Research Article

Homotopy Interior-Point Method for a General Multiobjective Programming Problem

X. Zhao,[1,2] S. G. Zhang,[1] and Q. H. Liu[3]

[1] *Department of Mathematics, Jilin University, Changchun 130001, China*

[2] *Department of Mathematics, Beihua University, Jilin 132013, China*

[3] *Institute of Applied Mathematics, Changchun University of Technology, Changchun 130012, China*

Correspondence should be addressed to Q. H. Liu, liuqinghuaijl@163.com

We present a combined homotopy interior-point method for a general multiobjective programming problem. For solving the KKT points of the multiobjective programming problem, the homotopy equation is constructed. We prove the existence and convergence of a smooth homotopy path from almost any initial interior point to a solution of the KKT system under some basic assumptions.

1. Introduction

We consider the following multiobjective programming problem:

$$\min f(x),$$
$$\text{s.t. } g(x) \leq 0, \qquad\qquad\qquad (\text{MOP})$$
$$h(x) = 0,$$

where $f = (f_1, f_2, \ldots, f_p)^T : R^n \rightarrow R^p$, $g = (g_1, g_2, \ldots, g_m)^T : R^n \rightarrow R^m$ and $h = (h_1, h_2, \ldots, h_s)^T : R^n \rightarrow R^s$.

Since Kellogg et al. [1] and Smale [2] have published their remarkable papers, more and more attention has been paid to the homotopy method. As a globally convergent method, the homotopy method now becomes an important tool for numerically solving complementary problem and nonlinear mathematical programming problem [3–5].

Among most interior-point methods for solving mathematical programming, one of the main ideas is numerically tracing the center path generated by the optimal solution set of the logarithmic barrier function. Usually, the strict convexity of the logarithmic barrier function or boundedness of the solution set is needed [6, 7]. Lin et al. [8] presented a new interior-point method, combined homotopy interior-point method (CHIP method), for convex nonlinear programming. Subsequently, Lin et al. [9] generalized the CHIP method to convex multi-objective programming with only inequality constraints. Recently, Song and Yao [10] generalized the combined homotopy interior-point method to the general multiobjective programming problem under the so-called normal cone condition. In that paper, they proved the existence of the homotopy path under the following assumptions:

(A_1) Ω_0 is nonempty and bounded;

(A_2) for all $x \in \Omega$, the vectors $\{\nabla g_i(x), i \in B(x), \nabla h_j(x), j \in J\}$ are linearly independent;

(A_3) for all $x \in \Omega$, $\{x + \sum_{i \in B(x)} u_i \nabla g_i(x) + \sum_{j \in J} z_j \nabla h_j(x) : z = (z_j) \in R^s, u_i \geq 0, i \in B(x)\} \bigcap \Omega = \{x\}$,

where $\Omega = \{x \in R^n \mid g(x) \leqq 0, h(x) = 0\}$, $\Omega_0 = \{x \in R^n \mid g(x) < 0, h(x) = 0\}$, and $B(x) = \{i \in \{1, 2, \ldots, m\} \mid g_i(x) = 0\}$.

In [10], the combined homotopy method was given as follows:

$$\overline{H}\left(\omega, \omega^{(0)}, t\right) = \begin{bmatrix} (1-t)\left(\nabla f(x)\lambda + \nabla g(x)u\right) + \nabla h(x)z + t\left(x - x^{(0)}\right) \\ h(x) \\ U \times g(x) - tU^{(0)} \times g\left(x^{(0)}\right) \\ (1-t)\left(1 - \sum_{i=1}^{p} \lambda_i\right)e - t\left(\lambda - \lambda^{(0)}\right) \end{bmatrix} = 0, \qquad (1.1)$$

where $x^{(0)} \in \Omega_0$, $u^{(0)} > 0$, $\lambda^{(0)} > 0$, and $\sum_{i=1}^{p} \lambda_i^{(0)} = 1$. However, the solution simply yields $\lambda = \lambda^{(0)}$ for all $t \in (0, 1]$. In fact, from the last equation, we have $p(1-t) + (pt - p - t) \sum_{i=1}^{p} \lambda_i + t = 0$. According to this, we know that $\lambda \equiv \lambda^0$ for all of $t \in [0, 1]$. That is, the method given in [10] just solves the single-objective programming problem.

The purpose of this paper is to generalize the combined homotopy interior-point method for a general multiobjective programming problem (MOP) under quasinorm cone condition that weakens the assumptions more than the ones in [10] and constructs a new homotopy equation which is much different and simpler the that one given in [9].

The paper is organized as follows. In Section 2, we recall some notations and pre-liminaries results. In Section 3, we construct a new combined homotopy mapping and prove the existence and convergence of a smooth homotopy path from almost any interior initial point to a KKT point of MOP under some assumptions. In Section 4, numerical results are given.

2. Some Definitions and Properties

Let R_+^n and R_{++}^n denote the nonnegative and positive orthants of R^n, respectively. For any two vectors $y = (y_1, y_2, \ldots, y_n)^T$ and $z = (z_1, z_2, \ldots, z_n)^T$ in R^n, we use the following conventions:

$$
\begin{aligned}
y = z, &\quad \text{iff } y_i = z_i, \ i = 1, 2, \ldots, n, \\
y < z, &\quad \text{iff } y_i < z_i, \ i = 1, 2, \ldots, n, \\
y \leqq z, &\quad \text{iff } y_i \leq z_i, \ i = 1, 2, \ldots, n, \\
y \leq z, &\quad \text{iff } y_i \leq z_i, y \neq z, \ i = 1, 2, \ldots, n.
\end{aligned}
\tag{2.1}
$$

Suppose that $f = (f_1, f_2, \ldots, f_p)^T : R^n \to R^p$, $g = (g_1, g_2, \ldots, g_m)^T : R^n \to R^m$, and $h = (h_1, h_2, \ldots, h_s)^T : R^n \to R^s$ are three times continuously differentiable functions. Let

$$
\Omega = \{x \in R^n \mid g(x) \leqq 0, h(x) = 0\}, \qquad \Omega_0 = \{x \in R^n \mid g(x) < 0, h(x) = 0\}, \qquad \partial\Omega = \frac{\Omega}{\Omega_0},
$$

$$
\Lambda^+ = \left\{ \lambda \in R_+^p \mid \sum_{i=1}^p \lambda_i = 1 \right\}, \qquad \Lambda^{++} = \left\{ \lambda \in R_{++}^p \mid \sum_{i=1}^p \lambda_i = 1 \right\},
$$

$$
I = \{1, 2, \ldots, m\}, \qquad J = \{1, 2, \ldots, s\},
\tag{2.2}
$$

and let

$$
B(x) = \{i \in \{1, 2, \ldots, m\} \mid g_i(x) = 0\}
\tag{2.3}
$$

denote the active index set at a given point.

Definition 2.1. A point $x \in \Omega$ is said to be an efficient solution to multiobjective programming problem (MOP), if there is no $y \in \Omega$ such that $f(y) \leq f(x)$ holds.

Definition 2.2. Let $U \subset R^n$ be an open set, and let $\varphi : U \to R^P$ be a smooth mapping. If Range $[\partial\varphi(x)/\partial x] = R^p$ for all $x \in \varphi^{-1}(y)$, then $y \in R^p$ is a regular value and $x \in R^n$ is a regular point.

Definition 2.3. Let $\eta_i : R^n \to R^n (i = 1, 2, \ldots, m)$. For any $x \in \Omega$, $\{\eta_i(x) : i \in B(x)\}$ is said to be positive linear independent with respect to $\nabla g(x)n$ and $\nabla h(x)$ if

$$
\nabla h(x)z + \sum_{i \in B(x)} (y_i \nabla g_i(x) + u_i \eta_i(x)) = 0, \quad z \in R^s, \ y_i \geq 0, \ u_i \geq 0 \ (i \in B(x))
\tag{2.4}
$$

implies that

$$
z = 0, \qquad y_i = 0, \qquad u_i = 0 \quad (i \in B(x)).
\tag{2.5}
$$

Lemma 2.4 (parametric form of the Sard theorem on a smooth manifold; see [11]). *Let Q, N and P be smooth manifolds of dimensions q, m and p. Respectively, let $\varphi : Q \times N \to P$ be a C^r map, where $r > \max\{0, m - p\}$. If $0 \in P$ is a regular value of φ, then for almost all $\alpha \in Q$, 0 is a regular value of $\varphi(\alpha, \cdot)$.*

Lemma 2.5 (inverse image theorem; see [12]). *If 0 is a regular value of the mapping $\varphi_\alpha(\cdot) \triangleq \varphi(\alpha, \cdot)$, then $\varphi_\alpha^{-1}(0)$ consists of some smooth manifolds.*

Lemma 2.6 (classification theorem of one-dimensional manifold; see [12]). *A one-dimensional smooth manifold is diffeomorphic to a unit circle or a unit interval.*

The following three basic assumptions are commonly used in this paper:

(C_1) *Ω_0 is nonempty and bounded;*

(C_2) *for any $x \in \Omega$, there exists a positive linear independent map $\eta(x)$ with respect to $\nabla g(x)$ and $\nabla h(x)$ such that, $\{\eta_i(x) : i \in B(x)\}$ is positive linear independent with respect to $\nabla g(x)$ and $\nabla h(x)$;*

(C_3) *for any $x \in \partial\Omega$, there exists a positive linear independent map $\eta(x)$ with respect to $\nabla g(x)$ and $\nabla h(x)$, such that*

$$\left\{ x + \sum_{i \in B(x)} u_i \eta_i(x) + \sum_{j \in J} z_j \nabla h_j(x) : z_j \in R, \ j \in \{J\}, \ u_i \geq 0, \ i \in B(x) \right\} \bigcap \Omega = \{x\} \tag{2.6}$$

(quasinormal cone condition).

Remark 2.7. If Ω satisfies assumptions (A_1)–(A_3), then it necessarily satisfies assumptions (C_1)–(C_3).

In fact, if we choose $\eta(x) = \nabla g(x)$, then it is easy to get the result. Clearly, if Ω satisfies assumptions (C_1)–(C_3), then it does not necessarily satisfies assumptions (A_1)–(A_3).

3. Main Results

Let $x \in \Omega$. We say that x is a KKT point of (MOP) if there exists $(\lambda, u, z) \in R_+^{p+m} \times R^s$, such that

$$\nabla f(x)\lambda + \nabla g(x)u + \nabla h(x)z = 0, \tag{3.1a}$$

$$U g(x) = 0, \tag{3.1b}$$

$$1 - \sum_{i=1}^{p} \lambda_i = 0, \tag{3.1c}$$

where $\nabla f(x) = (\nabla f_1(x), \dots, \nabla f_p(x)) \in R^{n \times p}$, $\nabla g(x) = (\nabla g_1(x), \dots, \nabla g_m(x)) \in R^{n \times m}$, $\nabla h(x) = (\nabla h_1(x), \dots, \nabla h_m(x)) \in R^{n \times s}$.

Meanwhile, the KKT system of MOP is (3.1a)–(3.1c).

For a convex multiobjective programming problem, the solution of the MOP can be obtained from the KKT system, and for a non-convex multi-objective programming problem, it is significant when we get a solution of the KKT system.

To solve the KKT system (3.1a)–(3.1c), we construct a homotopy equation as follows:

$$H\left(\omega,\omega^{(0)},t\right) = \begin{bmatrix} (1-t)\left(\nabla f(x)\lambda + \nabla g(x)u + t\eta(x)u^2\right) + \nabla h(x)z + t\left(x - x^{(0)}\right) \\ h(x) \\ U \times g(x) - tU^{(0)} \times g(x^{(0)}) \\ (1-t)\left(1 - \sum_{i=1}^{p}\lambda_i\right)e - t\left(\lambda^{5/2} - \left(\lambda^{(0)}\right)^{5/2}\right) \end{bmatrix} = 0, \quad (3.2)$$

where $\omega^{(0)} = (x^{(0)},\lambda^{(0)},u^{(0)},z^{(0)}) \in \Omega_0 \times \Lambda^{++} \times R_{++}^m \times \{0\}$, $\omega = (x,\lambda,u,z) \in R^n \times R^p \times R^m \times R^s$, $u^2 = (u_1^2,u_2^2\ldots,u_m^2)^T \in R^m$, $\lambda^{5/2} = (\lambda_1^{5/2},\lambda_2^{5/2},\ldots,\lambda_p^{5/2})^T \in R^p$, $U = \mathrm{diag}(u)$, $e = (1,1,\ldots,1)^T \in R^p$ and $t \in [0,1]$.

When $t = 1$, the homotopy equation (3.2) becomes

$$\nabla h(x)z + \left(x - x^{(0)}\right) = 0, \tag{3.3a}$$

$$h(x) = 0, \tag{3.3b}$$

$$Ug(x) - U^{(0)}g\left(x^{(0)}\right) = 0, \tag{3.3c}$$

$$\lambda^{5/2} = \left(\lambda^{(0)}\right)^{5/2}. \tag{3.3d}$$

By assumption (C_3), we get $z = 0$, $x = x^{(0)}$. Since $g(x^{(0)}) < 0$, (3.3c) implies that $u = u^{(0)}$. Equation (3.3d) shows that $\lambda = \lambda^{(0)}$. That is, the equation $H(\omega,\omega^{(0)},1) = 0$ with respect to ω has only one solution $\omega = \omega^{(0)} = (x^{(0)},\lambda^{(0)},u^{(0)},0)$.

When $t = 0$, $H(\omega,\omega^{(0)},t) = 0$ turns to the KKT system (3.1a)–(3.1c).

For a given $\omega^{(0)}$, rewrite $H(\omega,\omega^{(0)},t)$ as $H_{\omega^{(0)}}(\omega,t)$. The zero-point set of $H_{\omega^{(0)}}$ is

$$H_{\omega^{(0)}}^{-1} = \left\{(\omega,t) \in \Omega \times R_{++}^{p+m} \times R^s \times (0,1] : H\left(\omega,\omega^{(0)},t\right) = 0\right\}. \tag{3.4}$$

Theorem 3.1. *Suppose that H is defined as in (3.2) and let f, g, and h be three times continuously differentiable functions. In addition, let assumptions (C_1)–(C_3) hold, and let η_i be two times continuously differentiable function. Then, for almost all initial points $\omega^{(0)} \in \Omega_0 \times \Lambda^{++} \times R_{++}^m \times \{0\}$, 0 is a regular value of $H_{\omega^{(0)}}$ and $H_{\omega^{(0)}}^{-1}$ consists of some smooth curves. Among them, a smooth curve, say $\Gamma_{\omega^{(0)}}$, is starting from $(\omega^{(0)},1)$.*

Proof. Denote the Jacobi matrix of $H(\omega,\omega^{(0)},t)$ by $DH(\omega,\omega^{(0)},t)$. For any $\omega^{(0)} \in \Omega$ and $t \in [0,1]$, we have $DH(\omega,\omega^{(0)},t) = (\partial H/\partial\omega,\partial H/\partial\omega^{(0)},\partial H/\partial t)$. Now, we consider the submatrix of $DH(\omega,\omega^{(0)},t)$.

For any $(x, x^{(0)}, \lambda^{(0)}, u^{(0)}) \in R^n \times \Omega_0 \times \Lambda^{++} \times R^m_{++}$,

$$
\frac{\partial H}{\partial (x, x^{(0)}, \lambda^{(0)}, u^{(0)})} = \begin{bmatrix} Q & -tI_n & 0 & 0 \\ \nabla h(x)^T & 0 & 0 & 0 \\ U \nabla g(x)^T & -tU^{(0)} \nabla g(x^{(0)})^T & 0 & -t \operatorname{diag}(g(x^{(0)})) \\ 0 & 0 & \frac{5}{2} t (\lambda^{(0)})^{3/2} I_p & 0 \end{bmatrix}, \quad (3.5)
$$

where $Q = (1-t)(\sum_{i=1}^{p} \lambda_i \nabla^2 f_i(x) + \sum_{j=1}^{m} u_j \nabla^2 g_j(x) + t \sum_{j=1}^{m} u_j^2 \nabla \eta_j(x)) + \sum_{k=1}^{s} z_k \nabla^2 h_k(x) + tI_n$. We obtain that

$$
\operatorname{rank} \frac{\partial H}{\partial (x, x^{(0)}, \lambda^{(0)}, u^{(0)})} = n + p + m + s. \quad (3.6)
$$

That is, 0 is a regular value of H. By the parametric form of the Sard theorem, for almost all $\omega^{(0)} \in \Omega_0 \times \Lambda^{++} \times R^m_{++} \times \{0\}$, 0 is a regular value of $H_{\omega^{(0)}}$. By the inverse image theorem, $H_{\omega^{(0)}}^{-1}(0)$ consists of some smooth curves. Since $H(\omega^{(0)}, \omega^{(0)}, 1) = 0$, there must be a smooth curve, denoted by $\Gamma_{\omega^{(0)}}$, starting from $(\omega^{(0)}, 1)$. □

Theorem 3.2. *Let assumptions (C_1)-(C_2) hold. For a given $\omega^{(0)} = (x^{(0)}, \lambda^{(0)}, u^{(0)}, z^{(0)}) \in \Omega_0 \times \Lambda^{++} \times R^m_{++} \times \{0\}$, if 0 is a regular value of $H_{\omega^{(0)}}$, then the projection of the smooth curve $\Gamma_{\omega^{(0)}}$ on the component λ is bounded.*

Proof. Suppose that the conclusion does not hold. Since $(0, 1]$ is bounded, there exists a sequence $\{(\omega^{(k)}, t_k)\} \subset \Gamma_{\omega^{(0)}}$ such that

$$
t_k \longrightarrow t_*, \qquad \left\| \lambda^{(k)} \right\| \longrightarrow \infty. \quad (3.7)
$$

From the last equality of (3.2), we have

$$
\begin{pmatrix} 1 - t_k \\ 1 - t_k \\ \cdots \\ 1 - t_k \end{pmatrix} - \begin{pmatrix} (1-t_k)\lambda_1^{(k)} + (1-t_k)\sum_{i \neq 1} \lambda_i^{(k)} + t_k \left(\lambda_1^{(k)}\right)^{5/2} \\ (1-t_k)\lambda_2^{(k)} + (1-t_k)\sum_{i \neq 2} \lambda_i^{(k)} + t_k \left(\lambda_2^{(k)}\right)^{5/2} \\ \cdots \\ (1-t_k)\lambda_p^{(k)} + (1-t_k)\sum_{i \neq p} \lambda_i^{(k)} + t_k \left(\lambda_p^{(k)}\right)^{5/2} \end{pmatrix} - t_k \begin{pmatrix} \left(\lambda_1^{(0)}\right)^{5/2} \\ \left(\lambda_2^{(0)}\right)^{5/2} \\ \cdots \\ \left(\lambda_p^{(0)}\right)^{5/2} \end{pmatrix} = 0. \quad (3.8)
$$

If we assume $\|\lambda^{(k)}\| \to +\infty (k \to \infty)$, this hypothesis implies that

$$
\left\{ i \in \{1, 2, \dots, p\} : \lim_{k \to \infty} \lambda_i^{(k)} = \infty \right\} \neq \Phi. \quad (3.9)
$$

Since $t_k \to t_*$, $\lambda^{(k)} > 0$, it follows that the second part in the left-hand side of some equations in (3.8) tends to infinity as $k \to \infty$. But the other two parts are bounded. This is impossible. Thus, the component λ is bounded. □

Theorem 3.3. *Let f, g, and h be three times continuously differentiable functions. In addition, let assumptions (C_1)–(C_3) hold, and let η_i be two times continuously differentiable function. Then, for almost all of $\omega^{(0)} \in \Omega_0 \times \Lambda^{++} \times R_{++}^m \times \{0\}$, $H_{\omega^{(0)}}^{-1}(0)$ contains a smooth curve $\Gamma_{\omega^{(0)}} \subset \Omega \times R_+^p \times R_+^m \times R^s \times (0,1]$, which starts from $(\omega^{(0)}, 1)$. As $t \to 0$, the limit set $T \times \{0\} \subset \Omega \times \Lambda^+ \times R_+^m \times R^s \times \{0\}$ of $\Gamma_{\omega^{(0)}}$ is nonempty and every point in T is a solution of the KKT system (3.1a)–(3.1c).*

Proof. From the homotopy equation (3.2), it is easy to see that $\Gamma_{\omega^{(0)}} \subset \Omega \times R_+^p \times R_+^m \times R^s \times (0,1]$. By Theorem 3.1, for almost all $\omega^{(0)} \in \Omega_0 \times \Lambda^{++} \times R_{++}^m \times \{0\}$, 0 is a regular value of $H_{\omega^{(0)}}$ and $H_{\omega^{(0)}}^{-1}$ contains of a smooth curve $\Gamma_{\omega^{(0)}}$ starting from $(\omega^{(0)}, 1)$. By the classification theorem of one-dimensional smooth manifolds, $\Gamma_{\omega^{(0)}}$ is diffeomorphic to a unit circle or the unit interval $(0,1]$.

Noticing that

$$
\left| \frac{\partial H_{\omega^{(0)}}(\omega^{(0)}, 1)}{\partial \omega} \right| = \begin{vmatrix} I_n & 0 & 0 & \nabla h(x^{(0)}) \\ \nabla h(x^{(0)})^T & 0 & 0 & 0 \\ U^{(0)} \nabla g(x^{(0)})^T & 0 & \text{diag}(g(x^{(0)})) & 0 \\ 0 & -\dfrac{5}{2}(\lambda^{(0)})^{3/2} I_p & 0 & 0 \end{vmatrix}
$$
$$
= (-1)^s \left| \text{diag}(g(x^{(0)})) \right| \left\| -\frac{5}{2}(\lambda^{(0)})^{3/2} I_p \right\| \nabla h(x^{(0)})^T \nabla h(x^{(0)}) \right|,
$$

(3.10)

By assumption (C_2), we know that $|\nabla h(x^{(0)})^T \nabla h(x^{(0)})| \neq 0$. And because $g(x^{(0)}) < 0$, $\lambda^{(0)} \in \Lambda^{++}$, we obtain that $[\partial H_{\omega^{(0)}}(\omega^{(0)}, 1)/\partial \omega]$ is nonsingular. Therefore, the smooth curve $\Gamma_{\omega^{(0)}}$, which starts from $(\omega^{(0)}, 1)$, is diffeomorphic to $(0,1]$.

Let (ω^*, t_*) be a limit point of $\Gamma_{\omega^{(0)}}$. Only three cases are possible:

(a) $(\omega^*, t_*) \in \Omega \times \Lambda^+ \times R_+^m \times R^s \times \{0\}$,

(b) $(\omega^*, t_*) \in \partial(\Omega_0 \times R_+^{p+m}) \times R^s \times (0,1]$,

(c) $(\omega^*, t_*) \in \Omega \times R_+^{p+m} \times R^s \times \{1\}$.

Because $H(\omega^{(0)}, \omega^{(0)}, 1) = 0$ has a unique solution $(\omega^{(0)}, 1)$, case (c) will not happen.

In case (b), because Ω and $(0,1]$ are bounded sets and by assumption (C_2), for any $x \in \Omega$, there exists a positive linear independent map $\eta(x)$ with respect to $\nabla g(x)$ and $\nabla h(x)$ such that $\{\eta_i(x) : i \in B(x)\}$ is positive linear independent with respect to $\nabla g(x)$ and $\nabla h(x)$. From the first equality of (3.2), we get that the component z of $\Gamma_{\omega^{(0)}}$ is bounded.

If case (b) holds, then there exists a sequence $\{(\omega^{(k)}, t_k)\} \subset \Gamma_{\omega^{(0)}}$ such that

$$
\left\| (\omega^{(k)}, t_k) \right\| \longrightarrow \infty.
$$

(3.11)

Because Ω and $(0, 1]$ are bounded and by Theorem 3.2, there exists a sequence (denoted also by $\{(\omega^{(k)}, t_k)\} \subset \Gamma_{\omega^{(0)}}$) such that

$$x^{(k)} \longrightarrow x^*, \qquad \lambda^{(k)} \longrightarrow \lambda^*, \qquad \left\| u^{(k)} \right\| \longrightarrow \infty, \qquad z^{(k)} \longrightarrow z^*, \qquad t_k \longrightarrow t_*, \quad \text{as } k \longrightarrow \infty. \tag{3.12}$$

From the third equality of (3.2), we have

$$g\left(x^{(k)}\right) = t_k \left(U^{(k)}\right)^{-1} U^{(0)} g\left(x^{(0)}\right). \tag{3.13}$$

Hence, the active index set $B(x^*)$ is nonempty.

From the homotopy equation (3.2), it follows that

$$(1 - t_k)\left(\nabla f\left(x^{(k)}\right)\lambda^{(k)} + \nabla g\left(x^{(k)}\right)u^{(k)} + t_k \eta\left(x^{(k)}\right)\left(u^{(k)}\right)^2\right)$$
$$+ \nabla h\left(x^{(k)}\right)z^{(k)} + t_k\left(x^{(k)} - x^{(0)}\right) = 0. \tag{3.14}$$

(i) When $t_* = 1$, rewrite (3.14) as

$$\sum_{j \in B(x^*)} (1 - t_k)\left(\nabla g_j(x^{(k)})u_j^{(k)} + t_k \eta_j(x^{(k)})\left(u_j^{(k)}\right)^2\right) + \nabla h(x^{(k)})z^{(k)} + (x^{(k)} - x^{(0)})$$
$$= -(1 - t_k)\left[\sum_{j \notin B(x^*)} \nabla g_j(x^{(k)})u_j^{(k)} + t_k \eta_j(x^{(k)})\left(u_j^{(k)}\right)^2 + \nabla f(x^{(k)})\lambda^{(k)} - (x^{(k)} - x^{(0)})\right]. \tag{3.15}$$

Because $\{u_j^{(k)}\}, j \notin B(x^*)$, are bounded and by assumption (C_1), when $k \to \infty$, we observe that

$$\lim_{k \to \infty} \left(\sum_{j \in B(x^*)} (1 - t_k)\left(\nabla g_j\left(x^{(k)}\right)u_j^{(k)} + t_k \eta_j\left(x^{(k)}\right)\left(u_j^{(k)}\right)^2\right) + \nabla h\left(x^{(k)}\right)z^{(k)} + \left(x^{(k)} - x^{(0)}\right) = 0\right). \tag{3.16}$$

Using $x^{(k)} \to x^*$ and $z^{(k)} \to z^* (k \to \infty)$, we have from (3.16) that

$$\sum_{j \in B(x^*)} \left(\nabla g_j(x^*) \lim_{k \to \infty} (1 - t_k) \, u_j^{(k)} + \eta_j(x^*) \lim_{k \to \infty} (1 - t_k) \, t_k\left(u_j^{(k)}\right)^2\right) = -\left(\nabla h(x^*)z^* + x^* - x^{(0)}\right). \tag{3.17}$$

It is easy to see that the right-hand side of (3.17) is bounded. By assumption (C_2) and (3.17), we get

$$\lim_{k \to \infty} (1 - t_k) u_j^{(k)} = 0, \qquad \lim_{k \to \infty} (1 - t_k) t_k \left(u_j^{(k)} \right)^2 = \alpha_j, \quad j \in B(x^*), \tag{3.18}$$

where $\alpha_j \geq 0$.

Then, we have

$$x^{(0)} = x^* + \nabla h(x^*) z^* + \sum_{j \in B(x^*)} \alpha_j \eta_j(x^*), \tag{3.19}$$

which contradicts assumption (C_3).

(ii) When $t_* \in [0, 1)$, rewrite (3.14) as

$$\sum_{j \in B(x^*)} (1 - t_k) \left(\nabla g_j \left(x^{(k)} \right) u_j^{(k)} + t_k \eta_j \left(x^{(k)} \right) \left(u_j^{(k)} \right)^2 \right)$$

$$= -(1 - t_k) \left[\sum_{j \notin B(x^*)} \left(\nabla g_j \left(x^{(k)} \right) u_j^{(k)} + t_k \eta_j \left(x^{(k)} \right) \left(u_j^{(k)} \right)^2 \right) + \nabla f \left(x^{(k)} \right) \lambda^{(k)} \right] \tag{3.20}$$

$$- t_k \left(x^{(k)} - x^{(0)} \right) - \nabla h \left(x^{(k)} \right) z^{(k)}.$$

We know that, since Ω and $\{u_j^{(k)}\}$, $j \notin B(x^*)$ are bounded as $k \to \infty$, the right-hand side of (3.20) is bounded. But by assumptions (C_2) and (C_3), if $u_j^{(k)} \to \infty (j \in B(x^*))$, then the left-hand side of (3.20) is infinite, this is a contradiction.

As a conclusion, (a) is the only possible case, and ω^* is a solution of the KKT system (3.1a)–(3.1c).

Let s be the arc-length of $\Gamma_{\omega^{(0)}}$. We can parameterize $\Gamma_{\omega^{(0)}}$ with respect to s. $\qquad \square$

Theorem 3.4. *The homotopy path $\Gamma_{\omega^{(0)}}$ is determined by the following initial-value problem for the ordinary differential equation:*

$$DH_{\omega^{(0)}} (\omega(s), t(s)) \begin{bmatrix} \dot{\omega} \\ \dot{\mu} \end{bmatrix} = 0, \tag{3.21}$$

$$\omega(0) = \omega^{(0)}, \qquad t(0) = 1.$$

The component ω^* of the solution point $(\omega(s^*), t(s^*))$, for $t(s^*) = 0$, is a solution of the KKT system.

4. Algorithm and Example

The following algorithm describes the numerical computation for Pareto optimal solutions.

Algorithm 4.1 ((MOP) Euler-Newton method).

Step 1. Give an initial point $(\omega^{(0)}, 1) \in \Omega_0 \times R_{++}^{p+m} \times \{0\} \times \{1\}$, an initial step length $h_0 > 0$, and three small positive numbers $\varepsilon_1,\ \varepsilon_2,\ \varepsilon_3$. Let $k := 0$.

Step 2. Compute the direction $\gamma^{(k)}$ of the predictor step:

(a) compute a unit tangent vector $\xi^{(k)} \in R^{n+p+m+s+1}$ of Γ_{ω^0} at $(\omega^{(0)}, t_k)$;

(b) determine the direction $\gamma^{(k)}$ of the predictor step.

If the sign of the determinant $\begin{vmatrix} DH_{\omega^{(0)}}(\omega^{(k)}, t_k) \\ \xi^{(k)T} \end{vmatrix}$ is $(-1)^{p+m+s+pm+ps+ms+1}$, take $\gamma^{(k)} = \xi^{(k)}$.

If the sign of the determinant $\begin{vmatrix} DH_{\omega^{(0)}}(\omega^{(k)}, t_k) \\ \xi^{(k)T} \end{vmatrix}$ is $(-1)^{p+m+s+pm+ps+ms}$, take $\gamma^{(k)} = -\xi^{(k)}$.

Step 3. Compute a corrector point $(\omega^{(k+1)}, t_{k+1})$:

$$\left((\omega^*)^{(k)}, (t_*)_k \right) = \left(\omega^{(k)}, t_k \right) + h_k \gamma^{(k)},$$

$$\left(\omega^{(k+1)}, t_{k+1} \right) = \left((\omega^*)^{(k)}, (t_*)_k \right) - DH_{\omega^{(0)}} \left((\omega^*)^{(k)}, (t_*)_k \right)^+ H_{\omega^{(0)}} \left((\omega^*)^{(k)}, (t_*)_k \right), \tag{4.1}$$

where

$$DH_{\omega^{(0)}}(\omega, t)^+ = DH_{\omega^{(0)}}(\omega, t)^T \left(DH_{\omega^{(0)}}(\omega, t) DH_{\omega^{(0)}}(\omega, t)^T \right)^{-1} \tag{4.2}$$

is the Moore-Penrose inverse of $DH_{\omega^{(0)}}(\omega, t)$.

If $\|H_{\omega^{(0)}}(\omega^{(k+1)}, t_{k+1})\| \le \varepsilon_1$, let $h_{k+1} = \min\{h_0, 2h_k\}$ and go to Step 4.
If $\|H_{\omega^{(0)}}(\omega^{(k+1)}, t_{k+1})\| \in (\varepsilon_1, \varepsilon_2)$, let $h_{k+1} = h_k$ and go to Step 4.
If $\|H_{\omega^{(0)}}(\omega^{(k+1)}, t_{k+1})\| \ge \varepsilon_2$, let $h_{k+1} = \max\{(1/2)h_0, (1/2)h_k\}$ and go to Step 3.

Step 4. If $\omega^{(k+1)} \in \Omega \times R_+^{p+m} \times R^s$ and $t_{k+1} > \varepsilon_3$, let $k = k + 1$ and go to Step 2.
If $\omega^{(k+1)} \in \Omega \times R_+^{p+m} \times R^s$ and $t_{k+1} < -\varepsilon_3$, let $h_k := h_k(t_k/(t_k - t_{k+1}))$, go to Step 3, and recompute $(\omega^{(k+1)}, t_{k+1})$ for the initial point $(\omega^{(k)}, t_k)$.
If $\omega^{(k+1)} \notin \Omega \times R_+^{p+m} \times R^s$, let $h_k := (h_k/2)(t_k/(t_k - t_{k+1}))$, go to Step 3, and recompute $(\omega^{(k+1)}, t_{k+1})$ for the initial point $(\omega^{(k)}, t_k)$.
If $\omega^{(k+1)} \in \Omega \times R_+^{p+m} \times R^s$ and $|t_{k+1}| \le \varepsilon_3$, then stop.

Remark 4.2. In Algorithm 4.1, the arc length parameter is not computed explicitly. The tangent vector at a point on $\Gamma_{\omega^{(0)}}$ has two opposite directions, one (the positive direction) makes s increase and the other (the negative direction) makes s decrease. The negative direction will lead us back to the initial point, so we must go along the positive direction. The criterion in Step 2 (b) of Algorithm 4.1 that determines the positive direction is based on a basic theory of homotopy method, that is, the positive direction γ at any point (ω, t) on $\Gamma_{\omega^{(0)}}$ keeps the sign of the determinant $\begin{vmatrix} DH_{\omega^{(0)}}(\omega, t) \\ \gamma^T \end{vmatrix}$ invariant.

Table 1: Results of Example 4.3.

$x^{(0)}$	x^*	$f(x^*)$	$\|H_{\omega^*}\|$
(4.000, −1.000)	(7.1927, −2.0476)	(55.9269, 21.7710)	$1.0e^{-003}$
(3.000, 0.000)	(3.7552, −0.8690)	(14.8563, 1.3254)	$1.0e^{-013}$

Table 2: Results of Example 4.4.

$x^{(0)}$	x^*	$f(x^*)$	$\|H_{\omega^*}\|$
(1.0000, 2.0000, 0.0000, 1.0000, 1.0000)	(−1.3074, −2.8605)	(11.3566, −9.2942)	$1.0e^{-012}$
(−1.0000, −2.0000, 1.0000, 1.0000, 1.0000)	(−1.3073, −2.8603)	(11.3550, −9.2930)	$1.0e^{-003}$

Example 4.3. We have

$$
\begin{aligned}
\min \quad & f = \min\left\{ x_1^2 + x_2^2, (x_1 + 3)^2 + x_2^2 \right\}, \\
\text{s.t.} \quad & g_1(x) = (x_1 - 3)^2 + x_2^2 - 64 \leq 0, \\
& g_2(x) = (x_1 - 5)^2 + x_2^2 - 9 \leq 0, \\
& h(x) = x_1 - x_2^2 - 3 = 0.
\end{aligned}
\tag{4.3}
$$

The results are shown in Table 1.

Example 4.4. We have

$$
\begin{aligned}
\min f = \min\Bigg\{ & x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2, 3x_1 + 2x_2 - \frac{1}{3}x_3 + 0.01(x_4 - x_5)^3 \Bigg\}, \\
\text{s.t.} \ \ g_1(x) = & x_1^2 + x_2^2 + x_3^2 + x_4^2 - 10 \leq 0, \\
h_1(x) = & 4x_1 - 2x_2 + 0.8x_3 + 0.6x_4 + 0.5x_5^2 = 0, \\
h_2(x) = & x_1 + 2x_2 - x_3 - 0.5x_4 + x_5 - 2 = 0.
\end{aligned}
\tag{4.4}
$$

The results are shown in Table 2.

Acknowledgments

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