

## Research Article

# Positive Solutions for Nonlinear Differential Equations with Periodic Boundary Condition

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We study the existence of positive solutions for second-order nonlinear differential equations with nonseparated boundary conditions. Our nonlinearity may be singular in its dependent variable. The proof of the main result relies on a nonlinear alternative principle of Leray-Schauder. Recent results in the literature are generalized and significantly improved.

## 1. Introduction

In this paper, we establish the positive periodic solutions for the following singular differential equation:

$$-[p(x)y']' + q(x)y = f(x, y, y'), \quad 0 \leq x \leq T, \quad (1.1)$$

and boundary conditions

$$y(0) = y(T), \quad y^{[1]}(0) = y^{[1]}(T), \quad (1.2)$$

where  $p, q \in C(\mathbb{R}/T\mathbb{Z})$ , the nonlinearity  $f \in C((\mathbb{R}/T\mathbb{Z}) \times (0, \infty) \times \mathbb{R}, \mathbb{R})$ , and  $y^{[1]}(x) = p(x)y'(x)$  denotes the quasi-derivative of  $y(x)$ . We call boundary conditions (1.2) the periodic boundary conditions which are important representatives of nonseparated boundary

conditions. In particular, the nonlinearity may have a repulsive singularity at  $y = 0$ , which means that

$$\lim_{y \rightarrow 0^+} f(x, y, z) = +\infty, \quad \text{uniformly in } (x, z) \in \mathbb{R}^2. \quad (1.3)$$

Also  $f$  may take on negative values. Electrostatic or gravitational forces are the most important examples of singular interactions.

During the last few decades, the study of the existence of periodic solutions for singular differential equations have deserved the attention of many researchers [1–8]. Some classical tools have been used to study singular differential equations in the literature, including the degree theory [7–9], the method of upper and lower solutions [5, 10], Schauder’s fixed point theorem [2, 11, 12], some fixed point theorems in cones for completely continuous operators [13–15], and a nonlinear Leray-Schauder alternative principle [16–18].

However, the singular differential equation (1.1), in which the nonlinearity is dependent on the derivative and does not require  $f$  to be nonnegative, has not attracted much attention in the literature. There are not so many existence results for (1.1) even when the nonlinearity is independent of the derivative. In this paper, we try to fill this gap and establish the existence of positive  $T$ -periodic solutions of (1.1); proof of the existence of positive solutions is based on an application of a nonlinear alternative of Leray-Schauder, which has been used by many authors [16–18].

The rest of this paper is organized as follows. In Section 2, some preliminary results will be given, including a famous nonlinear alternative of Leray-Schauder type. In Section 3, we will state and prove the main results.

## 2. Preliminaries

Let us denote  $u(t)$  and  $v(t)$  by the solutions of the following homogeneous equations:

$$-[p(x)y']' + q(x)y = 0, \quad 0 \leq x \leq T, \quad (2.1)$$

satisfying the initial conditions

$$u(0) = 1, \quad u^{[1]}(0) = 0, \quad v(0) = 0, \quad v^{[1]}(0) = 1, \quad (2.2)$$

and set

$$D = u(T) + v^{[1]}(T) - 2. \quad (2.3)$$

Throughout this paper, we assume that (1.1) satisfies the following condition (2.4):

$$p(x) > 0, \quad q(x) > 0, \quad \int_0^T \frac{1}{p(x)} dx < \infty, \quad \int_0^T q(x) dx < \infty. \quad (2.4)$$

**Lemma 2.1** (see [19]). For the solution  $y(x)$  of the boundary value problem

$$\begin{aligned} -[p(x)y']' + q(x)y &= h(x), & 0 \leq x \leq T, \\ y(0) &= y(T), & y^{[1]}(0) = y^{[1]}(T), \end{aligned} \quad (2.5)$$

the formula

$$y(x) = \int_0^T G(x,s)h(s)ds, \quad x \in [0, T], \quad (2.6)$$

holds, where

$$\begin{aligned} G(x,s) &= \frac{v(T)}{D}u(x)u(s) - \frac{u^{[1]}(T)}{D}v(x)v(s) \\ &+ \begin{cases} \frac{v^{[1]}(T) - 1}{D}u(x)v(s) - \frac{u(T) - 1}{D}u(s)v(x), & 0 \leq s \leq x \leq T, \\ \frac{v^{[1]}(T) - 1}{D}u(s)v(x) - \frac{u(T) - 1}{D}u(x)v(s), & 0 \leq x \leq s \leq T, \end{cases} \end{aligned} \quad (2.7)$$

is the Green's function, and the number  $D$  is defined by (2.3).

**Lemma 2.2** (see [19]). Under condition (2.4), the Green's function  $G(x,s)$  of the boundary value problem (2.5) is positive, that is,  $G(x,s) > 0$ , for  $x, s \in [0, T]$ .

One denotes

$$A = \min_{0 \leq s, x \leq T} G(x,s), \quad B = \max_{0 \leq s, x \leq T} G(x,s), \quad \sigma = \frac{A}{B}. \quad (2.8)$$

Thus  $B > A > 0$  and  $0 < \sigma < 1$ .

**Remark 2.3.** If  $p(x) = 1$ ,  $q(x) = m^2 > 0$ , then the Green's function  $G(x,s)$  of the boundary value problem (2.5) has the form

$$G(x,s) = \begin{cases} \frac{e^{m(x-s)} + e^{m(T-x+s)}}{2m(e^{mT} - 1)}, & 0 \leq s \leq x \leq T, \\ \frac{e^{m(s-x)} + e^{m(T+x-s)}}{2m(e^{mT} - 1)}, & 0 \leq x \leq s \leq T. \end{cases} \quad (2.9)$$

It is obvious that  $G(x,s) > 0$  for  $0 \leq s, x \leq T$ , and a direct calculation shows that

$$A = \frac{e^{mT/2}}{m(e^{mT} - 1)}, \quad B = \frac{1 + e^{mT}}{2m(e^{mT} - 1)}, \quad \sigma = \frac{2e^{mT/2}}{1 + e^{mT}} < 1. \quad (2.10)$$

Let  $X = \mathbb{C}[0, T]$ , and we suppose that  $F : [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow [0, \infty)$  is a continuous function. Define an operator:

$$(Ty)(x) = \int_0^T G(x, s)F(s, y(s), z(s))ds \quad (2.11)$$

for  $y \in X$  and  $x \in [0, T]$ . It is easy to prove that  $T$  is continuous and completely continuous.

### 3. Main Results

In this section, we state and prove the new existence results for (1.1). In order to prove our main results, the following nonlinear alternative of Leray-Schauder is needed, which can be found in [20]. Let us define the function  $\omega(x) = \int_0^T G(x, s)ds$ . The usual  $L^1$ -norm over  $(0, T)$  is denoted by  $\|\cdot\|_1$ , and the supremum norm of  $\mathbb{C}[0, T]$  is denoted by  $\|\cdot\|$ .

**Lemma 3.1.** *Assume  $\Omega$  is a relatively compact subset of a convex set  $E$  in a normed space  $X$ . Let  $T : \overline{\Omega} \rightarrow E$  be a compact map with  $0 \in \Omega$ . Then one of the following two conclusions holds:*

- (i)  $T$  has at least one fixed point in  $\overline{\Omega}$ ;
- (ii) there exist  $u \in \partial\Omega$  and  $0 < \lambda < 1$  such that  $u = \lambda Tu$ .

Now we present our main existence result of positive solution to problem (1.1).

**Theorem 3.2.** *Suppose that (1.1) satisfies (2.4). Furthermore, assume that there exists a constant  $r > 0$  such that*

- (H<sub>1</sub>) *there exists a constant  $M > 0$  such that  $F(x, y, z) = f(x, y, z) + M \geq 0$  for all  $(x, y, z) \in [0, T] \times (0, r) \times \mathbb{R}$ ;*
- (H<sub>2</sub>) *there exist continuous, nonnegative functions  $g(y)$ ,  $h(y)$ , and  $\varrho(y)$  such that*

$$F(x, y, z) \leq (g(y) + h(y))\varrho(|z|), \quad \forall (x, y, z) \in [0, T] \times (0, r) \times \mathbb{R}, \quad (3.1)$$

*where  $g(y) > 0$  is nonincreasing,  $h(y)/g(y)$  is nondecreasing in  $(0, r]$  and  $\varrho(y)$  is nondecreasing in  $(0, \infty)$ ;*

- (H<sub>3</sub>) *there exist a nonincreasing positive continuous function  $g_0(y)$  on  $(0, \infty)$ , and a constant  $R_0 > 0$  such that  $F(x, y, z) \geq g_0(y)$  for  $(x, y, z) \in [0, T] \times (0, R_0] \times \infty$ , where  $g_0(y)$  satisfies  $\lim_{y \rightarrow 0^+} g_0(y) = +\infty$  and  $\lim_{y \rightarrow 0^+} \int_y^{R_0} g_0(u)du = +\infty$ ;*
- (H<sub>4</sub>) *the following inequalities hold:*

$$\sigma r > M\|\omega\|, \quad \frac{r}{g(\sigma r - M\|\omega\|)\{1 + h(r)/g(r)\}\varrho(L_1 r + L_2 T)} > \|\omega\|, \quad (3.2)$$

*where  $L_1 = 2\|q\|_1 / \min_{0 \leq x \leq T} p(x)$ ,  $L_2 = 2M / \min_{0 \leq x \leq T} p(x)$ .*

Then (1.1) has at least one positive  $T$ -periodic solution  $y$  with  $0 < \|y + M\omega\| \leq r$ .

*Proof.* Since  $(H_4)$  holds, let  $N_0 = \{n_0, n_0 + 1, \dots\}$ , and we can choose  $n_0 \in \{1, 2, \dots\}$  such that  $1/n_0 < \sigma r - M\|\omega\|$  and

$$\|\omega\|g(\sigma r - M\|\omega\|)\left\{1 + \frac{h(r)}{g(r)}\right\}Q(L_1r + L_2T) + \frac{1}{n_0} < r. \quad (3.3)$$

To show (1.1) has a positive solution, we should only show that

$$-[p(x)y']' + q(x)y = F(x, y(x) - M\omega(x), y'(x) - M\omega'(x)) \quad (3.4)$$

has a positive solution  $y$  satisfying (1.2). If it is right, then  $k(x) = y(x) - M\omega(x)$  is a solution of (1.1) since

$$\begin{aligned} -[p(x)k']' + q(x)k &= -[p(x)(y(x) - M\omega(x))]' + q(x)(y(x) - M\omega(x)) \\ &= F(x, y(x) - M\omega(x), y'(x) - M\omega'(x)) - M \\ &= f(x, y(x) - M\omega(x), y'(x) - M\omega'(x)) \\ &= f(x, k(x), k'(x)), \end{aligned} \quad (3.5)$$

where  $-[p(x)\omega']' + q(x)\omega(x) = 1$  is used.

Consider the family of equations:

$$-[p(x)y']' + q(x)y = \lambda F_n(x, y(x) - M\omega(x), y'(x) - M\omega'(x)) + \frac{q(x)}{n}, \quad (3.6)$$

where  $\lambda \in [0, 1]$ ,  $n \in N_0$ , and

$$F_n(x, y, z) = \begin{cases} F(x, y, z) & \text{if } y \geq \frac{1}{n}, \\ F\left(x, \frac{1}{n}, z\right) & \text{if } y \leq \frac{1}{n}. \end{cases} \quad (3.7)$$

Problem (3.6)–(1.2) is equivalent to the following fixed point of the operator equation:

$$y = T_n y, \quad (3.8)$$

where  $T_n$  is a continuous and completely continuous operator defined by

$$T_n y(x) = \lambda \int_0^T G(x, s) F_n(s, y(s) - M\omega(s), y'(s) - M\omega'(s)) ds + \frac{1}{n}, \quad (3.9)$$

and we used the fact

$$\int_0^T G(x, s) q(s) ds \equiv 1 \quad (\text{see Lemma 2.1 with } h = q). \quad (3.10)$$

Now we show  $\|y\| \neq r$  for any fixed point  $y$  of (3.8). If not, assume that  $y$  is a fixed point of (3.8) such that  $\|y\| = r$ . Note that

$$\begin{aligned}
 y(x) - \frac{1}{n} &= \lambda \int_0^T G(x, s) F_n(s, y(s) - M\omega(s), y'(s) - M\omega'(s)) ds \\
 &\geq \lambda A \int_0^T F_n(s, y(s) - M\omega(s), y'(s) - M\omega'(s)) ds \\
 &= \sigma B \lambda \int_0^T F_n(s, y(s) - M\omega(s), y'(s) - M\omega'(s)) ds \\
 &\geq \sigma \max_{x \in [0, T]} \left\{ \lambda \int_0^T G(x, s) F_n(s, y(s) - M\omega(s), y'(s) - M\omega'(s)) ds \right\} \\
 &= \sigma \left\| y - \frac{1}{n} \right\|.
 \end{aligned} \tag{3.11}$$

So we have

$$y(x) \geq \sigma \left\| y - \frac{1}{n} \right\| + \frac{1}{n} \geq \sigma \left( \|y\| - \frac{1}{n} \right) + \frac{1}{n} \geq \sigma r, \quad \text{for } 0 \leq x \leq T. \tag{3.12}$$

In order to pass the solutions of the truncation equation (3.6) (with  $\lambda = 1$ ) to that of the original equation (3.4), we need the fact that  $\|y'\|$  is bounded. Now we show that

$$\|y'\| \leq L_1 r \tag{3.13}$$

for a solution  $y(x)$  of (3.6).

Integrating (3.6) from 0 to  $T$  (with  $\lambda = 1$ ), we obtain

$$\int_0^T q(x) y(x) dx = \int_0^T \left[ F_n(x, y(x) - M\omega(x), y'(x) - M\omega'(x)) + \frac{q(x)}{n} \right] dx. \tag{3.14}$$

Since  $y(0) = y(T)$ , there exists  $x_0 \in [0, T]$  such that  $y'(x_0) = 0$ ; therefore,

$$\begin{aligned}
 |p(x)y'(x)| &= \left| \int_{x_0}^x (p(s)y'(s))' ds \right| \\
 &= \left| \int_{x_0}^x \left[ q(s)y(s) - F_n(s, y(s) - M\omega(s), y'(s) - M\omega'(s)) - \frac{q(s)}{n} \right] ds \right| \\
 &\leq \int_0^T \left[ q(s)y(s) + F_n(s, y(s) - M\omega(s), y'(s) - M\omega'(s)) + \frac{q(s)}{n} \right] ds \\
 &= 2 \int_0^T q(s)y(s) ds \\
 &\leq 2r \int_0^T q(s) ds.
 \end{aligned} \tag{3.15}$$

So,

$$|y'(x)| \leq \frac{2r \int_0^T q(s) ds}{p(x)} \leq \frac{2r \|q\|_1}{\min_{0 \leq x \leq T} p(x)} := L_1 r. \quad (3.16)$$

Similarly we have

$$|\omega'(x)| \leq \frac{2T}{\min_{0 \leq x \leq T} p(x)}. \quad (3.17)$$

By (3.12), we obtain  $y(x) - M\omega(x) \geq \sigma r - M\omega(x) \geq \sigma r - M\|\omega\| > 1/n_0 \geq 1/n$ . Thus from condition (H<sub>2</sub>)

$$\begin{aligned} y(x) &= \lambda \int_0^T G(x, s) F_n(s, y(s) - M\omega(s), y'(s) - M\omega'(s)) ds + \frac{1}{n} \\ &= \lambda \int_0^T G(x, s) F(s, y(s) - M\omega(s), y'(s) - M\omega'(s)) ds + \frac{1}{n} \\ &\leq \int_0^T G(x, s) F(s, y(s) - M\omega(s), y'(s) - M\omega'(s)) ds + \frac{1}{n} \\ &\leq \int_0^T G(x, s) g(y(s) - M\omega(s)) \left\{ 1 + \frac{h(y(s) - M\omega(s))}{g(y(s) - M\omega(s))} \right\} \varrho(|y'(s)| + M|\omega'(s)|) ds + \frac{1}{n} \\ &\leq g(\sigma r - M\|\omega\|) \left\{ 1 + \frac{h(r)}{g(r)} \right\} \varrho(L_1 r + L_2 T) \int_0^T G(t, s) ds + \frac{1}{n} \\ &\leq g(\sigma r - M\|\omega\|) \left\{ 1 + \frac{h(r)}{g(r)} \right\} \varrho(L_1 r + L_2 T) \|\omega\| + \frac{1}{n_0}. \end{aligned} \quad (3.18)$$

Therefore,

$$r = \|y\| \leq g(\sigma r - M\|\omega\|) \left\{ 1 + \frac{h(r)}{g(r)} \right\} \varrho(L_1 r + L_2 T) \|\omega\| + \frac{1}{n_0}. \quad (3.19)$$

This is a contradiction, so  $\|y\| \neq r$ .

Using Lemma 3.1, we know that

$$y = T_n y \quad (3.20)$$

has a fixed point, denoted by  $y_n$ , that is, equation

$$-[p(x)y']' + q(x)y = F_n(x, y(x) - M\omega(x), y'(x) - M\omega'(x)) + \frac{q(x)}{n} \quad (3.21)$$

has a periodic solution  $y_n$  with  $\|y_n\| < r$ . Using similar procedure to that of the proof of (3.13), we can prove that

$$\|y_n'\| \leq L_1 r. \quad (3.22)$$

In the next lemma, we will show that  $y_n(x) - M\omega(x)$  have a uniform positive lower bound, that is, there exists a constant  $\delta > 0$ , independent of  $n \in N_0$ , such that

$$y_n(x) - M\omega(x) \geq \delta \quad (3.23)$$

for all  $n \in N_0$ .

The fact  $\|y_n\| < r$  and  $\|y'_n\| \leq L_1 r$  shows that  $\{y_n\}_{n \in N_0}$  is a bounded and equicontinuous family on  $[0, T]$ . Thus the Arzela–Ascoli Theorem guarantees that  $\{y_n\}_{n \in N_0}$  has a subsequence,  $\{y_{n_i}\}_{i \in \mathbb{N}}$  converging uniformly on  $[0, T]$  to a function  $y \in X$ .  $F$  is uniformly continuous since  $y_n$  satisfies  $\delta + M\omega(x) \leq y_n(x) \leq r$  for all  $x \in [0, T]$ . Moreover,  $y_{n_i}$  satisfies the integral equation

$$y_{n_i}(x) = \int_0^T G(x, s) F_n(s, y_{n_i}(s) - M\omega(s), y'_{n_i}(s) - M\omega'(s)) ds + \frac{1}{n_i}. \quad (3.24)$$

Letting  $i \rightarrow \infty$ , we arrive at

$$y(x) = \int_0^T G(x, s) F_n(s, y(s) - M\omega(s), y'(s) - M\omega'(s)) ds. \quad (3.25)$$

Therefore,  $y$  is a positive periodic solution of (1.1) and satisfies  $0 < \|y + M\omega\| \leq r$ .  $\square$

**Lemma 3.3.** *There exists a constant  $\delta > 0$  such that any solution  $y_n$  of (3.6) (with  $\lambda = 1$ ) satisfies (3.23) for all  $n$  large enough.*

*Proof.* By condition (H<sub>3</sub>), there exist  $R_1 \in (0, R_0)$  and a continuous function  $\tilde{g}_0$  such that

$$F(x, y, y') - q(x)y \geq \tilde{g}_0(y) \geq \max\{M, r\|q\|\} \quad (3.26)$$

for all  $(x, y) \in [0, T] \times (0, R_1]$ , where  $\tilde{g}_0$  satisfies condition also like in (H<sub>3</sub>).

Choose  $n_1 \in N_0$  such that  $1/n_1 \leq R_1$ , and let  $N_1 = \{n_1, n_1 + 1, \dots\}$ . For  $n \in N_1$ , let

$$\alpha_n = \min_{0 \leq x \leq T} [y_n(x) - M\omega(x)], \quad \beta_n = \max_{0 \leq x \leq T} [y_n(x) - M\omega(x)]. \quad (3.27)$$

We first show that  $\beta_n > R_1$  for all  $n \in N_1$ . If not, assume that  $\beta_n \leq R_1$  for some  $n \in N_1$ .

If  $1/n \leq y_n(x) - M\omega(x) \leq R_1$ , we obtain from (3.26)

$$\begin{aligned} F_n(x, y_n(x) - M\omega(x), y'_n(x) - M\omega'(x)) &= F(x, y_n(x) - M\omega(x), y'_n(x) - M\omega'(x)) \\ &\geq q(x)(y_n(x) - M\omega(x)) + \tilde{g}_0(y_n(x) - M\omega(x)) \\ &\geq \tilde{g}_0(y_n(x) - M\omega(x)) > r\|q\|, \end{aligned} \quad (3.28)$$

and, if  $y_n(x) - M\omega(x) \leq 1/n$ , we obtain

$$F_n(x, y_n(x) - M\omega(x), y'_n(x)) = F\left(x, \frac{1}{n}, y'_n(x)\right) \geq \frac{q(x)}{n} + \tilde{g}_0\left(\frac{1}{n}\right) \geq \tilde{g}_0\left(\frac{1}{n}\right) > r\|q\|. \quad (3.29)$$



So we have

$$F_n(x, y_n(x) - M\omega(x), y'_n(x) - M\omega'(x)) > r\|q\|, \quad \text{for } \beta_n \leq R_1. \quad (3.30)$$

Integrating (3.6) (with  $\lambda = 1$ ) from 0 to  $T$ , we deduce that

$$\begin{aligned} 0 &= \int_0^T \left( -[p(x) y'_n(x)]' + q(x)y_n(x) - F_n(x, y_n(x) - M\omega(x), y'_n(x) - M\omega'(x)) - \frac{q(x)}{n} \right) dx \\ &= \int_0^T q(x)(y_n(x)) dx - \frac{1}{n} \int_0^T q(x) dx - \int_0^T F_n(x, y_n(x) - M\omega(x), y'_n(x) - M\omega'(x)) dx \\ &< \int_0^T q(x)y_n(x) dx - r\|q\|T \leq 0. \end{aligned} \quad (3.31)$$

This is a contradiction. Thus  $\beta_n > R_1$ , and we have

$$\|y_n - M\omega\| > R_1, \quad \forall n \in N_1. \quad (3.32)$$

To prove (3.23), we first show

$$y_n(x) \geq M\omega(x) + \frac{1}{n}, \quad 0 \leq x \leq T \text{ for } n \in N_1. \quad (3.33)$$

Let  $N_1 = P \cup Q$ ; here  $\alpha_n \geq R_1$  if  $n \in P$ , and  $\alpha_n < R_1$  if  $n \in Q$ . If  $n \in P$ , it is easy to verify (3.33) is satisfied. We now show (3.33) holds if  $n \in Q$ . If not, suppose there exists  $n \in Q$  with

$$\alpha_n = \min_{0 \leq x \leq T} [y_n(x) - M\omega(x)] = y_n(c_n) - M\omega(c_n) < \frac{1}{n} \quad (3.34)$$

for some  $c_n \in [0, T]$ . As  $\alpha_n = y_n(c_n) - M\omega(c_n) < R_1$ , by  $\beta_n > R_1$ , there exists  $c_n \in [0, T]$  (without loss of generality, we assume  $a_n < c_n$ ) such that  $y_n(a_n) = M\omega(a_n) + R_1$  and  $y_n(x) \leq M\omega(x) + R_1$  for  $a_n \leq x \leq c_n$ .

From (3.26), we easily show that

$$F_n(x, y(x) - M\omega(x), y'(x) - M\omega'(x)) > q(x)(y_n(x) - M\omega(x)) + M \quad \text{for } x \in [a_n, c_n]. \quad (3.35)$$

Using (3.6) (with  $\lambda = 1$ ) for  $y_n(x)$ , we have, for  $x \in [a_n, c_n]$ ,

$$\begin{aligned}
[-p(x)(y_n'(x) - M\omega'(x))]' &= -[p(x)(y_n'(x))] + M[p(x)(\omega'(x))]' \\
&= -q(x)y_n(x) + F_n(x, y_n(x) - M\omega(x), y_n'(x) - M\omega'(x)) \\
&\quad + \frac{q(x)}{n} - M[1 - q(x)\omega(x)] \\
&> -q(x)y_n(x) + q(x)(y_n(x) - M\omega(x)) + M \\
&\quad + \frac{q(x)}{n} - M[1 - q(x)\omega(x)] \\
&= \frac{q(x)}{n} \geq 0.
\end{aligned} \tag{3.36}$$

As  $y_n'(c_n) - M\omega'(c_n) = 0$ ,  $p(x) > 0$ , so  $y_n'(x) - M\omega'(x) < 0$  for all  $x \in [a_n, c_n)$ , and the function  $v_n := y_n - M\omega$  is strictly decreasing on  $[a_n, c_n]$ . We use  $\eta_n$  to denote the inverse function of  $y_n$  restricted to  $[a_n, c_n]$ . Thus there exists  $b_n \in (a_n, c_n)$  such that  $y_n(b_n) - M\omega(b_n) = 1/n$  and

$$y_n(x) - M\omega(x) \leq \frac{1}{n} \quad \text{for } c_n \geq x \geq b_n, \quad \frac{1}{n} \leq y_n(x) - M\omega(x) \leq R_1 \quad \text{for } b_n \geq x \geq a_n. \tag{3.37}$$

By using the method of substitution, we obtain

$$\begin{aligned}
\int_{1/n}^{R_1} F(\eta_n(v), v, v') dv &= \int_{b_n}^{a_n} F(x, y_n(x) - M\omega(x), y_n'(x) - M\omega'(x))(y_n'(x) - M\omega'(x)) dx \\
&= \int_{b_n}^{a_n} F_n(x, y_n(x) - M\omega(x), y_n'(x) - M\omega'(x))(y_n'(x) - M\omega'(x)) dx \\
&= \int_{b_n}^{a_n} \left( -[p(x)(y_n'(x))] + q(x)y_n'(x) - \frac{q(x)}{n} \right) (y_n'(x) - M\omega'(x)) dx \\
&= \int_{b_n}^{a_n} \left( -[p(x)(y_n'(x))] \right)' (y_n'(x) - M\omega'(x)) dx \\
&\quad + \int_{b_n}^{a_n} \left( q(x)y_n(x) - \frac{q(x)}{n} \right) (y_n'(x) - M\omega'(x)) dx.
\end{aligned} \tag{3.38}$$

By the facts  $\|y_n\| < r$ ,  $\|y_n'\|$  and  $\|\omega'\|$  are bounded, one can easily obtain that the second term is bounded. The first term is

$$\begin{aligned}
p(b_n)[y_n'(b_n)]^2 - p(a_n)[y_n'(a_n)]^2 + Mp(a_n)y_n'(a_n)\omega'(a_n) - Mp(b_n)y_n'(b_n)\omega'(b_n) \\
+ \int_{b_n}^{a_n} p(x)y_n'(x)y_n''(x)dx - M \int_{b_n}^{a_n} p(x)y_n'(x)\omega''(x)dx,
\end{aligned} \tag{3.39}$$

which is also bounded. As a consequence, there exists  $L > 0$  such that

$$\int_{1/n}^{R_1} F(\eta_n(y), y, y') dy \leq L. \quad (3.40)$$

On the other hand, by (H<sub>3</sub>), we can choose  $n_2 \in N_1$  large enough such that

$$\int_{1/n}^{R_1} F(\eta_n(y), y, y') dy \geq \int_{1/n}^{R_1} g_0(y) dy > L \quad (3.41)$$

for all  $n \in N_2 = \{n_2, n_2 + 1, \dots\}$ . This is a contradiction. So (3.33) holds.

Finally, we will show that (3.23) is right in  $n \in Q$ . Noticing estimate (3.33) and employing the method of substitution, we obtain

$$\begin{aligned} \int_{\alpha_n}^{R_1} F(\eta_n(y), y, y') dy &= \int_{c_n}^{\alpha_n} F(x, y_n(x) - M\omega(x), y'_n(x) - M\omega'(x))(y'_n(x) - M\omega'(x)) dx \\ &= \int_{c_n}^{\alpha_n} F_n(x, y_n(x) - M\omega(x), y'_n(x) - M\omega'(x))(y'_n(x) - M\omega'(x)) dx \\ &= \int_{c_n}^{\alpha_n} \left( -[p(x)y'_n(x)]' + q(x)y_n(x) - \frac{q(x)}{n} \right) (y'_n(x) - M\omega'(x)) dx. \end{aligned} \quad (3.42)$$

Obviously, the right-hand side of the above equality is bounded. On the other hand, by (H<sub>3</sub>),

$$\int_{\alpha_n}^{R_1} F(\eta_n(y), y, y') dy \geq \int_{\alpha_n}^{R_1} g_0(y) dy \longrightarrow +\infty \quad (3.43)$$

if  $\alpha_n \rightarrow 0^+$ . Thus we know that  $\alpha_n \geq \delta$  for some constant  $\delta > 0$ ; the proof is completed.  $\square$

**Corollary 3.4.** *Let the nonlinearity in (1.1) be*

$$f(x, y, z) = (1 + |z|^\gamma) \left( y^{-\alpha} + \mu y^\beta + e(x) \right), \quad 0 \leq x \leq T, \quad (3.44)$$

where  $\alpha > 0$ ,  $\beta, \gamma \geq 0$ ,  $e(x) \in C[0, T]$ , and  $\mu > 0$  is a positive parameter,

- (i) if  $\beta + \gamma < 1$ , then (1.1) has at least one positive periodic solution for each  $\mu > 0$ ;
- (ii) if  $\beta + \gamma \geq 1$ , then (1.1) has at least one positive periodic solution for each  $0 < \mu < \mu^*$ , where  $\mu^*$  is some positive constant.

*Proof.* We will apply Theorem 3.2. Take

$$M = e_0 = \max_{0 \leq x \leq T} |e(x)|, \quad g(y) = y^{-\alpha}, \quad h(y) = \mu y^\beta + 2e_0, \quad \varrho(z) = 1 + |z|^\gamma. \quad (3.45)$$

Then conditions (H<sub>1</sub>)–(H<sub>3</sub>) are satisfied and the existence condition (H<sub>4</sub>) becomes

$$\mu < \frac{r(\sigma r - M\|\omega\|)^\alpha - \|\omega\|(1 + (L_1 r + L_2 T)^\gamma) - 2e_0\|\omega\|r^\alpha(1 + (L_1 r + L_2 T)^\gamma)}{r^{\alpha+\beta}\|\omega\|(1 + (L_1 r + L_2 T)^\gamma)} \quad (3.46)$$

for some  $r > 0$ . So (1.1) has at least one positive periodic solution for

$$0 < \mu < \mu^* := \sup_{r>0} \frac{r(\sigma r - M\|\omega\|)^\alpha - \|\omega\|(1 + (L_1 r + L_2 T)^\gamma) - 2e_0\|\omega\|r^\alpha(1 + (L_1 r + L_2 T)^\gamma)}{r^{\alpha+\beta}\|\omega\|(1 + (L_1 r + L_2 T)^\gamma)}. \quad (3.47)$$

Note that  $\mu^* = \infty$  if  $\beta + \gamma < 1$  and  $\mu^* < \infty$  if  $\beta + \gamma \geq 1$ . We have (i) and (ii).  $\square$

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