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Research Article

Bernstein Widths of Some Classes of Functions Defined by a Self-Adjoint Operator

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We consider the classes of periodic functions with formal self-adjoint linear differential operators $W_p(\mathcal{L}_r)$, which include the classical Sobolev class as its special case. Using the iterative method of Buslaev, with the help of the spectrum of linear differential equations, we determine the exact values of Bernstein width of the classes $W_p(\mathcal{L}_r)$ in the space L_q for $1 < p \le q < \infty$.

1. Introduction and Main Result

Let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}$, and $\mathbb{N}^+$ be the sets of all complex numbers, real numbers, integers, nonnegative integers, and positive integers, respectively.

Let $\mathbb{T}$ be the unit circle realized as the interval $[0, 2\pi]$ with the points 0 and 2π identified, and as usual, let $L_q := L_q[0, 2\pi]$ be the classical Lebesgue integral space of 2π-periodic real-valued functions with the usual norm $\|\cdot\|_q$, $1 \le q \le \infty$. Denote by $\widetilde{W}_p^r$ the Sobolev space of functions $x(\cdot)$ on $\mathbb{T}$ such that the $(r-1)$st derivative $x^{(r-1)}(\cdot)$ is absolutely continuous on $\mathbb{T}$ and $x^{(r)}(\cdot) \in L_p$, $r \in \mathbb{N}$. The corresponding Sobolev class is the set

$$W_p^r := \left\{ x(\cdot) : x \in \widetilde{W}_p^r, \left\| x^{(r)}(\cdot) \right\|_p \le 1 \right\}. \tag{1.1}$$

Tikhomirov [1] introduced the notion of Bernstein width of a centrally symmetric set C in a normed space X. It is defined by the formula

$$b_n(C, X) := \sup_L \sup\{\lambda \ge 0 : L \cap \lambda BX \subset C\}, \tag{1.2}$$

where BX is the unit ball of X and the outer supremum is taken over all subspaces $L \subset X$ such that $\dim L \geq n+1$, $n \in \mathbb{N}$.

In particular, Tikhomirov posed the problem of finding the exact value of $b_n(C;X)$, where $C = W_p^r$ and $X = L_q$, $1 \leq p, q \leq \infty$. He also obtained the first results [1] for $p = q = \infty$ and $n = 2k-1$. Pinkus [2] found $b_{2n-1}(W_p^r; L_q)$, where $p = q = 1$. Later, Magaril-Il'yaev [3] obtained the exact value of $b_{2n-1}(W_p^r; L_q)$ for $1 < p = q < \infty$. The latest contribution to these fields is due to Buslaev et al. [4] who found the exact values of $b_{2n-1}(W_p^r; L_q)$ for all $1 < p \leq q < \infty$.

Let

$$\mathcal{L}_r(D) = D^r + a_{r-1}D^{r-1} + \cdots + a_1 D + a_0, \quad D = \frac{d}{dt}, \tag{1.3}$$

be an arbitrary linear differential operator of order r with constant real coefficients $a_0, a_1, \ldots, a_{r-1}$. Denote by p_r the characteristic polynomial of $\mathcal{L}_r(D)$. The linear differential operator $\mathcal{L}_r(D)$ will be called formal self-adjoint if $p_r(-t) = (-1)^r p_r(t)$ for each $t \in \mathbb{C}$.

We define the function classes $W_p(\mathcal{L}_r)$ as follows:

$$W_p(\mathcal{L}_r) = \left\{ x(\cdot) : x^{r-1} \in AC_{2\pi}, \|\mathcal{L}_r(D)x(\cdot)\|_p \leq 1 \right\}, \tag{1.4}$$

where $1 \leq p \leq \infty$.

In this paper, we consider some classes of periodic functions with formal self-adjoint linear differential operators $W_p(\mathcal{L}_r)$, which include the classical Sobolev class as its special case. Using the iterative method of Buslaev and Tikhomirov [5], with the help of the spectrum of linear differential equations, we determine the exact values of Bernstein width of the classes $W_p(\mathcal{L}_r)$ in the space L_q for $1 < p \leq q < \infty$. The results of Buslaev et al. [4] are extended to the classes (1.4) defined by differential operators (1.3).

We define Q_q to be the nonlinear transformation

$$(Q_q f)(t) := |f(t)|^{q-1} \operatorname{sign} f(t). \tag{1.5}$$

The main result of this paper is the following.

Theorem 1.1. *Let $\mathcal{L}_r(D)$ be an arbitrary formal self-adjoint linear differential operator given by* (1.3), *and $n, r \in \mathbb{N}$, $1 < p \leq q < \infty$. Then*

$$b_{2n-1}(W_p(\mathcal{L}_r); L_q) = \lambda_{2n} := \lambda_{2n}(p, q, \mathcal{L}_r), \tag{1.6}$$

where λ_{2n} is that eigenvalue λ of the boundary value problem

$$\begin{gathered} \mathcal{L}_r(D)y(t) = (-1)^r \lambda^{-q} (Q_q x)(t), \\ y(t) = (Q_p \mathcal{L}_r(D)x)(t), \\ x^{(j)}(0) = x^{(j)}(2\pi), \quad y^{(j)}(0) = y^{(j)}(2\pi), \quad j = 0, 1, \ldots, n-1, \end{gathered} \tag{1.7}$$

for which the corresponding eigenfunction $x(\cdot) = x_{2n}(\cdot)$ *has only* $2n$ *simple zeros on* $\mathbb{T}$ *and is normalized by the condition* $\|\mathcal{L}_r(D)x(\cdot)\|_p = 1$.

2. Proof of the Theorem

First we introduce some notations and formulate auxiliary statements.

Let $\mathcal{L}_r(D)$ be an arbitrary linear differential operator (1.3). Denote the 2π-periodic kernel of $\mathcal{L}_r(D)$ by

$$\operatorname{Ker}\mathcal{L}_r(D) = \{x(\cdot) \in C^r(\mathbb{T}) : \mathcal{L}_r(D)x(t) \equiv 0\}. \tag{2.1}$$

Let $\mu(0 \le \mu \le r)$ be the dimension of $\operatorname{Ker}\mathcal{L}_r(D)$ and $\{\varphi_i, \dots, \varphi_\mu\}$ an arbitrary basis in $\operatorname{Ker}\mathcal{L}_r(D)$.

$Z_c(f)$ denotes the number of zeros of f on a period, counting multiplicity, and $S_c(f)$ is the cyclic sign change count for a piecewise continuous, 2π-Periodic function f [2]. Following, $(x(\cdot), \lambda)$ is called the spectral pair of (1.7) if the function $x(\cdot)$ is normalized by the condition $\|\mathcal{L}_r(D)x(\cdot)\|_p = 1$. The set of all spectral pairs is denoted by $\mathrm{SP}(p, q, \mathcal{L}_r)$. Define the spectral classes $\mathrm{SP}_{2k}(p, q, \mathcal{L}_r)$ as

$$\mathrm{SP}_{2k}(p, q, \mathcal{L}_r) = \left\{(x(\cdot), \lambda) \in \mathrm{SP}(p, q, \mathcal{L}_r) : S_c(x(\cdot)) = 2k\right\}. \tag{2.2}$$

Let $\widehat{x}_{2n}(\cdot)$ be the solution of the extremal problem as follows:

$$\begin{gathered}\int_0^{\pi/2n} |\mathrm{X}(t)|^q dt \longrightarrow \sup, \\ \int_0^{\pi/2n} |\mathcal{L}_r(D)\mathrm{X}(t)|^p dt \le 1, \\ x^{(k)}\left(\frac{\left((\pi/2n) + (-1)^{k+1}(\pi/2n)\right)}{2}\right) = 0, \quad k = 0, 1, \dots, n-1\end{gathered} \tag{2.3}$$

and the function $x_{2n}(\cdot)$ is such that $x_{2n}(t) = -x_{2n}(t - \pi/n)$ for all $t \in \mathbb{T}$

$$x_{2n}(t) := \begin{cases} \widehat{x}_{2n}(t), & 0 \le t \le \dfrac{\pi}{2n}, \\ \widehat{x}_{2n}\left(\dfrac{\pi}{n} - t\right), & \dfrac{\pi}{2n} < t \le \dfrac{\pi}{n}. \end{cases} \tag{2.4}$$

Let us extend periodically the function $x_{2n}(t)$ onto $\mathbb{R}$ and normalize the obtained function as it is required in the definition of spectral pairs. From what has been done above, we get a function $x_{2n}(t)$ belonging to $\mathrm{SP}_{2n}(p, q, \mathcal{L}_r)$. Furthermore, by [6], which any other function from $\mathrm{SP}_{2n}(p, q, \mathcal{L}_r)$ differs from $x_{2n}(\cdot)$ only in the sign and in a shift of its argument, and there exists a number $N \in \mathbb{N}^+$ such that for every $n \ge N$, all zeros of $x_{2n}(\cdot)$ are simple, equidistant with a step equal to π/n, and $S_c(x_{2n}) = S_c(\mathcal{L}_r(D)x_{2n}) = 2n$. We denote the set of zeros (= sign variations) of $\mathcal{L}_r(D)x_{2n}$ on the period by $Q_{2n} = (\tau_1, \dots, \tau_{2n})$. Let

$$G_r(t) = \frac{1}{2\pi}\sum_{k\notin\Lambda}\frac{e^{ikt}}{p_r(ik)}, \tag{2.5}$$

where $\Lambda = \{k \in \mathbb{Z} : p_r(ik) = 0\}$ and i is the imaginary unit.

The 2π-periodic G-splines are defined as elements of the linear space

$$S(Q_{2n}, G_r) = \operatorname{span}\{\varphi_1(t), \dots, \varphi_\mu(t), G_r(t-\tau_1), \dots, G_r(t-\tau_{2n})\}. \tag{2.6}$$

As was proved in [7], if $n \geq N$, then $\dim S(Q_{2n}, G_r) = 2n$.

We assume (shifting $x(\cdot)$ if necessary) that $\mathcal{L}_r(D)\widehat{x}_{2n}(\cdot)$ is positive on $(-\pi, \pi + \pi/n)$. Let $L_{2n} := L_{2n}(r, p, q)$ denote the space of functions of the form

$$x(t) = \sum_{j=1}^{\mu} a_j\varphi_j(t) + \frac{1}{\pi}\int_{\mathbb{T}} G_r(t-\tau)\left(\sum_{i=1}^{2n} b_i y_i(\tau)\right)d\tau, \tag{2.7}$$

where $a_1, \dots, a_\mu,\ b_1, \dots, b_{2n} \in \mathbb{R}$, $\sum_{i=1}^{2n} b_i = 0$, $y_i(\cdot) = \chi_i(\cdot)\mathcal{L}_r(D)x_{2n}(\cdot - (i-1)\pi/n)$, and $\chi_i(\cdot)$ is the characteristic function of the interval $\Delta_i := [-\pi + (i-1)\pi/n, -\pi + i\pi/n]$, $1 \leq i \leq 2n$. Obviously, $\dim L_{2n} = 2n$ and $L_{2n} \subset W_p(\mathcal{L}_r)$.

Let us now consider exact estimate of Bernstein n-width. This was introduced in [1]. We reformulate the definition for a linear operator P mapping X to Y.

Definition 2.1 (see [2, page 149]). Let $P \in L(X, Y)$. Then the Bernstein n-width is defined by

$$b_n(P(X), Y) = \sup_{X_{n+1}} \inf_{\substack{Px\in X_{n+1}\\ Px\neq 0}} \frac{\|Px\|_Y}{\|x\|_X}, \tag{2.8}$$

where X_{n+1} is any subspace of span $\{Px : x \in X\}$ of dimension $\geq n+1$.

2.1. *Lower Estimate of Bernstein n-Width*

Consider the extremal problem

$$\frac{\|x(\cdot)\|_q^q}{\|\mathcal{L}_r(D)x(\cdot)\|_p^p} \longrightarrow \inf, \quad x(\cdot) \in L_{2n} \tag{2.9}$$

and denote the value of this problem by α^q. Let us show that $\alpha \geq \lambda_n$; this will imply the desired lower bound for b_{2n-1}. Let $x(\cdot) \in L_{2n}$, then

$$\|\mathcal{L}_r(D)x(\cdot)\|_p^p = \sum_{i=1}^{2n}\int_{\Delta_i}\left|\sum_{i=1}^{2n} b_i y_i(t)\right|^p dt = \sum_{i=1}^{2n}\int_{\Delta_i}|b_i|^p|\mathcal{L}_r(D)x_n(t)|^p dt = \frac{1}{2n}\sum_{i=1}^{2n}|b_i|^p \tag{2.10}$$

and by setting

$$z_i(\cdot) := \frac{1}{\pi}\int_{\mathbb{T}} G_r(\cdot-\tau)y_i(\tau)d\tau, \quad i=1,2,\ldots,2n, \tag{2.11}$$

we reduce problem (2.9) to the form

$$\frac{\left\|\sum_{j=1}^{\mu} a_j\varphi_j(\cdot)+\sum_{i=1}^{2n} b_i z_i(\cdot)\right\|_q^q}{1/2n\sum_{i=1}^{2n}|b_i|^p} \longrightarrow \inf, \quad a_1,\ldots,a_\mu,b_1,\ldots,b_{2n}\in\mathbb{R}. \tag{2.12}$$

This is a smooth finite-dimensional problem. It has a solution $(\overline{a}_1,\ldots\overline{a}_\mu,\overline{b}_1,\ldots,\overline{b}_{2n})$ and, $(\overline{b}_1,\ldots,\overline{b}_{2n})\neq\vec{0}$. According to the Lagrange multiplier rule, there exists a $\eta\in\mathbb{R}$ such that the derivatives of the function $(a_1,\ldots,a_\mu,b_1,\ldots,b_{2n}) \to g(a_1,\ldots,a_\mu,b_1,\ldots,b_{2n})+\eta(b_1+b_2+\cdots+b_{2n})$ (where $g(\cdot)$ is the function being minimized in (2.12)) with respect to $a_1,\ldots,a_\mu,b_1,\ldots,b_{2n}$ at the point $(\overline{a}_1,\ldots,\overline{a}_\mu,\overline{b}_1,\ldots,\overline{b}_{2n})$ are equal to zero. This leads to the relations

$$\int_{\mathbb{T}}\varphi_j(t)\left(Q_q\overline{x}\right)(t)dt=0, \quad j=1,\ldots,\mu, \tag{2.13}$$

$$\int_{\mathbb{T}} z_i(t)\left(Q_q\overline{x}\right)(t)dt=\frac{1}{2n}\frac{\left\|\overline{x(\cdot)}\right\|_q^q}{\|\mathcal{L}_r(D)\overline{x}(\cdot)\|_p^p}Q_p\overline{b}_i, \quad i=1,\ldots,2n, \tag{2.14}$$

where $\overline{x}(\cdot)=\sum_{j=1}^{\mu}\overline{a}_j\varphi_j(t)+\sum_{i=1}^{2n}\overline{b}_i z_i(\cdot)$.

We remark that $g(a_1,\ldots,a_\mu,b_1,\ldots,b_{2n})=g(da_1,\ldots,da_\mu,db_1,\ldots,db_{2n})$ for any $d\neq 0$, and hence the vector $(d\overline{a}_1,\ldots,d\overline{a}_\mu,d\overline{b}_1,\ldots,d\overline{b}_{2n})$ is also a solution of (2.12). Thus, it can be assumed that $|\overline{b}_i|\le 1$, $i=1,\ldots,2n$ and $\overline{b}_{i_0}=(-1)^{i_0+1}$ for some i_0, $1\le i_0\le 2n$.

Let

$$\widetilde{x}_{2n}(t)=\sum_{j=1}^{\mu}a_j^{\star}\varphi_j(t)+\sum_{i=1}^{2n}(-1)^{i+1}z_i(t) \tag{2.15}$$

and $\widetilde{x}_{2n}$ satisfies (1.7). Let $a^{\star}=(a_1^{\star},\ldots,a_{2n}^{\star})$ and let $b^{\star}=(1,-1,\ldots,1,-1)\in\mathbb{R}^{2n}$. It follows from the definitions of $\widetilde{x}_{2n}(\cdot)$ and $\overline{x}(\cdot)$ that

$$\mathcal{L}_r(D)\widetilde{x}_{2n}(t)-\mathcal{L}_r(D)\overline{x}(t)=\sum_{\substack{i=1\\ i\neq i_0}}^{2n}\left((-1)^{i+1}-\overline{b}_i\right)\chi_i(t)\mathcal{L}_r(D)x_{2n}\left(t-\frac{(i-1)\pi}{n}\right), \tag{2.16}$$

and hence $S_c(\mathcal{L}_r(D)\widetilde{x}_{2n}(\cdot),\ \mathcal{L}_r(D)\overline{x}(\cdot))$ has at most $2n-2$ sign changes. Then, by Rolle's theorem, $S_c(\mathcal{L}_r(D)\widetilde{x}_{2n}(\cdot)-\mathcal{L}_r(D)\overline{x}(\cdot))\le 2n-2$. For any $a,b\in\mathbb{R}$, $\operatorname{sign}(a+b)=\operatorname{sign}(Q_pa+Q_pb)$; therefore

$$S_c\left(\left(Q_q\widetilde{x}_{2n}\right)(\cdot)-\left(Q_q\overline{x}\right)(\cdot)\right)\le 2n-2. \tag{2.17}$$

In addition, since $\widetilde{x}_{2n}$ is 2π-periodic solution of the linear differential equation, $\mathcal{L}_r(D)y(t) = (-1)^r\lambda^{-q}(Q_q x)(t)$ and $\varphi_j(t) \in \operatorname{Ker}\mathcal{L}_r(D)$. Then, by [8, page 94], we have

$$\int_{\mathbb{T}} \varphi_j(t)\big(Q_q\widetilde{x}\big)(t)dt = 0, \quad j = 1,\dots,\mu. \tag{2.18}$$

If we now multiply both sides of (2.15) by $(Q_q\widetilde{x}_{2n})(t)$ and integrate over the interval Δ_i, $1 \le i \le 2n$, we get

$$\int_{\Delta_i} z_i(t)\big(Q_q\widetilde{x}_{2n}\big)(t)dt = (-1)^{i+1}\int_{\Delta_i} |\widetilde{x}_{2n}(t)|^q dt = (-1)^{i+1}\frac{\lambda_{2n}^q}{2n}, \tag{2.19}$$

due to $\int_{\mathbb{T}} z_i(t)(Q_q\widetilde{x}_{2n})(t)dt = \int_{\Delta_i} z_i(t)(Q_q\widetilde{x}_{2n})(t)dt$. Therefore, we have

$$\int_{\mathbb{T}} z_i(t)\big(Q_q\widetilde{x}_{2n}\big)(t)dt = (-1)^{i+1}\frac{\lambda_{2n}^q}{2n}, \quad i = 1,\dots,2n. \tag{2.20}$$

Changing the order of integration and using (2.14) and (2.20), we get that

$$\begin{aligned}
&\int_{\Delta_i} \mathcal{L}_r(D)x_{2n}\left(t - \frac{(i-1)\pi}{n}\right)\left(\frac{1}{\pi}\int_{\mathbb{T}} G_r(t-\tau)\big(\big(Q_q\widetilde{x}_{2n}\big)(\tau) - \big(Q_q\overline{x}\big)(\tau)\big)d\tau\right)dt \\
&\quad= \int_{\mathbb{T}} z_i(t)\big(\big(Q_q\widetilde{x}_{2n}\big)(t) - \big(Q_q\overline{x}\big)(t)\big)dt = \frac{(-1)^r}{2n}\left((-1)^{i+1}\lambda_{2n}^q - \frac{\left\|\overline{x(\cdot)}\right\|_q^q}{\|(\mathcal{L}_r(D)\overline{x})(\cdot)\|_p^p}Q_p\overline{b}_i\right) \\
&\quad= \frac{(-1)^r}{2n}\Big((-1)^{i+1}\lambda_{2n}^q - \alpha^q\|(\mathcal{L}_r(D)\overline{x})(\cdot)\|^{q-p}Q_p\overline{b_i}\Big), \quad i = 1,\dots,2n.
\end{aligned} \tag{2.21}$$

Denote by $f(\cdot)$ the factor multiply $\mathcal{L}_r(D)x_{2n}(t - (i-1)\pi/n)$ in the integral in the left-hand side of this equality. Since $\|(\mathcal{L}_r(D)\overline{x})(\cdot)\|_p \le 1$ and hence $\|(\mathcal{L}_r(D)\overline{x})(\cdot)\|_p^{q-p} \le 1$ for $p \le q$, if we assume that $\lambda_{2n} > \alpha$, then we arrive at the relations

$$\operatorname{sign}\int_{\Delta_i} \mathcal{L}_r(D)x_{2n}\left(t - \frac{(i-1)\pi}{n}\right)f(\cdot)dt = (-1)^{r+i+1}, \quad i = 1,\dots,2n. \tag{2.22}$$

Suppose for definiteness that $\mathcal{L}_r(D)x_{2n}(t - (i-1)\pi/n) > 0$ interior to Δ_i, $i = 1,\dots,2n$. Then it follows from (2.22) that there are points $t_i \in \Delta_i$ such that $\operatorname{sign} f(t_i) = (-1)^{i+1}$, $i = 1,\dots,2n$, that is, $S_c(f(\cdot)) \ge 2n-1$. But $f(\cdot)$ is periodic, and hence $S_c(f(\cdot)) \ge 2n$; therefore, $S_c(\mathcal{L}_r(D)f(\cdot)) \ge 2n$. Further, $\mathcal{L}_r(D)f(\cdot) = (Q_q\widetilde{x}_{2n})(t) - (Q_q\overline{x})(t)$, that is, $S_c((Q_q\widetilde{x}_{2n})(t) - (Q_q\overline{x})(t)) \ge 2n$.

We have arrived at a contradiction to (2.17), and hence $\lambda_{2n} \le \alpha$. Thus $b_{2n-1}(W_p(\mathcal{L}_r); L_q) \ge \lambda_{2n}$.

2.2. *Upper Estimate of Bernstein n-Width*

Assume the contrary: $b_{2n-1}(W_p(\mathcal{L}_r); L_q) > \lambda_{2n}$, $(1 < p \le q < \infty)$. Then, by definition, there exists a linearly independent system of $2n$ functions $L_{2n} := \operatorname{span}\{f_1, \dots, f_{2n}\} \subset L_q$ and number $\gamma > \lambda_{2n}$ such that $L_{2n} \cap \gamma S(L_q) \subseteq \mathcal{L}_r(D)$, or equivalently,

$$\min_{x(\cdot)\in L_{2n}} \frac{\|x(\cdot)\|_q}{\|\mathcal{L}_r(D)x(\cdot)\|_p} \ge \gamma > \lambda_{2n}. \tag{2.23}$$

Let us assign a vector $c \in \mathbb{R}^{2n}$ to each function $x(\cdot) \in L_{2n}$ by the following rule:

$$x(\cdot) \longrightarrow c = (c_1, \dots, c_{2n}) \in \mathbb{R}^{2n}, \quad \text{where } x(\cdot) = \sum_{j=1}^{2n} c_j f_j(\cdot). \tag{2.24}$$

Then inequality (2.23) acquires the form

$$\min_{c\in\mathbb{R}^{2n}\setminus\{0\}} \frac{\left\|\sum_{j=1}^{2n} c_j f_j(\cdot)\right\|_q}{\left\|\sum_{j=1}^{2n} c_j \mathcal{L}_r(D) f_j(\cdot)\right\|_p} \ge \gamma > \lambda_{2n}. \tag{2.25}$$

Let $c_0 = 0$. Consider the sphere S^{2n-1} in the space $\mathbb{R}^{2n}$ with radius 2π, that is,

$$S^{2n-1} := \left\{ c : c = (c_1, \dots, c_{2n}) \in \mathbb{R}^{2n},\ \|c\| = \sum_{j=1}^{2n} |c_j| = 2\pi \right\}. \tag{2.26}$$

To every vector $c \in \mathbb{R}^{2n}$ we assign function $u(t, c)$ defined by

$$u(t,c) = \begin{cases} (2\pi)^{-1/p} \operatorname{sign} c_k, & \text{for } t \in (t_{k-1}, t_k),\ k = 1, \dots, 2n, \\ 0, & \text{for } t = t_k,\ k = 1, \dots, 2n-1, \end{cases} \tag{2.27}$$

where $t_0 = 0$, $t_k = \sum_{i=1}^{k} |c_i|$, $k = 1, \dots, 2n$, and the extended 2π-periodically onto $\mathbb{R}$.

An analog of the Buslaev iteration process [5] is constructed in the following way: the function $x(t, c)$ is found as a periodic solution of the linear differential equation $\mathcal{L}_r(D)x_0 = u$; then the periodic functions $\{x_k(t, c)\}_{k\in\mathbb{N}^+}$ are successively determined from the differential equations

$$\begin{aligned} \mathcal{L}_r(D)x_k(t) &= (Q_{p'} y_k)(t), \\ \mathcal{L}_r(D)y_k(t) &= (-1)^r \mu_{k-1}^{-q} (Q_q x_{k-1})(t), \end{aligned} \tag{2.28}$$

where $p' = p/(p-1)$, and the constants $\{\mu_k : k = 0, \ldots\}$ are uniquely determined by the conditions

$$\|\mathcal{L}_r(D)x_k\|_p = 1, \qquad \left(Q_q x_k\right)(t) \perp \operatorname{Ker}\mathcal{L}_r(D), \qquad \left(Q_{p'} y_k\right)(t) \perp \operatorname{Ker}\mathcal{L}_r(D). \tag{2.29}$$

By analogy with the reasoning in [5], we can prove the following assertions.

(i) The iteration procedure (2.28)-(2.29) is well defined; the sequences $\{\mu_k\}_{k\in\mathbb{N}}$ are monotone nondecreasing and converge to an eigenvalue $\lambda(c) > 0$ of the problem (1.7).

(ii) The sequence $\{x_k(\cdot, c)\}_{k\in\mathbb{N}}$ has a subsequence that is convergent to an eigenfunction $x(\cdot, c)$ of the problem (1.7), with $\lambda(c) = \|x(\cdot, c)\|_p$.

(iii) For any $k \in \mathbb{N}$ there exists a $\widehat{c} \in S^{2n-1}$ such that $x_k(\cdot, \widehat{c})$ has at least $2n$ zeros $(Z_c(x_k(\cdot, \widehat{c})) \geq 2n)$ on $\mathbb{T}$.

(iv) In the set of spectral pairs $(\lambda(c), x(\cdot, c))$, there exists a pair $(\lambda(\widehat{c}), x(\cdot, \widehat{c}))$ such that $S_c(x(\cdot, \widehat{c}) = 2N \geq 2n$.

Items (i) and (ii) can be proved in the same way as Lemmas 1 and 2 of [5, Sections 6 and 10]. Item (iii) follows from the Borsuk theorem [10], which states that there exists a $\widehat{c} \in S^{2n-1}$ such that $Z_c(x_k(\cdot, \widehat{c})) \geq 2n-1$, but since the function $x_k(\cdot, \widehat{c})$ is periodic, we actually have $Z_c(x_k(\cdot, \widehat{c})) \geq 2n$. Finally, in item (iv), by (ii) and (iii), $Z_c(x(\cdot, \widehat{c})) \geq 2n$. In view of $x(\cdot, \widehat{c})$ zeros are simple; therefore, $S_c(x(\cdot, \widehat{c})) \geq 2n$.

Note that [8] the linear differential equation $\mathcal{L}_r(D)f = g$ has a 2π-periodic solution if and only if $\int_{\mathbb{T}} g(t)v(t)dt = 0$, where $v(\cdot) \in \operatorname{Ker}\mathcal{L}_r(D)$ and g is an integrable 2π-periodic function. Using the method similar to [5, 11], it is not difficult to show that spectral pairs of (1.7) are unique and spectral value λ_n is monotone decreasing for n; it follows that

$$\lambda(\widehat{c}) = \lambda_{2N} \leq \lambda_{2n}. \tag{2.30}$$

Therefore, by virtue of items (i), (ii), and (2.30), we obtain

$$\begin{aligned}\min_{c\in\mathbb{R}^{2n}\setminus\{0\}} \frac{\left\|\sum_{j=1}^{2n} c_j f_j(\cdot)\right\|_q}{\left\|\sum_{j=1}^{2n} c_j \mathcal{L}_r(D) f_j(\cdot)\right\|_p} &\leq \frac{\left\|\sum_{j=1}^{2n} \widehat{c}_j f_j(\cdot)\right\|_q}{\left\|\sum_{j=1}^{2n} \widehat{c}_j \mathcal{L}_r(D) f_j(\cdot)\right\|_p} \\ &\leq \frac{\|x_k(\cdot, \widehat{c})\|_q}{\|\mathcal{L}_r(D)x_k(\cdot, \widehat{c})\|_p} \leq \lambda(\widehat{c}) \leq \lambda_{2n},\end{aligned} \tag{2.31}$$

which contradicts (2.25). Hence $b_{2n-1}(W_p(\mathcal{L}_r); L_q) \leq \lambda_{2n}$. Thus, the upper bound is proved. This completes the proof of the theorem.

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