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*Research Article*

# A Note on Some Generalized Closed Sets in Bitopological Spaces Associated to Digraphs

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Many investigations are undergoing of the relationship between topological spaces and graph theory. The aim of this short communication is to study the nature and properties of some generalized closed sets in the bitopological spaces associated to the digraph. In particular, some relations between generalized closed sets in the bitopological spaces associated to the digraph are characterized.

## 1. Introduction

Concerning the applications of bitopological spaces, there are many approaches to the sets equipped with two topologies of which one may occasionally be finer than the other in analysis, potential theory, directed graphs, and general topology. Lukeš [1] formulated certain new methods to be used in discussing fine topologies, especially in analysis and potential theory in 1977 and one of the properties introduced by him is Lusin-Menchoff property of the fine topologies. This is the initiative to the study of various problems in analysis and potential theory with bitopological spaces.

Brelot [2] compared the notion of a regular point of a set with that of a stable point of a compact set for an analogous Dirichlet problem and thus arrived at a general notion of thinness in classical potential theory.

Bhargava and Ahlborn [3] investigated certain tieups between the theory of directed graphs and point set topology. They obtained several theorems relating connectedness and accessibility properties of a directed graph to the properties of the topology associated to that digraph. Further, they investigated these topologies in terms of closure, kernal, and core operators. This work extended to ceriatn aspects of work done by Bhargava in [4].

Evans et al. [5] proved that there is a one-to-one correspondence between the labelled topologies on $n$ points and labelled transitive digraph with $n$ vertices. Anderson and

Chartrand [6] investigated the lattice graph of the topologies to the transitive digraphs. In particular, they characterized those transitive digraphs whose topologies have isomorphic lattice graphs.

In theoretical development of bitopological spaces [7], several generalized closed sets have been introduced already. Fukutake [8] defined one kind of semiopen sets in bitopological spaces and studied their properties in 1989. Also, he introduced generalized closed sets and pairwise generalized closure operator [9] in bitopological spaces in 1986. A set $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is $\tau_i\tau_j$-generalized closed set (briefly $\tau_i\tau_j$-$g$ closed) [10] if $\tau_j$-cl$(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\tau_i$-open in $X$, $i, j = 1, 2$ and $i \neq j$. Also, he defined a new closure operator and strongly pairwise $T_{1/2}$-space. Further study on semiopen sets had been made by Bose [11] and Maheshwari and Prasad [12].

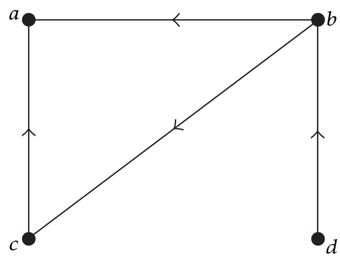
Semi generalized closed sets and generalized semiclosed sets are extended to bitopological settings by Khedr and Al-saadi [13]. They proved that the union of two $ij$-sg closed sets need not be $ij$-sg closed. This is an unexpected result. Also, they defined that the $ij$-semi generalized closure of a subset $A$ of a space $X$ is the intersection of all $ij$-sg closed sets containing $A$ and is denoted by $ij$-sgcl$(A)$. Rao and Mariasingam [14] defined and studied regular generalized closed sets in bitopological settings. Rao and Kannan [15] introduced semi star generalized closed sets in bitopological spaces in the year 2005. $(\tau_1, \tau_2)^*$-semi star generalized closed sets [16], regular generalized star star closed sets [17], semi star generalized closed sets [18], and the survey on Levine's generalized closed sets [19] had been studied in bitopological spaces in 2010, 2011, 2012, 2012, respectively.

The aim of this short communication is to study the nature and properties of some generalized closed sets in the bitopological spaces associated to the digraph. In particular, some relations between generalized closed sets in the bitopological spaces associated to the digraph are characterized.

## 2. Preliminaries

A digraph is an ordered pair $(X, \Gamma)$, where $X$ is a set and $\Gamma$ is a binary relation on $X$. A topology may be determined on a set $X$ by suitably defining subsets of $X$ to be open with respect to the digraph $(X, \Gamma)$. A set $A$ of the digraph $(X, \Gamma)$ is open if there does not exist an edge from $A^C$ to $A$. In other words, a set $A$ of the digraph $(X, \Gamma)$ is open if $p_i \in A^C$ and $p_j \in A$ imply that $p_ip_j \notin \Gamma$. A set $A$ of the digraph $(X, \Gamma)$ is closed if $A^C$ is open. Consequently, a set $A$ of the digraph $(X, \Gamma)$ is closed if there does not exist an edge from $A$ to $A^C$. Equivalently, a set $A$ of the digraph $(X, \Gamma)$ is closed if $p_i \in A$ and $p_j \in A^C$ imply that $p_ip_j \notin \Gamma$. Thus, each digraph $(X, \Gamma)$ determines a unique topological space $(X, \tau_\Gamma^+)$, where $\tau_\Gamma^+ = \{A : A \subseteq X \text{ of } (X, \Gamma) \text{ is open}\}$. Moreover, $(X, \tau_\Gamma^+)$ has completely additive closure. That is, the intersection of any number of open sets is open.

For example, consider the following digraph $(X, \Gamma)$, where $X = \{a, b, c, d\}$.



Then the topology associated to the above digraph is $\tau_{\Gamma}^{+} = \{\phi, X, \{d\}, \{b,d\}, \{b,c,d\}\}$.

Consequently, $\{A : A \subseteq X$ and there does not exist an edge from $A$ to $A^C$ in $(X,\Gamma)\}$ forms the topology on $X$ and it is denoted by $\tau_{\Gamma}^{-}$. Hence, we have a unique topological space $(X,\tau_{\Gamma}^{-})$. Thus, the topology associated to the digraph is $\tau_{\Gamma}^{-} = \{\phi, X, \{a\}, \{a,c\}, \{a,b,c\}\}$.

Now, we are comfortable to define the bitopological space $(X,\tau_{\Gamma}^{+},\tau_{\Gamma}^{-})$ with the help of these two unique topologies $\tau_{\Gamma}^{+},\tau_{\Gamma}^{-}$ associated to the digraph $(X,\Gamma)$, where $\tau_{\Gamma}^{+},\tau_{\Gamma}^{-}$ are the right and left associated topologies. Also, the topology $\tau_{\Gamma}^{+}$ is called the dual topology to $\tau_{\Gamma}^{-}$ and vise versa so that for every set $A \subseteq X$, the set $\tau_{\Gamma}^{+}$-cl$(A)$ is the least $\tau_{\Gamma}^{-}$-open set containing $A$ and the set $\tau_{\Gamma}^{-}$-cl$(A)$ is the least $\tau_{\Gamma}^{+}$-open set containing $A$. For any set $A \subseteq X$ of the digraph $(X,\Gamma)$, the closure of $A$ with respect to $\tau_{\Gamma}^{+}$ is defined by $\tau_{\Gamma}^{+}$-cl$(A) = \{p_j : p_j$ is accessible from $p_i$ for some $p_i \in A\}$. In digraph, $\tau_{\Gamma}^{+}$-cl$[\{c\}] = \{a,c\}$, since $a$ is the only point accessible from $c$. Also, $\tau_{\Gamma}^{-}$-cl$[\{c\}] = \{b,c,d\}$.

To retain the standard notation in the recent trend, $(X,\tau_1,\tau_2)$ will denote the bitopological space $(X,\tau_{\Gamma}^{+},\tau_{\Gamma}^{-})$. A set $A$ is semiopen [20] in a topological space $(X,\tau)$ if $A \subseteq \mathrm{cl}[\mathrm{int}(A)]$ and the complements of semiopen sets are called semiclosed sets. $\tau_j$-scl$(A)$ and $\tau_j$-cl$(A)$ represent the semiclosure and closure of a set $A$ with respect to the topology $\tau_j$, respectively, and they are defined by intersection of all $\tau_j$-semiclosed and $\tau_j$-closed sets containing $A$, respectively. Co $\tau_j$ represents the complements of members of $\tau_j$. Moreover, a set $A$ of a bitopological space $(X,\tau_1,\tau_2)$ is $\tau_i\tau_j$-semi generalized closed (resp., $\tau_i\tau_j$-generalized semiclosed, $\tau_i\tau_j$-semi star generalized closed [21–23]) if $\tau_j$-scl$(A) \subseteq U$ (resp., $\tau_j$-scl$(A) \subseteq U$, $\tau_j$-cl$(A) \subseteq U$) whenever $A \subseteq U$ and $U$ is $\tau_i$-semiopen (resp., $\tau_i$-open, $\tau_i$-semiopen) in $X$, $i,j = 1,2$ and $i \neq j$.

$\tau_i\tau_j$-semi generalized closed sets, $\tau_i\tau_j$-generalized semiclosed sets, and $\tau_i\tau_j$-semi star generalized closed sets are denoted by $\tau_i\tau_j$-sg closed sets, $\tau_i\tau_j$-gs closed sets, and $\tau_i\tau_j$-$s^*g$ closed sets, respectively.

## 3. Relations between Some Generalized Closed Sets

In this section, we discuss some relations between generalized closed sets in the bitopological spaces associated to the digraphs.

$\tau_1$-open (resp., $\tau_2$-open) sets and $\tau_i\tau_j$-$s^*g$ closed sets are independent for $i,j = 1,2$ and $i \neq j$ in general. For example, let $X = \{a,b,c\}$, $\tau_1 = \{\phi, X, \{a\}\}$, $\tau_2 = \{\phi, X, \{a\}, \{a,c\}\}$. Then $\{a\}$ is $\tau_1$-open but neither $\tau_1\tau_2$-$s^*g$ closed nor $\tau_2\tau_1$-$s^*g$ closed in X. Also, $\{b,c\}$ is both $\tau_1\tau_2$-$s^*g$ closed and $\tau_2\tau_1$-$s^*g$ closed, but not $\tau_1$-open in X. Similarly, $\{a,c\}$ is $\tau_2$-open but neither $\tau_1\tau_2$-$s^*g$ closed nor $\tau_2\tau_1$-$s^*g$ closed in X. Also $\{b,c\}$ is both $\tau_1\tau_2$-$s^*g$ closed and $\tau_2\tau_1$-$s^*g$ closed, but not $\tau_2$-open in X.

Similarly, $\tau_1$-closed (resp., $\tau_2$-closed) sets and $\tau_i\tau_j$-$s^*g$ closed sets are independent for $i,j$ =1,2 and $i \neq j$ in general. Since every $\tau_i = \mathrm{co}\,\tau_j$ in a bitopological space $(X,\tau_1,\tau_2)$ is associated to the digraph $(X,\Gamma)$ and every $\tau_i$-open set is $\tau_i\tau_j$-$s^*g$ open in every bitopological space X, we have every $\tau_j$-closed set is $\tau_i\tau_j$-$s^*g$ open in X for $i,j = 1,2$ and $i \neq j$. Also, every $\tau_j$-closed set is $\tau_i\tau_j$-$s^*g$ closed in X and hence every $\tau_i$-open set is $\tau_i\tau_j$-$s^*g$ closed in X associated to the digraph $(X,\Gamma)$ for $i,j = 1,2$ and $i \neq j$.

Suppose that $A$ is $\tau_i$-open in X. Then $A^C$ is $\tau_i$-closed and hence it is $\tau_j\tau_i$-closed in X. Also $A$ is $\tau_j$-closed and hence $A^C$ is $\tau_j$-open in X. This implies that $A$ is $\tau_j\tau_i$-closed in X associated to the digraph $(X,\Gamma)$ for $i,j = 1,2$ and $i \neq j$. So we have the following.

**Theorem 3.1.** *Every $\tau_1$-open (resp., $\tau_2$-open) set is both $\tau_i\tau_j$-$s^*g$ closed and $\tau_i\tau_j$-$s^*g$ open in X associated to the digraph $(X, \Gamma)$ for $i, j = 1, 2$ and $i \neq j$.*

**Theorem 3.2.** *Every $\tau_1$-closed (resp., $\tau_2$-closed) set is both $\tau_i\tau_j$-$s^*g$ closed and $\tau_i\tau_j$-$s^*g$ open in X associated to the digraph $(X, \Gamma)$ for $i, j = 1, 2$ and $i \neq j$.*

Since every $\tau_i\tau_j$-$s^*g$ closed (resp., $\tau_i\tau_j$-$s^*g$ open) sets are $\tau_i\tau_j$-$g$ closed, $\tau_i\tau_j$-sg closed and $\tau_i\tau_j$-gs closed (resp., $\tau_i\tau_j$-$g$ open, $\tau_i\tau_j$-sg open and $\tau_i\tau_j$-gs open) in $X$, one can obtain the following:

**Theorem 3.3.** *Every member of both $\tau_1$ and $\tau_2$ is $\tau_i\tau_j$-$g$ closed, $\tau_i\tau_j$-$sg$ closed, $\tau_i\tau_j$-$gs$ closed, $\tau_i\tau_j$-$g$ open, $\tau_i\tau_j$-$sg$ open and $\tau_i\tau_j$-$gs$ open, in X associated to the digraph $(X, \Gamma)$ for $i, j = 1, 2$ and $i \neq j$.*

A subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is $\tau_i\tau_j$-nowhere dense (resp., $\tau_i\tau_j$-somewhere dense) if $\tau_i$-int$[\tau_j$-cl$(A)] = \phi$ (resp., $\tau_i$-int$[\tau_j$-cl$(A)] \neq \phi$). Clearly, $\tau_i\tau_j$-nowhere dense sets and $\tau_i\tau_j$-$s^*g$ closed sets are independent for $i, j = 1, 2$ and $i \neq j$ in general. For example, let $X = \{a, b, c\}, \tau_1 = \{\phi, X, \{a\}\}, \tau_2 = \{\phi, X, \{a\}, \{b, c\}\}$. Then $\{a\}$ is $\tau_1\tau_2$-$s^*g$ closed but not $\tau_1\tau_2$-nowhere dense in X. Also, $\{b\}$ is $\tau_1\tau_2$- nowhere dense but not $\tau_1\tau_2$-$s^*g$ closed in X.

Suppose that $A$ is $\tau_i\tau_j$-nowhere dense in a bitopological space $(X, \tau_1, \tau_2)$ associated to the digraph $(X, \Gamma)$. Then $\tau_i$-int$[\tau_j$-cl$(A)] = \phi$. Since $\tau_i$ = co, $\tau_j$, one has $\tau_j$-cl$(A) = \phi$. This implies that $A = \phi$. Hence, $A$ is $\tau_i\tau_j$-$g$ closed, $\tau_i\tau_j$-$sg$ closed, $\tau_i\tau_j$-$gs$ closed, $\tau_i\tau_j$-$s^*g$ closed, $\tau_i\tau_j$-$g$ open, $\tau_i\tau_j$-$sg$ open, $\tau_i\tau_j$-$gs$ open, and $\tau_i\tau_j$-$s^*g$ open in X associated to the digraph $(X, \Gamma)$ for $i, j = 1, 2$ and $i \neq j$.

Therefore, one can conclude that every nonempty $\tau_i\tau_j$-$g$ closed (resp., $\tau_i\tau_j$-$sg$ closed, $\tau_i\tau_j$-$gs$ closed, $\tau_i\tau_j$-$s^*g$ closed, $\tau_i\tau_j$-$g$ open, $\tau_i\tau_j$-$sg$ open, $\tau_i\tau_j$-$gs$ open, and $\tau_i\tau_j$-$s^*g$ open) set is $\tau_i\tau_j$-somewhere dense in $X$ associated to the digraph $(X, \Gamma)$ for $i, j = 1, 2$ and $i \neq j$.

Since the set $\tau_j$-cl$(A)$ is the least $\tau_i$-open set containing $A$ in the bitopological space X associated to the digraph $(X, \Gamma)$, $\tau_j$-cl$(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\tau_i$-open, for $i, j = 1, 2$ and $i \neq j$. Hence every subset $A \subseteq X$ of the digraph $(X, \Gamma)$ is $\tau_i\tau_j$-$g$ closed and hence $\tau_i\tau_j$-$g$ open.

## 4. Conclusion

Thus, we have discussed the nature and properties of some generalized closed sets in the bitopological spaces associated to the digraphs in this short communication. This may be a new beginning for further research on the study of generalized closed sets in the bitopological spaces associated to the directed graphs. Hence, further research may be undertaken towards this direction. That is, one may take further research to find the suitable way of defining the bitopological spaces associated to the digraphs by using bitopological generalized closed sets such that there is a one-to-one correspondence between them. It may also lead to the new properties of separation axioms on these spaces.

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