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Research Article

A General Iterative Method for a Nonexpansive Semigroup in Banach Spaces with Gauge Functions

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We study strong convergence of the sequence generated by implicit and explicit general iterative methods for a one-parameter nonexpansive semigroup in a reflexive Banach space which admits the duality mapping J_φ, where φ is a gauge function on $[0,\infty)$. Our results improve and extend those announced by G. Marino and H.-K. Xu (2006) and many authors.

1. Introduction

Let E be a real Banach space and E^* the dual space of E. Let K be a nonempty, closed, and convex subset of E. A (one-parameter) nonexpansive semigroup is a family $\mathfrak{F} = \{T(t) : t \geq 0\}$ of self-mappings of K such that

(i) $T(0)x = x$ for all $x \in K$,

(ii) $T(t+s)x = T(t)T(s)x$ for all $t, s \geq 0$ and $x \in K$,

(iii) for each $x \in K$, the mapping $T(\cdot)x$ is continuous,

(iv) for each $t \geq 0$, $T(t)$ is nonexpansive, that is,

$$\|T(t)x - T(t)y\| \leq \|x - y\|, \quad \forall x, y \in K. \tag{1.1}$$

We denote F by the common fixed points set of $\mathfrak{F}$, that is, $F := \bigcap_{t\geq 0} F(T(t))$.

In 1967, Halpern [1] introduced the following classical iteration for a nonexpansive mapping $T : K \to K$ in a real Hilbert space:

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) T x_n, \quad n \geq 0, \tag{1.2}$$

where $\{\alpha_n\} \subset (0,1)$ and $u \in K$.

In 1977, Lions [2] obtained a strong convergence provide the real sequence $\{\alpha_n\}$ satisfies the following conditions:

C1: $\lim_{n\to\infty}\alpha_n = 0$; C2: $\sum_{n=0}^{\infty} \alpha_n = \infty$; C3: $\lim_{n\to\infty}(\alpha_n - \alpha_{n-1})/\alpha_n^2 = 0$.

Reich [3] also extended the result of Halpern from Hilbert spaces to uniformly smooth Banach spaces. However, both Halpern's and Lion's conditions imposed on the real sequence $\{\alpha_n\}$ excluded the canonical choice $\alpha_n = 1/(n+1)$.

In 1992, Wittmann [4] proved that the sequence $\{x_n\}$ converges strongly to a fixed point of T if $\{\alpha_n\}$ satisfies the following conditions:

C1: $\lim_{n\to\infty}\alpha_n = 0$; C2: $\sum_{n=0}^{\infty} \alpha_n = \infty$; C3: $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

Shioji and Takahashi [5] extended Wittmann's result to real Banach spaces with uniformly Gâteaux differentiable norms and in which each nonempty closed convex and bounded subset has the fixed point property for nonexpansive mappings. The concept of the Halpern iterative scheme has been widely used to approximate the fixed points for nonexpansive mappings (see, e.g., [6–12] and the reference cited therein).

Let $f : K \to K$ be a contraction. In 2000, Moudafi [13] introduced the explicit viscosity approximation method for a nonexpansive mapping T as follows:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T x_n, \quad n \geq 0, \tag{1.3}$$

where $\alpha_n \in (0,1)$. Xu [14] also studied the iteration process (1.3) in uniformly smooth Banach spaces.

Let A be a strongly positive bounded linear operator on a real Hilbert space H, that is, there is a constant $\overline{\gamma} > 0$ such that

$$\langle Ax, x \rangle \geq \overline{\gamma} \|x\|^2, \quad \forall x \in H. \tag{1.4}$$

A typical problem is to minimize a quadratic function over the fixed points set of a nonexpansive mapping on a Hilbert space H:

$$\min_{x \in C} \frac{1}{2} \langle Ax, x \rangle - \langle x, b \rangle, \tag{1.5}$$

where C is the fixed points set of a nonexpansive mapping T on H and b is a given point in H.

In 2006, Marino and Xu [15] introduced the following general iterative method for a nonexpansive mapping T in a Hilbert space H:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A) T x_n, \quad n \geq 1, \tag{1.6}$$

where $\{\alpha_n\} \subset (0,1)$, f is a contraction on H, and A is a strongly positive bounded linear operator on H. They proved that the sequence $\{x_n\}$ generated by (1.6) converges strongly to a fixed point $x^* \in F(T)$ which also solves the variational inequality

$$\langle (A-\gamma f)x^*, x-x^* \rangle \geq 0, \quad \forall x \in F(T), \tag{1.7}$$

which is the optimality condition for the minimization problem: $\min_{x\in C}(1/2)\langle Ax, x\rangle - h(x)$, where h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for $x \in H$).

Suzuki [16] first introduced the following implicit viscosity method for a nonexpansive semigroup $\{T(t) : t \geq 0\}$ in a Hilbert space:

$$x_n = \alpha_n u + (1-\alpha_n)T(t_n)x_n, \quad n \geq 1, \tag{1.8}$$

where $\{\alpha_n\} \subset (0,1)$ and $u \in K$. He proved strong convergence of iteration (1.8) under suitable conditions. Subsequently, Xu [17] extended Suzuki's [16] result from a Hilbert space to a uniformly convex Banach space which admits a weakly sequentially continuous normalized duality mapping.

Motivated by Chen and Song [18], in 2007, Chen and He [19] investigated the implicit and explicit viscosity methods for a nonexpansive semigroup without integral in a reflexive Banach space which admits a weakly sequentially continuous normalized duality mapping:

$$x_n = \alpha_n f(x_n) + (1-\alpha_n)T(t_n)x_n, \quad n \geq 1, \tag{1.9}$$

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T(t_n)x_n, \quad n \geq 1, \tag{1.10}$$

where $\{\alpha_n\} \subset (0,1)$.

In 2008, Song and Xu [20] also studied the iterations (1.9) and (1.10) in a reflexive and strictly convex Banach space with a Gâteaux differentiable norm. Subsequently, Cholamjiak and Suantai [21] extended Song and Xu's results to a Banach space which admits duality mapping with a gauge function. Wangkeeree and Kamraksa [22] and Wangkeeree et al. [23] obtained the convergence results concerning the duality mapping with a gauge function in Banach spaces. The convergence of iterations for a nonexpansive semigroup and nonlinear mappings has been studied by many authors (see, e.g., [24–38]).

Let E be a real reflexive Banach space which admits the duality mapping J_φ with a gauge φ. Let $\{T(t) : t \geq 0\}$ be a nonexpansive semigroup on E. Recall that an operator A is said to be *strongly positive* if there exists a constant $\overline{\gamma} > 0$ such that

$$\begin{gathered}\langle Ax, J_\varphi(x) \rangle \geq \overline{\gamma}\|x\|\varphi(\|x\|), \\ \|\alpha I - \beta A\| = \sup_{\|x\|\leq 1} \left| \langle (\alpha I - \beta A)x, J_\varphi(x) \rangle \right|,\end{gathered} \tag{1.11}$$

where $\alpha \in [0,1]$ and $\beta \in [-1,1]$.

Motivated by Chen and Song [18], Chen and He [19], Marino and Xu [15], Colao et al. [39], and Wangkeeree et al. [23], we study strong convergence of the following general iterative methods:

$$x_n = \alpha_n \gamma f(x_n) + (I - \alpha_n A)T(t_n)x_n, \quad n \geq 1, \tag{1.12}$$

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)T(t_n)x_n, \quad n \geq 1, \tag{1.13}$$

where $\{\alpha_n\} \subset (0,1)$, f is a contraction on E and A is a positive bounded linear operator on E.

2. Preliminaries

A Banach space E is called *strictly convex* if $\|x+y\|/2 < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. A Banach space E is called *uniformly convex* if for each $\epsilon > 0$ there is a $\delta > 0$ such that for $x, y \in E$ with $\|x\|, \|y\| \leq 1$ and $\|x-y\| \geq \epsilon$, $\|x+y\| \leq 2(1-\delta)$ holds. The *modulus of convexity* of E is defined by

$$\delta_E(\epsilon) = \inf\left\{1 - \left\|\frac{1}{2}(x+y)\right\| : \|x\|, \|y\| \leq 1, \ \|x-y\| \geq \epsilon\right\}, \tag{2.1}$$

for all $\epsilon \in [0,2]$. E is uniformly convex if $\delta_E(0) = 0$, and $\delta_E(\epsilon) > 0$ for all $0 < \epsilon \leq 2$. It is known that every uniformly convex Banach space is strictly convex and reflexive. Let $S(E) = \{x \in E : \|x\| = 1\}$. Then the norm of E is said to be *Gâteaux differentiable* if

$$\lim_{t \to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{2.2}$$

exists for each $x, y \in S(E)$. In this case E is called *smooth*. The norm of E is said to be *Fréchet differentiable* if for each $x \in S(E)$, the limit is attained uniformly for $y \in S(E)$. The norm of E is called *uniformly Fréchet differentiable*, if the limit is attained uniformly for $x, y \in S(E)$. It is well known that (uniformly) *Fréchet differentiability* of the norm of E implies (uniformly) *Gâteaux differentiability* of the norm of E.

Let $\rho_E : [0,\infty) \to [0,\infty)$ be the *modulus of smoothness* of E defined by

$$\rho_E(t) = \sup\left\{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : \ x \in S(E), \quad \|y\| \leq t\right\}. \tag{2.3}$$

A Banach space E is called *uniformly smooth* if $\rho_E(t)/t \to 0$ as $t \to 0$. See [40–42] for more details.

We need the following definitions and results which can be found in [40, 41, 43].

Definition 2.1. A continuous strictly increasing function $\varphi : [0,\infty) \to [0,\infty)$ is said to be gauge function if $\varphi(0) = 0$ and $\lim_{t\to\infty}\varphi(t) = \infty$.

Definition 2.2. Let E be a normed space and φ a gauge function. Then the mapping $J_\varphi : E \to 2^{E^*}$ defined by

$$J_\varphi(x) = \{f^* \in E^* : \langle x, f^* \rangle = \|x\|\varphi(\|x\|), \ \|f^*\| = \varphi(\|x\|)\}, \quad x \in E, \tag{2.4}$$

is called the duality mapping with gauge function φ.

In the particular case $\varphi(t) = t$, the duality mapping $J_\varphi = J$ is called the normalized duality mapping.

In the case $\varphi(t) = t^{q-1}$, $q > 1$, the duality mapping $J_\varphi = J_q$ is called the generalized duality mapping. It follows from the definition that $J_\varphi(x) = \varphi(\|x\|)/\|x\|J(x)$ and $J_q(x) = \|x\|^{q-2}J(x)$, $q > 1$.

Remark 2.3. For the gauge function φ, the function $\Phi : [0,\infty) \to [0,\infty)$ defined by

$$\Phi(t) = \int_0^t \varphi(s)ds \tag{2.5}$$

is a continuous convex and strictly increasing function on $[0,\infty)$. Therefore, Φ has a continuous inverse function Φ^{-1}.

It is noted that if $0 \le k \le 1$, then $\varphi(kx) \le \varphi(x)$. Further

$$\Phi(kt) = \int_0^{kt} \varphi(s)ds = k\int_0^t \varphi(kx)dx \le k\int_0^t \varphi(x)dx = k\Phi(t). \tag{2.6}$$

Remark 2.4. For each x in a Banach space E, $J_\varphi(x) = \partial\Phi(\|x\|)$, where ∂ denotes the subdifferential.

We also know the following facts:

(i) J_φ is a nonempty, closed, and convex set in E^* for each $x \in E$,

(ii) J_φ is a function when E^* is strictly convex,

(iii) If J_φ is single-valued, then

$$\begin{gathered} J_\varphi(\lambda x) = \frac{sign(\lambda)\varphi(\|\lambda x\|)}{\varphi(\|x\|)} J_\varphi(x), \qquad \forall x \in E, \lambda \in \mathbb{R}, \\ \langle x - y, J_\varphi(x) - J_\varphi(y)\rangle \ge (\varphi(\|x\|) - \varphi(\|y\|))(\|x\| - \|y\|), \qquad \forall x, y \in E. \end{gathered} \tag{2.7}$$

Following Browder [43], we say that a Banach space E has a weakly continuous duality mapping if there exists a gauge φ for which the duality mapping J_φ is single-valued and continuous from the weak topology to the weak* topology, that is, for any $\{x_n\}$ with $x_n \rightharpoonup x$, the sequence $\{J_\varphi(x_n)\}$ converges weakly* to $J_\varphi(x)$. It is known that the space ℓ^p has a weakly continuous duality mapping with a gauge function $\varphi(t) = t^{p-1}$ for all $1 < p < \infty$. Moreover, φ is invariant on $[0,1]$.

Lemma 2.5 (See [44]). *Assume that a Banach space E has a weakly continuous duality mapping J_φ with gauge φ.*

(i) *For all $x, y \in E$, the following inequality holds:*

$$\Phi(\|x+y\|) \leq \Phi(\|x\|) + \langle y, J_\varphi(x+y)\rangle. \tag{2.8}$$

In particular, for all $x, y \in E$,

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, J(x+y)\rangle. \tag{2.9}$$

(ii) *Assume that a sequence $\{x_n\}$ in E converges weakly to a point $x \in E$. Then the following holds:*

$$\limsup_{n\to\infty} \Phi(\|x_n - y\|) = \limsup_{n\to\infty} \Phi(\|x_n - x\|) + \Phi(\|x - y\|) \tag{2.10}$$

for all $x, y \in E$.

Lemma 2.6 (See [23]). *Assume that a Banach space E has a weakly continuous duality mapping J_φ with gauge φ. Let A be a strongly positive bounded linear operator on E with coefficient $\overline{\gamma} > 0$ and $0 < \rho \leq \varphi(1)\|A\|^{-1}$. Then $\|I - \rho A\| \leq \varphi(1)(1 - \rho\overline{\gamma})$.*

Lemma 2.7 (See [12]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\gamma_n)a_n + \gamma_n\delta_n, \quad n \geq 1, \tag{2.11}$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that
(a) $\sum_{n=1}^{\infty} \gamma_n = \infty$; (b) $\limsup_{n\to\infty} \delta_n \leq 0$ or $\sum_{n=1}^{\infty} |\gamma_n\delta_n| < \infty$.
Then $\lim_{n\to\infty} a_n = 0$.

3. Implicit Iteration Scheme

In this section, we prove a strong convergence theorem of an implicit iterative method (1.12).

Theorem 3.1. *Let E be a reflexive which admits a weakly continuous duality mapping J_φ with gauge φ such that φ is invariant on $[0,1]$. Let $\mathfrak{F} = \{T(t) : t \geq 0\}$ be a nonexpansive semigroup on E such that $F \neq \emptyset$. Let f be a contraction on E with the coefficient $\alpha \in (0,1)$ and A a strongly positive bounded linear operator with coefficient $\overline{\gamma} > 0$ and $0 < \gamma < \overline{\gamma}\varphi(1)/\alpha$. Let $\{\alpha_n\}$ and $\{t_n\}$ be real sequences satisfying $0 < \alpha_n < 1$, $t_n > 0$ and $\lim_{n\to\infty} t_n = \lim_{n\to\infty} \alpha_n/t_n = 0$. Then $\{x_n\}$ defined by (1.12) converges strongly to $q \in F$ which solves the following variational inequality:*

$$\langle (A - \gamma f)(q), J_\varphi(q - w)\rangle \leq 0, \quad \forall w \in F. \tag{3.1}$$

Proof. First, we prove the uniqueness of the solution to the variational inequality (3.1) in F. Suppose that $p, q \in F$ satisfy (3.1), so we have

$$\begin{aligned} &\langle (A-\gamma f)(p), J_\varphi(p-q)\rangle \le 0, \\ &\langle (A-\gamma f)(q), J_\varphi(q-p)\rangle \le 0. \end{aligned} \tag{3.2}$$

Adding the above inequalities, we get

$$\langle A(p) - A(q) - \gamma(f(p) - f(q)), J_\varphi(p-q)\rangle \le 0. \tag{3.3}$$

This shows that

$$\langle A(p-q), J_\varphi(p-q)\rangle \le \gamma\langle f(p) - f(q), J_\varphi(p-q)\rangle, \tag{3.4}$$

which implies by the strong positivity of A

$$\overline{\gamma}\|p-q\|\varphi(\|p-q\|) \le \langle A(p-q), J_\varphi(p-q)\rangle \le \gamma\alpha\|p-q\|\varphi(\|p-q\|). \tag{3.5}$$

Since φ is invariant on $[0,1]$,

$$\varphi(1)\overline{\gamma}\|p-q\|\varphi(\|p-q\|) \le \gamma\alpha\|p-q\|\varphi(\|p-q\|). \tag{3.6}$$

It follows that

$$(\varphi(1)\overline{\gamma} - \gamma\alpha)\|p-q\|\varphi(\|p-q\|) \le 0. \tag{3.7}$$

Therefore $p = q$ since $0 < \gamma < (\overline{\gamma}\varphi(1))/\alpha$.

We next prove that $\{x_n\}$ is bounded. For each $w \in F$, by Lemma 2.6, we have

$$\begin{aligned} \|x_n - w\| &= \|\alpha_n\gamma f(x_n) + (I-\alpha_n A)T(t_n)x_n - w\| \\ &= \|(I-\alpha_n A)T(t_n)x_n - (I-\alpha_n A)w + \alpha_n(\gamma f(x_n) - A(w))\| \\ &\le \varphi(1)(1-\alpha_n\overline{\gamma})\|x_n - w\| + \alpha_n(\gamma\alpha\|x_n - w\| + \|\gamma f(w) - A(w)\|) \\ &\le \|x_n - w\| - \alpha_n\varphi(1)\overline{\gamma}\|x_n - w\| + \alpha_n\gamma\alpha\|x_n - w\| + \alpha_n\|\gamma f(w) - A(w)\|, \end{aligned} \tag{3.8}$$

which yields

$$\|x_n - w\| \le \frac{1}{\varphi(1)\overline{\gamma} - \gamma\alpha}\|\gamma f(w) - A(w)\|. \tag{3.9}$$

Hence $\{x_n\}$ is bounded. So are $\{f(x_n)\}$ and $\{AT(t_n)x_n\}$.

We next prove that $\{x_n\}$ is relatively sequentially compact. By the reflexivity of E and the boundedness of $\{x_n\}$, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and a point p in E such that $x_{n_j} \rightharpoonup p$ as $j \to \infty$. Now we show that $p \in F$. Put $x_j = x_{n_j}$, $\beta_j = \alpha_{n_j}$ and $s_j = t_{n_j}$ for $j \in \mathbb{N}$, fix $t > 0$. We see that

$$\begin{aligned}
\|x_j - T(t)p\| &\le \sum_{k=0}^{[t/s_j]-1} \|T((k+1)s_j)x_j - T(ks_j)x_{j+1}\| \\
&\quad + \left\|T\left(\left[\frac{t}{s_j}\right]s_j\right)x_j - T\left(\left[\frac{t}{s_j}\right]s_j\right)p\right\| + \left\|T\left(\left[\frac{t}{s_j}\right]s_j\right)p - T(t)p\right\| \\
&\le \left[\frac{t}{s_j}\right]\|T(s_j)x_j - x_j\| + \|x_j - p\| + \left\|T\left(t - \left[\frac{t}{s_j}\right]s_j\right)p - p\right\| \\
&= \left[\frac{t}{s_j}\right]\beta_j\|AT(s_j)x_j - \gamma f(x_j)\| + \|x_j - p\| + \left\|T\left(t - \left[\frac{t}{s_j}\right]s_j\right)p - p\right\| \\
&\le \frac{t\beta_j}{s_j}\|AT(s_j)x_j - \gamma f(x_j)\| + \|x_j - p\| \\
&\quad + \max\{\|T(s)p - p\| : 0 \le s \le s_j\}.
\end{aligned} \tag{3.10}$$

So we have

$$\limsup_{j\to\infty}\Phi(\|x_j - T(t)p\|) \le \limsup_{j\to\infty}\Phi(\|x_j - p\|). \tag{3.11}$$

On the other hand, by Lemma 2.5 (ii), we have

$$\limsup_{j\to\infty}\Phi(\|x_j - T(t)p\|) = \limsup_{j\to\infty}\Phi(\|x_j - p\|) + \Phi(\|T(t)p - p\|). \tag{3.12}$$

Combining (3.11) and (3.12), we have

$$\Phi(\|T(t)p - p\|) \le 0. \tag{3.13}$$

This implies that $p \in F$. Further, we see that

$$\begin{aligned}
\|x_j - p\|\varphi(\|x_j - p\|) &= \langle x_j - p, J_\varphi(x_j - p)\rangle \\
&= \langle (I - \beta_j A)T(s_j)x_j - (I - \beta_j A)p, J_\varphi(x_j - p)\rangle \\
&\quad + \beta_j\langle \gamma f(x_j) - \gamma f(p), J_\varphi(x_j - p)\rangle + \beta_j\langle \gamma f(p) - A(p), J_\varphi(x_j - p)\rangle \\
&\le \varphi(1)(1 - \beta_j\overline{\gamma})\|x_j - p\|\varphi(\|x_j - p\|) \\
&\quad + \beta_j\gamma\alpha\|x_j - p\|\varphi(\|x_j - p\|) + \beta_j\langle \gamma f(p) - A(p), J_\varphi(x_j - p)\rangle.
\end{aligned} \tag{3.14}$$

So we have

$$\|x_j - p\|\varphi(\|x_j - p\|) \leq \frac{1}{\varphi(1)\overline{\gamma} - \gamma\alpha}\langle \gamma f(p) - A(p), J_\varphi(x_j - p)\rangle. \tag{3.15}$$

By the definition of Φ, it is easily seen that

$$\Phi(\|x_j - p\|) \leq \|x_j - p\|\varphi(\|x_j - p\|). \tag{3.16}$$

Hence

$$\Phi(\|x_j - p\|) \leq \frac{1}{\varphi(1)\overline{\gamma} - \gamma\alpha}\langle \gamma f(p) - A(p), J_\varphi(x_j - p)\rangle. \tag{3.17}$$

Therefore $\Phi(\|x_j - p\|) \to 0$ as $j \to \infty$ since J_φ is weakly continuous; consequently, $x_j \to p$ as $j \to \infty$ by the continuity of Φ. Hence $\{x_n\}$ is relatively sequentially compact.

Finally, we prove that p is a solution in F to the variational inequality (3.1). For any $w \in F$, we see that

$$\begin{aligned}
\langle (I - T(t_n))x_n - (I - T(t_n))w, J_\varphi(x_n - w)\rangle &= \langle x_n - w, J_\varphi(x_n - w)\rangle \\
&\quad - \langle T(t_n)x_n - T(t_n)w, J_\varphi(x_n - w)\rangle \\
&\geq \|x_n - w\|\varphi\|x_n - w\| \\
&\quad - \|T(t_n)x_n - T(t_n)w\|\|J_\varphi(x_n - w)\| \\
&\geq \|x_n - w\|\varphi\|x_n - w\| \\
&\quad - \|x_n - w\|\|J_\varphi(x_n - w)\| \\
&= 0.
\end{aligned} \tag{3.18}$$

On the other hand, we have

$$(A - \gamma f)(x_n) = -\frac{1}{\alpha_n}(I - \alpha_n A)(I - T(t_n))x_n, \tag{3.19}$$

which implies

$$\begin{aligned}
\langle (A - \gamma f)(x_n), J_\varphi(x_n - w)\rangle &= -\frac{1}{\alpha_n}\langle (I - T(t_n))x_n - (I - T(t_n))w, J_\varphi(x_n - w)\rangle \\
&\quad + \langle A(I - T(t_n))x_n, J_\varphi(x_n - w)\rangle \\
&\leq \langle A(I - T(t_n))x_n, J_\varphi(x_n - w)\rangle.
\end{aligned} \tag{3.20}$$

Observe

$$\|x_j - T(s_j)x_j\| = \beta_j\|\gamma f(x_j) - AT(s_j)x_j\| \to 0, \tag{3.21}$$

as $j \to \infty$. Replacing n by n_j and letting $j \to \infty$ in (3.20), we obtain

$$\langle (A-\gamma f)(p), J_\varphi(p-w)\rangle \leq 0, \quad \forall w \in F. \tag{3.22}$$

So $p \in F$ is a solution of variational inequality (3.1); and hence $p = q$ by the uniqueness. In a summary, we have proved that $\{x_n\}$ is relatively sequentially compact and each cluster point of $\{x_n\}$ (as $n \to \infty$) equals q. Therefore $x_n \to q$ as $n \to \infty$. This completes the proof. □

4. Explicit Iteration Scheme

In this section, utilizing the implicit version in Theorem 3.1, we consider the explicit one in a reflexive Banach space which admits the duality mapping J_φ.

Theorem 4.1. *Let E be a reflexive Banach space which admits a weakly continuous duality mapping J_φ with gauge φ such that φ is invariant on [0,1]. Let $\{T(t) : t \geq 0\}$ be a nonexpansive semigroup on E such that $F \neq \emptyset$. Let f be a contraction on E with the coefficient $\alpha \in (0,1)$ and A a strongly positive bounded linear operator with coefficient $\overline{\gamma} > 0$ and $0 < \gamma < \overline{\gamma}\,\varphi(1)/\alpha$. Let $\{\alpha_n\}$ and $\{t_n\}$ be real sequences satisfying $0 < \alpha_n < 1$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $t_n > 0$ and $\lim_{n\to\infty} t_n = \lim_{n\to\infty} \alpha_n/t_n = 0$. Then $\{x_n\}$ defined by* (1.13) *converges strongly to $q \in F$ which also solves the variational inequality* (3.1).

Proof. Since $\alpha_n \to 0$, we may assume that $\alpha_n < \varphi(1)\|A\|^{-1}$ and $1-\alpha_n(\varphi(1)\overline{\gamma}-\gamma\alpha) > 0$ for all n. First we prove that $\{x_n\}$ is bounded. For each $w \in F$, by Lemma 2.6, we have

$$\begin{aligned}
\|x_{n+1}-w\| &= \left\|\alpha_n\gamma f(x_n) + (I-\alpha_n A)T(t_n)x_n - w\right\| \\
&= \left\|(I-\alpha_n A)T(t_n)x_n - (I-\alpha_n A)w + \alpha_n\left(\gamma f(x_n) - A(w)\right)\right\| \\
&\leq \varphi(1)\left(1-\alpha_n\overline{\gamma}\right)\|x_n - w\| + \alpha_n\gamma\alpha\|x_n-w\| + \alpha_n\left\|\gamma f(w) - A(w)\right\| \\
&= \left(\varphi(1) - \alpha_n\left(\varphi(1)\overline{\gamma}-\gamma\alpha\right)\right)\|x_n-w\| + \alpha_n\left\|\gamma f(w)-A(w)\right\| \\
&\leq \left(1-\alpha_n\left(\varphi(1)\overline{\gamma}-\gamma\alpha\right)\right)\|x_n-w\| + \alpha_n\left(\varphi(1)\overline{\gamma}-\gamma\alpha\right)\frac{\left\|\gamma f(w)-A(w)\right\|}{\varphi(1)\overline{\gamma}-\gamma\alpha}.
\end{aligned} \tag{4.1}$$

It follows from induction that

$$\|x_{n+1}-w\| \leq \max\left\{\|x_1-w\|, \frac{\left\|\gamma f(w)-A(w)\right\|}{\varphi(1)\overline{\gamma}-\gamma\alpha}\right\}, \quad n \geq 1. \tag{4.2}$$

Thus $\{x_n\}$ is bounded, and hence so are $\{f(x_n)\}$ and $\{AT(t_n)x_n\}$. From Theorem 3.1, there is a unique solution $q \in F$ to the following variational inequality:

$$\langle (A-\gamma f)q, J_\varphi(q-w)\rangle \leq 0, \quad \forall w \in F. \tag{4.3}$$

Next we prove that

$$\limsup_{n\to\infty}\langle (A-\gamma f)q, J_\varphi(q-x_{n+1})\rangle \le 0. \tag{4.4}$$

Indeed, we can choose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle (A-\gamma f)q, J_\varphi(q-x_n)\rangle = \limsup_{j\to\infty}\left\langle (A-\gamma f)q, J_\varphi\left(q-x_{n_j}\right)\right\rangle. \tag{4.5}$$

Further, we can assume that $x_{n_j} \rightharpoonup p \in E$ by the reflexivity of E and the boundedness of $\{x_n\}$. Now we show that $p\in F$. Put $x_j = x_{n_j}, \beta_j = \alpha_{n_j}$ and $s_j = t_{n_j}$ for $j\in\mathbb{N}$, fix $t>0$. We obtain

$$\begin{aligned}
\|x_{j+1}-T(t)p\| &\le \sum_{k=0}^{[t/s_j]-1}\|T((k+1)s_j)x_j - T(ks_j)x_{j+1}\| \\
&\quad + \left\|T\left(\left[\frac{t}{s_j}\right]s_j\right)x_j - T\left(\left[\frac{t}{s_j}\right]s_j\right)p\right\| + \left\|T\left(\left[\frac{t}{s_j}\right]s_j\right)p - T(t)p\right\| \\
&\le \left[\frac{t}{s_j}\right]\|T(s_j)x_j - x_{j+1}\| + \|x_j-p\| + \left\|T\left(t-\left[\frac{t}{s_j}\right]s_j\right)p-p\right\| \\
&= \left[\frac{t}{s_j}\right]\beta_j\|AT(s_j)x_j - \gamma f(x_j)\| + \|x_j-p\| + \left\|T\left(t-\left[\frac{t}{s_j}\right]s_j\right)p-p\right\| \\
&\le \frac{t\beta_j}{s_j}\|AT(s_j)x_j-\gamma f(x_j)\| + \|x_j-p\| \\
&\quad + \max\{\|T(s)p-p\| : 0\le s\le s_j\}.
\end{aligned} \tag{4.6}$$

It follows that $\limsup_{n\to\infty}\Phi(\|x_j - T(t)p\|) \le \limsup_{n\to\infty}\Phi(\|x_j-p\|)$. From Lemma 2.5 (ii) we have

$$\limsup_{n\to\infty}\Phi(\|x_j-T(t)p\|) = \limsup_{n\to\infty}\Phi(\|x_j-p\|) + \Phi(\|T(t)p-p\|). \tag{4.7}$$

So we have $\Phi(\|T(t)p-p\|) \le 0$ and hence $p\in F$. Since the duality mapping J_φ is weakly sequentially continuous,

$$\begin{aligned}
\limsup_{n\to\infty}\langle (A-\gamma f)q, J_\varphi(q-x_{n+1})\rangle &= \limsup_{j\to\infty}\left\langle (A-\gamma f)q, J_\varphi\left(q-x_{n_j+1}\right)\right\rangle \\
&= \langle (A-\gamma f)q, J_\varphi(q-p)\rangle \le 0.
\end{aligned} \tag{4.8}$$

Finally, we show that $x_n \to q$. From Lemma 2.5 (i), we have

$$\begin{aligned}
\Phi(\|x_{n+1}-q\|) &= \Phi(\|(I-\alpha_n A)T(t_n)x_n-(I-\alpha_n A)q+\alpha_n(\gamma f(x_n)-\gamma f(q)) \\
&\quad +\alpha_n(\gamma f(q)-A(q))\|) \\
&\le \Phi(\|(I-\alpha_n A)(T(t_n)x_n-q)+\alpha_n(\gamma f(x_n)-\gamma f(q))\|) \\
&\quad + \alpha_n\langle \gamma f(q)-A(q), J_\varphi(x_{n+1}-q)\rangle \\
&\le \Phi(\varphi(1)(1-\alpha_n\overline{\gamma})\|x_n-q\|+\alpha_n\gamma\alpha\|x_n-q\|) \\
&\quad + \alpha_n\langle \gamma f(q)-A(q), J_\varphi(x_{n+1}-q)\rangle \\
&= \Phi((\varphi(1)-\alpha_n(\varphi(1)\overline{\gamma}-\gamma\alpha))\|x_n-q\|) \\
&\quad + \alpha_n\langle \gamma f(q)-A(q), J_\varphi(x_{n+1}-q)\rangle \\
&\le (1-\alpha_n(\varphi(1)\overline{\gamma}-\gamma\alpha))\Phi(\|x_n-q\|) \\
&\quad + \alpha_n\langle \gamma f(q)-A(q), J_\varphi(x_{n+1}-q)\rangle.
\end{aligned} \tag{4.9}$$

Note that $\sum_{n=1}^{\infty}\alpha_n=\infty$ and $\limsup_{n\to\infty}\langle \gamma f(q)-A(q), J_\varphi(x_{n+1}-q)\rangle \le 0$. Using Lemma 2.7, we have $x_n \to q$ as $n \to \infty$ by the continuity of Φ. This completes the proof. □

Remark 4.2. Theorems 3.1 and 4.1 improve and extend the main results proved in [15] in the following senses:

(i) from a nonexpansive mapping to a nonexpansive semigroup,

(ii) from a real Hilbert space to a reflexive Banach space which admits a weakly continuous duality mapping with gauge functions.

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