin^6\psi \sin^4\chi \sin^2\theta + r^6 r_{,\psi}^2 \sin^6\psi \sin^4\chi \sin^2\theta + r^6 r_{,\chi}^2 \sin^4\psi \sin^4\chi \sin^2\theta + r^6 r_{,\theta}^2 \sin^4\psi \sin^2\chi \sin^2\theta + r^6 r_{,\phi}^2 \sin^4\psi \sin^2\chi \Big)^{1/2} d\psi\, d\chi\, d\theta\, d\phi. \tag{C.2}$$

Then the variational principle (2.8) becomes

$$\delta \int \left[\Sigma + \lambda \frac{1}{5} r^5 \sin^3\psi \sin^2\chi \sin\theta \right] d\psi\, d\chi\, d\theta\, d\phi = 0, \tag{C.3}$$

where

$$\Sigma = \Big(r^8 \sin^6\psi \sin^4\chi \sin^2\theta + r^6 r_{,\psi}^2 \sin^6\psi \sin^4\chi \sin^2\theta + r^6 r_{,\chi}^2 \sin^4\psi \sin^4\chi \sin^2\theta + r^6 r_{,\theta}^2 \sin\psi^4 \sin^2\chi \sin^2\theta + r^6 r_{,\phi} \sin^4\psi \sin^2\chi \Big)^{1/2}. \tag{C.4}$$

Thus, the Lagrangian is

$$L = \Sigma + \lambda \frac{1}{5} r^5 \sin^3\psi \sin^2\chi \sin\theta. \tag{C.5}$$

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