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Research Article

Extinction and Permanence of a General Predator-Prey System with Impulsive Perturbations

Xianning Liu[1] and Lansun Chen[2]

[1] *Key Laboratory of Eco-Environments in Three Gorges Reservoir Region (Ministry of Education), School of Mathematics and Statistics, Southwest University, Chongqing 400715, China*
[2] *Institute of Mathematics, Academy of Mathematics and System Sciences, Beijing 100080, China*

Correspondence should be addressed to Xianning Liu, liuxn@swu.edu.cn

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A general predator-prey system is studied in a scheme where there is periodic impulsive perturbations. This scheme has the potential to protect the predator from extinction but under some conditions may also serve to lead to extinction of the prey. Conditions for extinction and permanence are obtained via the comparison methods involving monotone theory of impulsive systems and multiple Liapunov functions, which establish explicit bounds on solutions. The existence of a positive periodic solution is also studied by the bifurcation theory. Application is given to a Lotka-Volterra predator-prey system with periodic impulsive immigration of the predator. It is shown that the results are quite different from the corresponding system without impulsive immigration, where extinction of the prey can never be achieved. The prey will be extinct or permanent independent of whether the system without impulsive effect immigration is permanent or not. The model and its results suggest an approach of pest control which proves more effective than the classical one.

1. Introduction

Systems of differential equations with impulses are found in almost every domain of applied sciences. They generally describe phenomena which are subject to short-time perturbations or instantaneous changes. That is why in recent years these systems have been the object of many investigations [1–7], in which an abundance of basic theories has been developed. Systematic accounts of the subject can be found in [1, 3]. Some impulsive equations have been recently introduced in population dynamics in relation to impulsive birth [8], chemotherapeutic treatment [9], and pulse vaccination [10] of disease and the impulses could also be due to invasion or stocking and harvesting of species [11, 12].

The Lotka-Volterra system is a fundamental one to model the population dynamics. It can describe the basic interactions between species such as cooperation, competition, and predator-prey. It can be extended in many ways: Wang and Chen [13] considered stage-structure for the predator; Xiao and Chen [14] introduced diseases for the prey; while Xu and Chen [15] focused on the functional responses and diffusions of the predator. With regard to impulsive effects Lakmeche and Arino [9] studied a two-dimensional competing Lotka-Volterra system with impulses arising from chemotherapeutic treatment where the stability of a trivial periodic solution was studied and conditions for the existence of a positive periodic solution bifurcating from the trivial one were established. The Lotka-Volterra predator-prey system

$$\begin{aligned} \dot{x}_1 &= x_1(r_1 - a_{11}x_1 - a_{12}x_2), \\ \dot{x}_2 &= x_2(-r_2 + a_{21}x_1) \end{aligned} \tag{1.1}$$

can be developed by introducing a constant periodic impulsive immigration for the predator. That is

$$\begin{gathered} \dot{x}_1 = x_1(r_1 - a_{11}x_1 - a_{12}x_2), \quad \dot{x}_2 = x_2(-r_2 + a_{21}x_1), \quad t \neq n\tau, \\ x_1(t^+) = x_1(t), \quad x_2(t^+) = x_2(t) + b, \quad t = n\tau, \\ x(0^+) = x_0 = (x_{01}, x_{02}), \end{gathered} \tag{1.2}$$

where $x_1(t)$, $x_2(t)$ are the densities of the prey and predator at time t, respectively, r_1 is the intrinsic growth rate of prey, r_2 is the death rate of predator, a_{11} is the rate of intraspecific competition or density dependence, a_{12} is the per capita rate of predation of the predator, a_{21} denotes the product of the per capita rate of predation and the rate of conversing prey into predator, τ is the period of the impulsive immigration effect. This immigration could be artificially planting of predator in order to protect it from extinction. It could also be short-time invasion of predator as a disaster for the prey. For example, this has often been seen in recent years that a large amount of locusts may invade into some areas and cause damages to other species in the northwestern China of Xinjiang province and Inner Mongolia.

Usually biological pest control requires the introduction of a predator decreasing the pest population to an acceptable level as referred in [16, 17] and the references cited therein. It provides only short-term results as after some time this kind of predator-prey system will reach its coexisting equilibrium no matter how large the initial density of the predator is. In this case, system (1.1) can serve as a model of pest control, which will be called classical approach in this paper. However, the dynamics of system (1.1) are very simple. Either there is a positive equilibrium ($r_1/a_{11} > r_2/a_{21}$), in which case it is global asymptotically stable. Or there is no positive equilibrium ($r_1/a_{11} \leq r_2/a_{21}$), in which case $x_2(t)$ goes extinct and $x_1(t)$ tends to r_1/a_{11}, the capacity of the prey. In each case, the prey can never become extinct. That is why the classical approach of this kind in pest control is not so effective. System (1.2) serves as a different approach of biological pest control, in which predator is released impulsively.

Besides this, Liu et al. [18] considered the following system which also includes impulsive chemical pest control of pesticide using into system (1.2):

$$\begin{gathered} \dot{x}_1 = x_1(r_1 - a_{11}x_1 - a_{12}x_2), \quad \dot{x}_2 = x_2(-r_2 + a_{21}x_1), \quad t \neq n\tau, \\ x_1(t^+) = (1 - p_1)x_1(t), \quad x_2(t^+) = (1 - p_2)x_2(t) + b, \quad t = n\tau, \\ x(0^+) = x_0 = (x_{01}, x_{02}), \end{gathered} \tag{1.3}$$

where $0 \leq p_i < 1$, $i = 1, 2$.

However, in order to check the effect of pest control, it is important to study the extinction and permanence of such kind of impulsive systems as (1.2) and (1.3). We will consider the following general impulsive predator-prey system, which includes (1.2) and (1.3) as special cases:

$$\begin{gathered} \dot{x}_1 = x_1 f_1(x_1, x_2), \quad \dot{x}_2 = x_2 f_2(x_1, x_2), \quad t \neq n\tau, \\ x_1(t^+) = I_1(x_1(t)), \quad x_2(t^+) = I_2(x_2(t)), \quad t = n\tau, \\ x(0^+) = x_0 = (x_{01}, x_{02}). \end{gathered} \tag{1.4}$$

Ballinger and Liu [19] established some conditions to guarantee permanence of a general impulsive system by the method of Liapunov function and applied their results to the impulsive Lotka-Volterra system; however, their conditions include the existence of a positive equilibrium of the corresponding system without impulses. In Liu et al. [18], Liu and Chen [20], and Zhang et al. [21], extinction and permanence of impulsive prey-predator systems with different functional responses were established via comparison. But their methods and results depended on solving a prey eradicated periodic solution explicitly and obtaining its global asymptotical attractivity directly which is impossible for general systems. We will study the permanence of system (1.4) through some techniques of comparison methods involving monotone theory of impulsive systems and multiple Liapunov functions, which establish explicit bounds on solutions. The existence and global attractivity of the prey eradicated periodic solution are ensured by monotone theory of impulsive systems and extinction and permanence are obtained by generalizing the comparison skills to study the properties of solutions near boundary. Compared to [18, 20, 21], the model (1.4) and results in this paper have the following advantages.

(i) Both the functional response and impulsive effect are in general functions which can be applied in many different settings.

(ii) Extinction and permanence results do not depend on solving the boundary system and obtaining a trivial periodic solution and explicitly.

Our permanence results also known as practical persistence are stronger than permanence. Motivated by the approach of Wang and Ma [22], Cao and Gard [23] introduced the idea and methods, which were developed further in the context of reaction-diffusion models by Cantrell and Cosner [24]. And a discussion of how the methods are applied to various sorts of ecological models, including some discrete models, was given by Cosner [25]. Applying our results to (1.2), we can see that system (1.2) may be permanent or have at least one species reaching extinction, independent of whether system (1.1) is permanent or not.

The organization of this paper is as follows. In the next section, we introduce notations and definitions which will be used in this paper and give some basic assumptions on system (1.4). In Section 3, we present extinction and permanence results of system (1.4) and study the existence of a positive periodic solution by means of bifurcation theory. In Section 4, we apply our results to system (1.2) and interpret the biological meanings. And in the last section, we discuss our methods and results.

2. Notations and Definitions

In this section, we agree on some notations which will prove useful and give some definitions.

Let $R_+ = [0, \infty)$, $R_+^2 = \{x \in R^2 \mid x \geq 0\}$, and N be the set of all nonnegative integers. Denote by $F = (F_1, F_2)$ the map defined by the right hand of system (1.4). Let $V_0 = \{V : R_+ \times R_+^2 \mapsto R_+ \mid V$ is continuous on $(n\tau, (n+1)\tau] \times R_+^2$ and $\lim_{(t,y)\to(n\tau,x), t>n\tau} V(t, y) = V(n\tau^+, x)$ exists$\}$.

Definition 2.1. $V \in V_0$, then for $(t, x) \in (n\tau, (n+1)\tau] \times R_+^2$, the upper right derivative of $V(t, x)$ with respect to the impulsive differential system (1.4) is defined as

$$D^+V(t, x) = \limsup_{h\to 0^+} \frac{1}{h}[V(t + h, x + hF(x)) - V(t, x)]. \tag{2.1}$$

We will assume the following basic conditions for system (1.4) hold throughout this paper.

(A1) $f_i : \times R_+^2 \mapsto R_+$ is differentiable and $\partial f_i/\partial x_i \leq 0$, $i = 1, 2$, $\partial f_1/\partial x_2 \leq 0$, $\partial f_2/\partial x_1 \geq 0$.

(A2) $I_i : R_+ \mapsto R_+$ is continuous, $I_i(0) \geq 0$, $I_i(u) > 0$ for $u > 0$, and I_i is nondecreasing for $i = 1, 2$.

The solution of system (1.4) is a piecewise continuous function $x : R_+ \mapsto R_+^2$, $x(t)$ is continuous on $(n\tau, (n + 1)\tau]$, $n \in N$, and $x(n\tau^+) = \lim_{t\to n\tau^+} x(t)$ exists. Obviously the smoothness properties of f_i guarantee the global existence and uniqueness of solutions of system (1.4) (see [1, 3] for details on fundamental properties of impulsive systems). (A1) shows that x_1 and x_2 can be the densities of the prey and the predator at time t, respectively, and both the species are density dependent. With (A2), we can see that impulsive perturbations cannot make any species disappear instantly or in limited time interval. Since $\dot{x}_i(t) = 0$ whenever $x_i(t) = 0$, $i = 1, 2$, $t \neq n\tau$, $n \in N$ and $x_1(n\tau^+) = I_1(x_1(n\tau))$, $x_2(n\tau^+) = I_2(x_2(n\tau))$, $n \in N$, by (A2), we have the following lemma.

Lemma 2.2. *Suppose $x(t)$ is a solution of (1.4) with $x(0^+) \geq 0$, then $x(t) \geq 0$ for all $t \geq 0$. And further $x(t) > 0$, $t \geq 0$ if $x(0^+) > 0$.*

Definition 2.3. System (1.4) is said to be permanent if there exist constants $M \geq m > 0$ such that $m \leq x_i(t) \leq M$, $i = 1, 2$ for all t sufficiently large, where $x(t)$ is any solution of (1.4) with $x(0^+) > 0$.

We will use a basic comparison result from in [3, Theorem 3.1.1]. For convenience, we state it in our notations.

Suppose that $g : R_+ \times R_+ \mapsto R_+$ satisfies the following condition (H).

(H) g is continuous in $(n\tau, (n+1)\tau] \times R_+$ and for $x \in R_+$, $n \in N$, $\lim_{(t,y)\to(n\tau^+,x)} g(t,y) = g(n\tau^+, x)$ exists.

Lemma 2.4. *Let $V \in V_0$. Assume that*

$$\begin{aligned} D^+V(t,x) &\le g(t, V(t,x)), \quad t \ne n\tau, \\ V(t, x(t^+)) &\le \psi_n(V(t, x(t))), \quad t = n\tau, \end{aligned} \tag{2.2}$$

where $g : R_+ \times R_+ \mapsto R_+$ satisfies (H) and $\psi_n : R_+ \mapsto R_+$ is nondecreasing. Let $r(t)$ be the maximal solution of the scalar impulsive differential equation

$$\begin{aligned} \dot{u} &= g(t,u), \quad t \ne n\tau, \\ u(t^+) &= \psi_n(u(t)), \quad t = n\tau, \\ u(0^+) &= u_0 \end{aligned} \tag{2.3}$$

existing on $[0,\infty)$. Then, $V(0^+, x_0) \le u_0$ implies that $V(t, x(t)) \le r(t)$, $t \ge 0$, where $x(t)$ is any solution of (1.4).

Similar result can be obtained when all the directions of the inequalities in (2.2) are reversed. Note that if we have some smoothness conditions of g to guarantee the existence and uniqueness of solutions for (2.3), then $r(t)$ is exactly the unique solution of (2.3).

3. Main Results

In this section, we will establish conditions for the extinction and permanence of system (1.4) and study the bifurcation of a positive periodic solution for system (1.4).

3.1. Uniformly Ultimate Upper Boundary

Firstly, we establish conditions for that all solutions of (1.4) are uniformly bounded above. This is usually valid from the biological interpreting of the system. Mathematically, it is easy to be achieved by using the method of Liapunov functions and the comparison results of Lemma 2.4. For example, we give one set of such conditions here.

Theorem 3.1. *Suppose that the following condition* (H1) *holds.*

(H1) *There exists $V(t) = V(t,x)$, $V \in V_0$ such that the following conditions hold.*

(i) *$V(t,x) \ge c_1x_1 + c_2x_2$, for some $c_1, c_2 > 0$;*

(ii)

$$\begin{aligned} D^+V(t, x(t)) &\le -\lambda V(t) + K, \quad t \ne n\tau, \\ V(t, x(t^+)) &\le V(t, x(t)) + b, \quad t = n\tau, \end{aligned} \tag{3.1}$$

where λ, K, b are positive constants.

Then, system (1.4) *is ultimately upper bounded.*

Proof. Let $V(0^+) = V(0, x(0^+))$. By (i), it suffices to prove that $V(t)$ is ultimately upper bounded. In view of (ii), this is similar to the proof of [18, Lemma 3.2]. Thus we omit it here. The proof is complete. □

3.2. Prey Eradicated Periodic Solution

To study the prey eradicated periodic solution, we consider the following scalar impulsive system, which will also serve as an comparison system for studying the permanence of system (1.4):

$$\begin{aligned} \dot{u} &= u f_2(\varepsilon, u), \quad t \neq n\tau, \\ u(t^+) &= I_2(u(t)), \quad t = n\tau, \\ u(0^+) &= u_0 > 0. \end{aligned} \tag{3.2}$$

Since one-dimensional continuous differential system is naturally monotone system and I_2 is nondecreasing, the solutions of system (3.2) are also monotone with respect to initial values [1, the proof of Theorem 12.5].

Lemma 3.2. *Suppose that the following condition holds.*

(H2) *There exist positive constants a_2, b_2, and $\alpha_2$such that $I_2(u) \geq a_2 u + b_2$ for $0 \leq u \leq \alpha_2$.*

Then, there exists $\delta_2 > 0$ such that $u(\tau^+) \geq u_0$ for $u_0 \leq \delta_2$, where $u(t)$ is the solution of (3.2).

Proof. Let $\delta_2 = \min\{b_2, \alpha_2, \alpha_2 / \exp(\tau f_2(\varepsilon, 0))\}$ and $u_0 \leq \delta_2$. Obviously, $u(t)$ is positive for $t > 0$. By (A1), we have

$$\dot{u} \leq u f_2(\varepsilon, 0), \quad t \in (0, \tau). \tag{3.3}$$

Hence, $u(t) \leq u_0 \exp(t f_2(\varepsilon, 0)) \leq \delta_2 \exp(\tau f_2(\varepsilon, 0)) \leq \alpha_2$, $t \in (0, \tau]$ if $f_2(\varepsilon, 0) \geq 0$ and $u(t) \leq u_0 \exp(t f_2(\varepsilon, 0)) \leq u_0 \leq \delta_2 \leq \alpha_2$, $t \in (0, \tau]$ if $f_2(\varepsilon, 0) < 0$. By (3.2), we have

$$u(\tau) = u_0 \exp\left(\int_0^\tau f_2(\varepsilon, u(s)) ds\right). \tag{3.4}$$

Hence, by (H2),

$$u(\tau^+) \geq a_2 u_0 \exp\left(\int_0^\tau f_2(\varepsilon, u(s)) ds\right) + b_2 \geq b_2 \geq \delta_2 \geq u_0. \tag{3.5}$$

The proof is complete. □

Lemma 3.3. *Suppose that the following condition holds.*

(H3) *There exist positive constants* A_2, B_2, *and* β_2 *such that*

$$A_2 \exp\left(\tau f_2(0,0)\right) < 1, \quad f_2(0,u) < 0 \tag{3.6}$$

and $I_2(u) \le A_2 u + B_2$ *for* $u \ge \beta_2$.

Then, there exist $\varepsilon_2 > 0$, $M_2 > 0$ *such that* $u(\tau^+) \le u_0$ *for* $u_0 \ge M_2$, *where* $u(t)$ *is the solution of* (3.2) *with* $0 \le \varepsilon \le \varepsilon_2$.

Proof. By (3.6), there exists $\varepsilon_2 > 0$ such that

$$A_2 \exp\left(\tau f_2(\varepsilon,0)\right) < 1, \quad f_2(\varepsilon,u) < 0 \tag{3.7}$$

for $0 \le \varepsilon \le \varepsilon_2$ and $u \ge \beta_2$. Let $M = \max\{I_2(u) \mid 0 \le u \le \beta_2\}$ and $M_2 = \max\{M, \beta_2, B_2/(1 - A_2 \exp(\tau f_2(\varepsilon,0)))\} > 0$. Let $u_0 \ge M_2$ and $u(t)$ be the solution of (3.2) with $0 \le \varepsilon \le \varepsilon_2$. There are two cases for $u(t)$, $t \in (0,\tau]$.

Case 1. There exists $t_1 \in (0,\tau]$ such that $u(t_1) < \beta_2$.

Let $t^* = \inf\{t \in (0,\tau] \mid u(t) < \beta_2\}$. Then, $u(t^*) = \beta_2$. Since $f_2(\varepsilon,\beta_2) < 0$, we can conclude that $u(t) < \beta_2$, $t \in (t^*,\tau]$. Hence,

$$u(\tau^+) = I_2(u(\tau)) \le M \le M_2 \le u_0. \tag{3.8}$$

Case 2. $u(t) \ge \beta_2, t \in (0,\tau]$.

By (A1), we have.

$$\dot{u} \le u f_2(\varepsilon,0), \quad t \in (0,\tau). \tag{3.9}$$

Hence,

$$u(\tau) \le u_0 \exp\left(\tau f_2(\varepsilon,0)\right). \tag{3.10}$$

Therefore,

$$\begin{aligned} u(\tau^+) &\le A_2 u_0 \exp\left(\tau f_2(\varepsilon,0)\right) + B_2 \\ &= u_0 + B_2 - \left(1 - A_2 \exp\left(\tau f_2(\varepsilon,0)\right)\right) u_0 \\ &\le u_0. \end{aligned} \tag{3.11}$$

This completes the proof. □

Let $0 < u_0^1 \le \delta_2$ and $u_0^2 \ge \beta_2$ for δ_2, β_2 in Lemmas 3.2 and 3.3, respectively. Consider the solution $u(t)$ of (3.2) with $u_0 \in [u_0^1, u_0^2]$. By Lemma 2.4, similar to [12, Theorem 3.1], we can define a map

$$P : \left[u_0^1, u_0^2\right] \longmapsto \left[u_0^1, u_0^2\right], \qquad P(u_0) = I_2(u(\tau)) \tag{3.12}$$

and show P has a fixed point which corresponds to the initial value of a positive periodic solution of (3.2). Thus, we have the following theorem.

Theorem 3.4. *Suppose that* (H2) *and* (H3) *hold. Then, there exists $\varepsilon_2 > 0$ such that system* (3.2) *has a positive τ-periodic solution $u^\varepsilon(t)$ for each $0 \le \varepsilon \le \varepsilon_2$.*

Modify $v(0), v(T)$ to $v(0^+), v(T^+)$ and consider for the case with $T = \tau$ and $q = 1$ in the definition of lower and upper solutions of [26, Definition 3.1], the solution $u(t)$ of (3.2) with $0 < u_0 \le \delta_2$ ($u_0 \ge \beta_2$) here is factually also the lower solution (upper solution) of (3.2). Hence similar to [26, Theorem 3.6], the solutions of (3.2) with initial values $0 < u_0 \le \delta_2$ and $u_0 \ge \beta_2$ will tend to a positive τ-periodic solution of (3.2). If the positive τ-periodic solution of (3.2) is unique, then its global attractivity will also be established. In fact, for any solution $u(t)$ of (3.2) with initial value $u_0 > 0$, we can always find a lower solution $u_1(t)$ and an upper solution $u_2(t)$ with initial values $0 < u_1(0^+) = u_0^1 \le \alpha_2$ and $u_2(0^+) = u_0^2 \ge \beta_2$, respectively, such that $u_0^1 \le u_0 \le u_0^2$. Then since the solutions of system (3.2) are monotone with respect to initial values, $u_1(t) \le u(t) \le u_2(t)$. And since both $u_1(t)$ and $u_2(t)$ will tend to the unique positive τ-periodic solution of (3.2), so is $u(t)$. Using the theory of concave operators, [26, Theorem 3.8] established the uniqueness of positive periodic solution for a general n-dimensional monotone impulsive differential system. With the map P defined above, we can simply use that result here. We first give the definition of strongly concave operator on R_+^n.

Definition 3.5. An operator $U : R_+^n \mapsto R_+^n$ is strongly concave on R_+^n if for any $x \in R_+^n$, $x > 0$ and any number $s \in (0,1)$, there exists a positive number η such that $U(sx) \ge (1+\eta)sU(x)$.

Now let $u(t)$ be the solution of (3.2) with initial value $u_0 > 0$. Let $\overline{f}_2 = u f_2(\varepsilon, u)$. Define

$$F(u(t)) = \overline{f}_2(\varepsilon, u(t)) - D\overline{f}_2(\varepsilon, u(t))u(t), \tag{3.13}$$

where $D\overline{f}_2$ is the Jacobian matrix of $\overline{f}_2$ (here is just its derivative with respect to u since (3.2) is one-dimensional). By [26, Theorems 3.8 and 3.9], we have the following result.

Theorem 3.6. *Suppose that* (H2), (H3), *and the following (H4) hold.*

(H4) $F(u) > 0$ *for any $u > 0$ and I_2 is strongly concave or is linear.*

Then, there exists a$\varepsilon_2 > 0$ such that for each $0 \le \varepsilon \le \varepsilon_2$, system (3.2) *has a unique positive τ-periodic solution $u^\varepsilon(t)$, which is a global attractor for any solution $u(t)$ of* (3.2) *with $u_0 > 0$.*

Remark 3.7. The unique positive τ-periodic solution is corresponding to the unique positive fixed point of P. Since the map P here is one-dimensional, it could be possible to establish its uniqueness of positive fixed point directly in some cases. Then (H4), can be replaced by the following (H4′), which states the uniqueness directly.

(H4′) P has at most one positive fixed point.

3.3. Extinction of the Prey

Theorem 3.8. *Let* $(x_1(t), x_2(t))$ *be the solution of* (1.4). *Suppose that (H2)–(H4) and the following (H5) hold.*

(H5) *There exists* $A_1 > 0$ *such that* $I_1(x) \leq A_1 x$ *and*

$$A_1 \exp\left(\int_0^{\tau} f_1\left(0, u^0(s)\right) ds\right) < 1, \tag{3.14}$$

where $u^0(t)$ *is the unique positive periodic solution of* (3.2) *for* $\varepsilon = 0$.
Then $\lim_{t\to\infty} x_1(t) = 0$, $\lim_{t\to\infty} |x_2(t) - u^0(t)| = 0$.

Proof. By (3.14), the continuity of f_1, and the Lebesgue Theorem, we can choose an $\eta > 0$ sufficiently small such that $A_1 \exp(\sigma_\eta^0) < 1$, where $\sigma_\eta^0 = \int_0^{\tau} f_1(0, u^0(s) - \eta) ds$. Note that $\dot{x}_2 \geq x_2 f_2(0, x_2)$, consider (3.2) with $\varepsilon = 0$ and $u(0^+) = x_2(0^+)$. Let $V(t, x) = x_2(t)$. From Lemma 2.4 and Theorem 3.6, we have $x_2(t) \geq u(t)$ and $|u(t) - u^0(t)| \to 0$ as $t \to \infty$. Hence, there exists $t_1 > 0$ such that

$$x_2(t) \geq u(t) > u^0(t) - \eta \tag{3.15}$$

for all $t \geq t_1$. For simplification and without loss of generality, we may assume (3.15) holds for all $t \geq 0$. By (A1), we get

$$\dot{x}_1 \leq x_1 f_1\left(0, u^0(t) - \eta\right) \tag{3.16}$$

which leads to

$$\begin{aligned} x_1((n+1)\tau) &\leq x_1(n\tau^+) \exp\left(\int_{n\tau}^{(n+1)\tau} f_1\left(0, u^0(s) - \eta\right) ds\right) \\ &= x_1(n\tau^+) \exp\left(\sigma_\eta^0\right) \\ &\leq x_1(n\tau) A_1 \exp\left(\sigma_\eta^0\right). \end{aligned} \tag{3.17}$$

Hence $x_1(n\tau) \le x_1(\tau)(A_1 \exp(\sigma_\eta^0))^{n-1}$ and $x_1(n\tau) \to 0$ as $n \to \infty$. Therefore, $x_1(t) \to 0$ as $t \to \infty$ since

$$\begin{aligned} 0 < x_1(t) &= x_1(n\tau^+) \exp\left(\int_{n\tau}^{t} f_1\left(0, u^0(s) - \eta\right) ds\right) \\ &\le x_1(n\tau) A_1 \exp\left(\int_0^{\tau} \left| f_1\left(0, u^0(s) - \eta\right)\right| ds\right) \end{aligned} \tag{3.18}$$

for $n\tau < t \le (n+1)\tau$. Now, we prove that $|x_2(t) - u^0(t)| \to 0$ as $t \to \infty$. For $0 < \varepsilon \le \varepsilon_2$ in Theorem 3.6, there exists a $t_1 > 0$ such that $0 < x_1(t) < \varepsilon$, $t \ge t_1$. Without loss of generality, we may assume that $0 < x_1(t) < \varepsilon$ for all $t \ge 0$. Then, by (A1), we have

$$x_2 f_2(0, x_2) \le \dot{x}_2 \le x_2 f_2(\varepsilon, x_2). \tag{3.19}$$

From Lemma 2.4 and Theorem 3.6, we have $u_1(t) \le x_2(t) \le u_2(t)$ and $|u_1(t) - u^0(t)| \to 0$, $|u_2(t) - u^{\varepsilon}(t)| \to 0$ as $t \to \infty$, where $u_1(t)$ is solution of (3.2) with $\varepsilon = 0$, $u_1(0^+) = x_2(0^+)$ and $u_2(t)$ are solutions of (3.2) with $u_2(0^+) = x_2(0^+)$. Therefore, for any small enough $\eta > 0$, we have

$$u^0(t) - \eta < x_2(t) < u^{\varepsilon}(t) + \eta \tag{3.20}$$

for large t. Let $\varepsilon \to 0$, we get

$$u^0(t) - \eta < x_2(t) \le u^0(t) + \eta \tag{3.21}$$

for large t, which implies $\lim_{t\to\infty} |x_2(t) - u^0(t)| = 0$. The proof is complete. □

3.4. Permanence

Theorem 3.9. *Suppose that* (H1)–(H4) *and the following* (H6) *hold.*

(H6) *There exists* $a_1 > 0$ *such that* $I_1(x) \ge a_1 x$ *and*

$$a_1 \exp\left(\int_0^{\tau} f_1\left(0, u^0(s)\right) ds\right) > 1, \tag{3.22}$$

where $u^0(s)$ *is the unique positive periodic solution of* (3.2) *for* $\varepsilon = 0$.
Then, system (1.4) *is permanent.*

Proof. Let $x(t)$ be any solution of (1.4) with $x_0 \in \operatorname{int} R_+^2$. From Theorem 3.1, we may assume $x_i(t) \le M$, $t \ge 0$, $i = 1, 2$ where M is a positive constant independent of initial values. Since

$$\dot{x}_2 \ge x_2 f_2(0, x_2), \tag{3.23}$$

by Lemma 2.4 and Theorem 3.6, for sufficiently small $\eta_1 > 0$, $x_2(t) \geq u^0(t) - \eta_1$ for t large enough. Hence obviously there exists $m_2 > 0$ such that $x_2(t) \geq m_2$ for all t large enough. We shall next find an $m_1 > 0$ such that $x_1(t) \geq m_1$ for t large enough. We will do it in the following two steps.

(1) Let ε_2 be the positive constant in the conclusion of Theorem 3.6. By (3.22), the continuity of f_1, and the Lebesgue Theorem, we can choose $0 < m_3 \leq \varepsilon_2$, $\eta_1 > 0$ be small enough such that $\sigma = a_1 \exp(\int_0^\tau f_2(m_3, u^{m_3}(s) + \eta_1)ds) > 1$. We will prove $x_1(t) \leq m_3$ cannot hold for all $t \geq 0$. Otherwise,

$$\dot{x}_2 \leq x_2 f_2(m_3, x_2). \tag{3.24}$$

By Lemma 2.4 and Theorem 3.6, we have $x_2(t) \leq u(t)$ and $|u(t) - u^{m_3}(t)| \to 0$ as $t \to \infty$, where $u(t)$ is the solution of (3.2) with $\varepsilon = m_3$ and $u(0^+) = x_2(0^+)$. Therefore, there exist, a $T_1 > 0$ such that

$$x_2(t) \leq u(t) \leq u^{m_3}(t) + \eta_1, \tag{3.25}$$

$$\dot{x}_1 \geq x_1 f_1(x_1, u^{m_3}(t) + \eta_1) \tag{3.26}$$

for $t \geq T_1$.

Let $N_1 \in N$ and $N_1\tau \geq T_1$. Integrating (3.26) on $(n\tau, (n+1)\tau]$ for $n \geq N_1$, we have

$$\begin{aligned} x_1((n+1)\tau) &\geq x_1(n\tau^+) \exp\left(\int_{n\tau}^{(n+1)\tau} f_1(m_3, u^{m_3}(s) + \eta_1)ds\right) \\ &= x_1(n\tau^+) \exp\left(\int_0^\tau f_1(m_3, u^{m_3}(s) + \eta_1)ds\right) \\ &\geq x_1(n\tau) a_1 \exp\left(\int_0^\tau f_1(m_3, u^{m_3}(s) + \eta_1)ds\right) \\ &= x_1(n\tau)\sigma. \end{aligned} \tag{3.27}$$

Then, $x_1((N_1 + k)\tau) \geq x_1(N_1\tau)\sigma^k \to \infty$ as $k \to \infty$, which is a contradiction. Hence, there exists a $t_1 > 0$ such that $x_1(t_1) > m_3$.

(2) If $x_1(t) > m_3$ for all $t \geq t_1$, then our aim is obtained. Hence we need only to consider those solutions which leave the region $R = \{x \in R_+^2 \mid x_1 \leq m_3\}$ and reenter it again. Let $t^* = \inf_{t \geq t_1}\{x_1(t) \leq m_3\}$. Then, $x_1(t) > m_3$ for $t \in [t_1, t^*)$ and $x_1(t^*) \geq m_3$. Suppose $t^* \in (n_1\tau, (n_1+1)\tau]$, $n_1 \in N$. Let $u_M(t)$ be the solution of (3.2) with $\varepsilon = m_3$ and $u_M((n_1+1)\tau^+) = M$. Then there is $t_2 > 0$ such that $u_M(t) < u^{m_3}(t) + \eta_1$ for $t \geq t_2 + (n_1+1)\tau$. Note that t_2 is independent of $x(t)$. Select $n_2, n_3 \in N$ such that $n_2\tau > t_2$ and

$$a_1^{n_2} \exp(-|\sigma_1| + n_2\sigma_1)\sigma^{n_3} > 1, \tag{3.28}$$

where $\sigma_1 = \tau f_1(m_3, M)$. We claim there must be a $t_3 \in (t^*, (n_1+1+n_2+n_3)\tau]$ such that $x_1(t_3) > m_3$. Otherwise for $t \in (t^*, (n_1+1+n_2+n_3)\tau]$, $x_1(t) \le m_3$ and

$$\dot{x}_2 \le f_2(m_3, x_2). \tag{3.29}$$

By Lemma 2.4, we have

$$x_2(t) \le u_M(t), \quad t \in ((n_1+1)\tau, (n_1+1+n_2+n_3)\tau]. \tag{3.30}$$

Then,

$$x_2(t) \le u_M(t) < u^{m_3}(t) + \eta_1, \tag{3.31}$$

and (3.26) holds for $t \in ((n_1+1+n_2)\tau, (n_1+1+n_2+n_3)\tau]$. As in step 1, we have

$$x_1((n_1+1+n_2+n_3)\tau) \ge x_1((n_1+1+n_2)\tau)\sigma^{n_3}. \tag{3.32}$$

Since

$$\dot{x}_1 \ge x_1 f_1(m_3, M), \quad t \in (t^*, (n_1+1+n_2+n_3)\tau], \tag{3.33}$$

integrating it on $[t^*, (n_1+1+n_2)\tau]$, we have

$$x_1((n_1+1+n_2)\tau) \ge m_3 a_1^{n_2} \exp(-|\sigma_1| + n_2\sigma_1). \tag{3.34}$$

Thus, by (3.28), we have

$$x_1((n_1+1+n_2+n_3)\tau) \ge m_3 a_1^{n_2} \exp(-|\sigma_1| + n_2\sigma_1)\sigma^{n_3} > m_3, \tag{3.35}$$

which is a contradiction. Let $\bar{t} = \inf_{t \ge t^*}\{x_1(t) > m_3\}$. Then, $\bar{t} \in (t^*, (n_1+1+n_2+n_3)\tau]$. Denote $a = \min\{1, a_1^{n2+n_3}\}$. For $t \in (t^*, \bar{t}]$, we have

$$x_1(t) \ge x_1(t^*) a \exp(-(1+n_2+n_3)|\sigma_1|) \ge m_3 a \exp(-(1+n_2+n_3)|\sigma_1|) \triangleq m_1. \tag{3.36}$$

For $t > \bar{t}$, the same arguments can be continued since $x_1(\bar{t}^+) > m_3$. Hence $x_1(t) \ge m_1$ for all $t \ge t_1$. The proof is complete. □

3.5. Existence of a Positive Periodic Solution

Since system (1.4) may have a prey eradicated periodic solution, we can use the bifurcation theory in [9] to study the existence of positive periodic solution. System (1.4) can be rewritten as the following more general system:

$$\dot{x} = F(x), \quad t \neq n\tau, \\ x(t^+) = \Theta(x(t)), \quad t = n\tau, \tag{3.37}$$

where $F = (x_1 f_1(x), x_2 f_2(x))$, $\Theta = (I_1(x), I_2(x)) : R_+^2 \mapsto R_+^2$, are suitable smooth. For convenience and using the same notations in [9], we have exchanged the subscripts of x_1, x_2 and f_1, f_2 in system (1.4). Suppose

$$\dot{x}_1 = g(x_1) = F_1(x_1, 0), \quad t \neq n\tau, \\ x_1(t^+) = \theta(x_1(t)) = \Theta_1(x_1(t), 0), \quad t = n\tau \tag{3.38}$$

has a periodic solution $x_s(t)$. Thus, $\zeta = (x_s, 0)^T$ is a trivial periodic solution of (3.37). By studying the local stability of ζ and a standard computation of Floqet exponent, [9] establishes a bifurcation theory which gives the existence positive periodic solution of (3.37). The main idea of the process is to select the period τ as parameter and transform the problem of finding positive periodic solution into a fixed-point problem. Then, establish the conditions of the implicit function theorem. We will use their results to study the existence of positive periodic solution for (1.4). For simplification, we will only state some necessary notations and the bifurcation theorem of [9].

Let Φ be the flow associated to (3.37), we have $x(t) = \Phi(t, x_0)$, $0 < t \leq \tau$, where $x_0 = x(0^+)$. Now, we list following notations we will use from [9]:

$$d_0' = 1 - \frac{\partial\Theta_2}{\partial x_2}\frac{\partial\Phi_2}{\partial x_2}(\tau_0, x_0), \text{ where } \tau_0 \text{ is the root of } d_0' = 0,$$

$$a_0' = 1 - \left(\frac{\partial\Theta_1}{\partial x_1}\frac{\partial\Phi_1}{\partial x_1}\right)(\tau_0, x_0),$$

$$b_0' = -\left(\frac{\partial\Theta_1}{\partial x_1}\frac{\partial\Phi_1}{\partial x_2} + \frac{\partial\Theta_1}{\partial x_2}\frac{\partial\Phi_2}{\partial x_2}\right)(\tau_0, x_0),$$

$$\frac{\partial\Phi_1(t, x_0)}{\partial x_1} = \exp\left(\int_0^t \frac{\partial F_1(\zeta(r))}{\partial x_1}dr\right),$$

$$\frac{\partial\Phi_2(t, x_0)}{\partial x_2} = \exp\left(\int_0^t \frac{\partial F_2(\zeta(r))}{\partial x_2}dr\right),$$

$$\frac{\partial\Phi_1(t, x_0)}{\partial x_2} = \int_0^t \exp\left(\int_u^t \frac{\partial F_1(\zeta(r))}{\partial x_1}dr\right)\frac{\partial F_1(\zeta(u))}{\partial x_2}\exp\left(\int_0^u \frac{\partial F_2(\zeta(r))}{\partial x_2}dr\right)du,$$

$$\frac{\partial^2\Phi_2(t,x_0)}{\partial x_1\partial x_2} = \int_0^t \exp\left(\int_u^t \frac{\partial F_2(\zeta(r))}{\partial x_2}dr\right)\frac{\partial^2 F_2(\zeta(u))}{\partial x_1\partial x_2}\exp\left(\int_0^u \frac{\partial F_2(\zeta(r))}{\partial x_2}dr\right)du,$$

$$\begin{aligned}\frac{\partial^2\Phi_2(t,x_0)}{\partial x_2^2} &= \int_0^t \exp\left(\int_u^t \frac{\partial F_2(\zeta(r))}{\partial x_2}dr\right)\frac{\partial^2 F_2(\zeta(u))}{\partial x_2^2}\exp\left(\int_0^u \frac{\partial F_2(\zeta(r))}{\partial x_2}dr\right)du \\ &\quad + \int_0^t\left\{\exp\left(\int_u^t \frac{\partial F_2(\zeta(r))}{\partial x_2}dr\right)\frac{\partial^2 F_2(\zeta(u))}{\partial x_2\partial x_1}\right\} \\ &\quad \times\left\{\int_0^u \exp\left(\int_p^u \frac{\partial F_1(\zeta(r))}{\partial x_1}dr\right)\frac{\partial F_1(\zeta(u))}{\partial x_2}\exp\left(\int_0^p \frac{\partial F_2(\zeta(r))}{\partial x_2}dr\right)dp\right\}du,\end{aligned}$$

$$\frac{\partial^2\Phi_2(t,x_0)}{\partial\overline{\tau}\partial x_2} = \frac{\partial F_2(\zeta(t))}{\partial x_2}\exp\left(\int_0^t \frac{\partial F_2(\zeta(r))}{\partial x_2}dr\right),$$

$$\frac{\partial\Phi_1(\tau_0,x_0)}{\partial\tau} = \dot{x}_s(\tau_0)$$

$$\begin{aligned}B &= \frac{\partial^2\Theta_2}{\partial x_1\partial x_2}\left(\frac{\partial\Phi_1(\tau_0,x_0)}{\partial\overline{\tau}} + \frac{\partial\Phi_1(\tau_0,x_0)}{\partial x_1}\frac{1}{a_0'}\frac{\partial\Theta_1}{\partial x_1}\frac{\partial\Phi_1(\tau_0,x_0)}{\partial t\overline{a}u}\right)\frac{\partial\Phi_2(\tau_0,x_0)}{\partial x_2} \\ &\quad - \frac{\partial\Theta_2}{\partial x_2}\left(\frac{\partial^2\Phi_2(\tau_0,x_0)}{\partial\overline{\tau}\partial x_2} + \frac{\partial^2\Phi_2(\tau_0,x_0)}{\partial x_1\partial x_2}\frac{1}{a_0'}\frac{\partial\Theta_1}{\partial x_1}\frac{\partial\Phi_1(\tau_0,x_0)}{\partial\overline{\tau}}\right), \\ C &= 2\frac{\partial^2\Theta_2}{\partial x_1\partial x_2}\left(-\frac{b_0'}{a_0'}\frac{\partial\Phi_1(\tau_0,x_0)}{\partial x_1} + \frac{\partial\Phi_1(\tau_0,x_0)}{\partial x_2}\right)\times\frac{\partial\Phi_2(\tau_0,x_0)}{\partial x_2} \\ &\quad - \frac{\partial^2\Theta_2}{\partial x_2^2}\left(\frac{\partial\Phi_2(\tau_0,x_0)}{\partial x_2}\right)^2 + 2\frac{\partial\Theta_2}{\partial x_2}\frac{b_0'}{a_0'}\frac{\partial^2\Phi_2(\tau_0,x_0)}{\partial x_2\partial x_1} - \frac{\partial\Theta_2}{\partial x_2}\frac{\partial^2\Phi_2(\tau_0,x_0)}{\partial x_2^2}.\end{aligned} \tag{3.39}$$

Theorem 3.10 (see [9]). *If* $|1-a_0'|<1$ *and* $d_0'=0$, *then one has the following:*

(a) *If* $BC\neq 0$, *then there is a bifurcation of nontrivial periodic solution. Moreover, there is a bifurcation of supercritical case if* $BC<0$ *and a subcritical case if* $BC>0$.

(b) *If* $BC=0$, *then there is an undetermined case.*

4. Application

System (1.2) developed the Lotka-Volterra predator-prey system with periodic constant impulsive immigration effect on the predator which is quite natural. For example, we can use the impulsive effects for the purpose of protecting the predator or eliminating the prey. Similar results could be achieved by impulsive invasion of predator. Applying the results in Section 3, we first establish conditions both for the system to be permanent and driving the prey to extinction.

Theorem 4.1. *There exists a constant* $M>0$ *such that* $x_i(t)\leq M$, $i=1,2$ *for each solution* $x(t)$ *of* (1.2) *with all* t *large enough.*

Proof. Suppose $x(t)$ is any solution of (1.2). Let $V(t) = V(t, x(t)) = a_{21}x_1(t) + a_{12}x_2(t)$. Let $0 < \lambda \leq r_2$. Then, when $t \neq n\tau$,

$$D^+V(t) + \lambda V(t) = a_{21}x_1(r_1 + \lambda - a_{11}x_1) - a_{12}(r_2 - \lambda)x_2 \leq a_{21}x_1(r_1 + \lambda - a_{11}x_1) \leq K, \tag{4.1}$$

for some positive constant K. When $t = n\tau$, $V(n\tau^+) = V(n\tau) + a_{12}b$. Thus, (H1) is satisfied, the conclusion comes from Theorem 3.1. The proof is complete. □

Theorem 4.2. *Let $x(t)$ be any solution of* (1.2). *Then,* $\lim_{t\to\infty}x_1(t) = 0$, $\lim_{t\to\infty}|x_2(t) - x_2^*(t)| = 0$ *if* $b > r_1r_2\tau/a_{12}$, *where* $x_2^*(t) = b\exp(-r_2(t - n\tau))/(1 - \exp(-r_2\tau))$, $t \in (n\tau, (n+1)\tau]$, $n \in N$, $x_2^*(0^+) = b/(1 - \exp(-r_2\tau))$.

Proof. Consider system (3.2) with the f_2 and I_2 taking the forms in (1.2). Since $I_2(u) = u + b$, $f_2(0,0) = f_2(0,u) = -r_2$, obviously, (H2) and (H3) are satisfied. Clearly $x_2^*(t)$ is a positive τ periodic solution when $\varepsilon = 0$. Note that $P(u) = u\exp(-r_2\tau) + b$, it has a unique positive fixed point since $P(0) > 0, \lim_{u\to\infty}(P(u) - u) = -\infty$ and $P(u) - u$ is strictly decreasing. Since $I_1(x_1) = x_1$ and

$$\exp\left(\int_0^\tau f_1(0, x_2^*(s))ds\right) = \exp\left(\int_0^\tau (r_1 - a_{12}x_2^*(s))ds\right) = \exp\left(r_1\tau - \frac{a_{12}b}{r_2}\right) < 1 \tag{4.2}$$

when $b > r_1r_2\tau/a_{12}$, (H5) is satisfied. Thus, the results follow from Theorem 3.8 and Remark 3.7. The proof is complete. □

Theorem 4.3. *System* (1.2) *is permanent if* $b < r_1r_2\tau/a_{12}$.

Proof. Let $x(t)$ be any solution of (1.2) with $x_0 \in \operatorname{int} R_+^2$. Obviously, (H1)–(H3) and (H4′) are satisfied. Since $I_1(x_1) = x_1$ and

$$\exp\left(\int_0^\tau f_1(0, x_2^*(s))ds\right) = \exp\left(\int_0^\tau (r_1 - a_{12}x_2^*(s))ds\right) = \exp\left(r_1\tau - \frac{a_{12}b}{r_2}\right) > 1 \tag{4.3}$$

when $b < r_1r_2\tau/a_{12}$, (H6) is also satisfied. Hence the result follows from Theorem 3.9 and Remark 3.7. The proof is complete. □

Next, we show that system (1.2) has positive periodic solution when it is permanent. Note (1.2) has a trivial periodic solution $(0, x_2^*(t))^T$. We also exchange the subscripts of x_1 and x_2 as in Theorem 3.10. Thus,

$$\begin{aligned} F_1(x_1, x_2) &= x_1(-r_2 + a_{21}x_2), \\ F_2(x_1, x_2) &= x_2(r_1 - a_{11}x_2 - a_{12}x_1), \\ \Theta_1(x_1, x_2) &= x_1 + b, \\ \Theta_2(x_1, x_2) &= x_2, \\ \zeta(t) &= (x_s(t), 0)^T = (x_2^*(t), 0). \end{aligned} \tag{4.4}$$

Then, we can compute that

$$\begin{aligned}
a_0' &= 1 - \exp(-r_2\tau_0) > 0, \\
b_0' &= -\frac{a_{21}b\exp(-r_2\tau_0)}{1-\exp(-r_2\tau_0)}\int_0^{\tau_0}\exp\left(r_1u - \frac{a_{12}b(1-\exp(-r_2u))}{r_2(1-\exp(-r_2\tau_0))}\right)du < 0, \\
\frac{\partial^2\Phi_2(\tau_0,x_0)}{\partial\overline{\tau}\partial x_2} &= r_1 - a_{12}\frac{b\exp(-r_2\tau_0)}{1-\exp(-r_2\tau_0)}, \\
\frac{\partial^2\Phi_2(\tau_0,x_0)}{\partial x_1\partial x_2} &= -a_{12}\tau_0 < 0, \\
\frac{\partial^2\Phi_2(\tau_0,x_0)}{\partial x_2^2} &= -a_{11}\tau_0 \\
&\quad - \int_0^{\tau_0}\left\{\frac{a_{12}a_{21}b\exp(-r_2u)\exp(\mathbb{R})}{1-\exp(-r_2\tau_0)}\right\} \\
&\quad \times\left\{\int_0^u \exp\left(r_1p - \frac{a_{12}b(1-\exp(-r_2p))}{r_2(1-\exp(-r_2\tau_0))}\right)dp\right\}du < 0, \\
\frac{\partial\Phi_1(\tau_0,x_0)}{\partial\overline{\tau}} &= \dot{x}_s(\tau_0) = -\frac{r_2b\exp(-r_2\tau_0)}{1-\exp(-r_2\tau_0)} < 0,
\end{aligned} \tag{4.5}$$

where $\mathbb{R}$ denotes $r_1(\tau_0 - u) - a_{12}b(\exp(-r_2u) - \exp(-r_2\tau_0))/(r_2(1-\exp(-r_2\tau_0)))$.

Since $\partial\Theta_i/\partial x_j = 0$, $i \neq j$, $\partial\Theta_i/\partial x_i = 1$, $\partial^2\Theta_i/\partial x_1\partial x_2 = 0$, $i = 1,2$ and $\partial^2\Theta_2/\partial x_2^2 = 0$, it is easy to verify that $C > 0$ and

$$B = -\left(r_1 - a_{12}\frac{b\exp(-r_2\tau_0)}{1-\exp(-r_2\tau_0)} + \frac{a_{12}\tau_0r_2b\exp(-r_2\tau_0)}{(1-\exp(-r_2\tau_0))^2}\right). \tag{4.6}$$

To determine the sign of B, let $\phi(t) = r_1 - a_{12}b\exp(-r_2t)/(1-\exp(-r_2\tau_0))$. We have $d\phi/dt = r_2a_{12}b\exp(-r_2t)/(1-\exp(-r_2\tau_0)) > 0$. Thus, we conclude that $\phi(\tau_0) > 0$ since $\int_0^{\tau_0}\phi(t)dt = r_1\tau_0 - a_{12}b/r_2 = 0$ and $\phi(t)$ is strictly increasing. Therefore, we have $B < 0$ from (4.6) and the following result.

Theorem 4.4. *System* (1.2) *has a supercritical bifurcation of positive periodic solution at the point* $\tau_0 = a_{12}b/(r_1r_2)$, *that is, system* (1.2) *has a positive periodic solution if* $\tau > \tau_0$ *and is close to* τ_0, *where* τ_0 *is the root of* $d_0' = 0$.

Remark 4.5. From Theorems 4.2 and 4.3, we know that the trivial periodic solution ζ is a global attractor if $\tau < \tau_0 = a_{12}b(r_1r_2)$ and is unstable if $\tau > \tau_0 = a_{12}b(r_1r_2)$. Thus, the bifurcation, if it exists, should be a supercritical one and the positive periodic solution is stable.

To find the biological implications of the results for system (1.2), we now evaluate the average density of the species when system (1.2) has a positive periodic solution.

Theorem 4.6. *Suppose system* (1.2) *has a positive periodic solution* $\widehat{x}(t)$ *with* $\widehat{x}(0^+) = \widehat{x}_0 = (\widehat{x}_{01}, \widehat{x}_{02})^T$. *Then,* $(1/\tau)\int_0^\tau \widehat{x}_1(t)dt < \min\{r_1/a_{11}, r_2/a_{21}\}$ *and* $(1/\tau)\int_0^\tau \widehat{x}_2(t)dt < r_1/a_{12}$.

Proof. Since $\widehat{x}(t)$ is a positive periodic solution of system (1.2), we have $\widehat{x}(0^+) = \widehat{x}(\tau^+)$ which gives

$$\widehat{x}_{01} = \widehat{x}_{01} \exp\left(r_1\tau - a_{11}\int_0^\tau \widehat{x}_1(t)dt - a_{12}\int_0^\tau \widehat{x}_2(t)dt\right),$$
$$\widehat{x}_{02} = \widehat{x}_{02} \exp\left(-r_2\tau + a_{21}\int_0^\tau \widehat{x}_1(t)dt\right) + b. \tag{4.7}$$

Hence, the conclusion is quite clear and the proof is complete. □

To end this section, we explain the biological implications for the results of (1.2). Let us recall the condition $b = r_1 r_2 \tau / a_{12}$. It can be rewritten as $r_1\tau = a_{12}b/r_2$ or $\int_{n\tau}^{(n+1)\tau} r_1 dt = \int_{n\tau}^{(n+1)\tau} a_{12}x_2^*(t)dt$. If there is no prey or its density is very small, the density of the predator is $x_2^*(t)$. It is clear that $a_{12}b/r_2$ is the amount of prey that the predator can eat in τ period of time and $r_1\tau$ means the increasing amount of prey in such a period of time. Thus, $r_1r_2\tau/a_{12}$ can be interpreted as the amount of immigrating predator which the prey could supply for with its increment during τ period of time. Then, it is easy to understand that the prey will go extinct when $b > r_1r_2\tau/a_{12}$ and should be permanent otherwise.

It is the impulsive immigration of the predator that makes the dynamics of system (1.2) quite different from that of system (1.1). We note that the conditions in Theorems 4.2 and 4.3 have no relations with that for permanence or extinction of the corresponding system (1.1), which means that system (1.2) can be permanent or extinct no matter whether (1.1) is permanent or not. Therefore, our results suggest a biological approach in pest control by adding some amount of predator impulsively after a fixed period of time. If the amount is large enough, it can drive the pest to extinction which the classical approach can never achieve. When the magnitude of the impulse, b, is not too large, system (1.2) is permanent and has a positive periodic solution. Our further numeric results show that the periodic solution is a global attractor. In this case, if we use the classical way, the density of prey will tend to either r_1/a_{11} or r_2/a_{21}. As Theorem 4.6 shows, the average density of the prey is smaller than each of them which means that this approach is still better than the classical one.

5. Discussion

In this paper, we established some conditions of extinction and permanence for a general impulsive predator-prey system. These two concepts are important for a biological system and are useful in protecting the diversities of species. As a simple application, we applied the results to the Lotka-Volterra predator-prey system with periodic constant impulsive immigration effect on the predator. Similarly, our results can also be applied to the models in [18, 20, 21] and we obtain all results therein directly. The analysis process of are the same as that of Section 4. The methods and results in [18, 20, 21] are based on explicitly solving the prey eradicated periodic solution. Different from this, the existence and global attractivity of the prey-eradication periodic solution are ensured by monotone theory and all the conditions in this paper are given for the parameters or the functions in the right-hand side of system (1.4). This makes our results may be easily applied to some more general predator-prey systems with different functional responses and nonlinear impulsive perturbations. Since both system (1.2) and (1.3) do not include density dependent of predator, we can check that the map P has a unique positive fixed point directly, which ensures the uniqueness of prey-eradication periodic solution. If we add the density dependent to the (1.1), that is, consider

$$\dot{x}_1 = x_1(r_1 - a_{11}x_1 - a_{12}x_2),$$
$$\dot{x}_2 = x_2(-r_2 + a_{21}x_1 - a_{22}x_2), \tag{5.1}$$

where all the parameters are positive. Let f_2 be the right-hand side of the second equation. As defined in Section 3, we can compute that

$$F(u(t)) = f_2(\varepsilon, u(t)) - Df_2(\varepsilon, u(t))u(t) = a_{22}u(t). \tag{5.2}$$

Thus, (H4) may be easily satisfied and it is also easy to apply our results to the above Lotka-Volterra system when introducing some more general practical impulsive effects.

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