Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2012, Article ID 492951, 20 pages
doi:10.1155/2012/492951

Research Article

An Iterative Algorithm for the Generalized Reflexive Solution of the Matrix Equations $AXB = E$, $CXD = F$

Deqin Chen,[1] Feng Yin,[1] and Guang-Xin Huang[2]

[1] *School of Science, Sichuan University of Science and Engineering, Zigong 643000, China*
[2] *College of Management Science, Key Laboratory of Geomathematics in Sichuan, Chengdu University of Technology, Chengdu 610059, China*

Correspondence should be addressed to Deqin Chen, chendeqin@suse.edu.cn

Received 21 December 2011; Revised 30 April 2012; Accepted 16 May 2012

Academic Editor: Jinyun Yuan

An iterative algorithm is constructed to solve the linear matrix equation pair $AXB = E$, $CXD = F$ over generalized reflexive matrix X. When the matrix equation pair $AXB = E$, $CXD = F$ is consistent over generalized reflexive matrix X, for any generalized reflexive initial iterative matrix X_1, the generalized reflexive solution can be obtained by the iterative algorithm within finite iterative steps in the absence of round-off errors. The unique least-norm generalized reflexive iterative solution of the matrix equation pair can be derived when an appropriate initial iterative matrix is chosen. Furthermore, the optimal approximate solution of $AXB = E$, $CXD = F$ for a given generalized reflexive matrix X_0 can be derived by finding the least-norm generalized reflexive solution of a new corresponding matrix equation pair $A\widetilde{X}B = \widetilde{E}$, $C\widetilde{X}D = \widetilde{F}$ with $\widetilde{E} = E - AX_0B$, $\widetilde{F} = F - CX_0D$. Finally, several numerical examples are given to illustrate that our iterative algorithm is effective.

1. Introduction

Let $\mathcal{R}^{m\times n}$ denote the set of all m-by-n real matrices. I_n denotes the n order identity matrix. Let $P \in \mathcal{R}^{m\times m}$ and $Q \in \mathcal{R}^{n\times n}$ be two real generalized reflection matrices, that is, $P^T = P$, $P^2 = I_m$, $Q^T = Q$, $Q^2 = I_n$. A matrix $A \in \mathcal{R}^{m\times n}$ is called generalized reflexive matrix with respect to the matrix pair (P, Q) if $PAQ = A$. For more properties and applications on generalized reflexive matrix, we refer to [1, 2]. The set of all m-by-n real generalized reflexive matrices with respect to matrix pair (P, Q) is denoted by $\mathcal{R}_r^{m\times n}(P, Q)$. We denote by the superscripts T the transpose of a matrix. In matrix space $\mathcal{R}^{m\times n}$, define inner product as $\mathrm{tr}(B^TA) = \mathrm{trace}(B^TA)$ for all $A, B \in \mathcal{R}^{m\times n}$; $\|A\|$ represents the Frobenius norm of A. $\mathcal{R}(A)$ represents the column space of A. vec$(\cdot)$ represents the vector operator; that is, $\text{vec}(A) = (a_1^T, a_2^T, \ldots, a_n^T)^T \in \mathcal{R}^{mn}$ for the matrix $A = (a_1, a_2, \ldots, a_n) \in \mathcal{R}^{m\times n}$, $a_i \in R^m$, $i = 1, 2, \ldots, n$. $A \otimes B$ stands for the Kronecker product of matrices A and B.

In this paper, we will consider the following two problems.

Problem 1. For given matrices $A \in \mathcal{R}^{p\times m}$, $B \in \mathcal{R}^{n\times q}$, $C \in \mathcal{R}^{s\times m}$, $D \in \mathcal{R}^{n\times t}$, $E \in \mathcal{R}^{p\times q}$, $F \in \mathcal{R}^{s\times t}$, find matrix $X \in \mathcal{R}_r^{m\times n}(P, Q)$ such that

$$AXB = E, \qquad CXD = F. \tag{1.1}$$

Problem 2. When Problem 1 is consistent, let S_E denote the set of the generalized reflexive solutions of Problem 1. For a given matrix $X_0 \in \mathcal{R}_r^{m\times n}(P, Q)$, find $\widehat{X} \in S_E$ such that

$$\left\|\widehat{X} - X_0\right\| = \min_{X\in S_E}\|X - X_0\|. \tag{1.2}$$

The matrix equation pair (1.1) may arise in many areas of control and system theory. Dehghan and Hajarian [3] presented some examples to show a motivation for studying (1.1). Problem 2 occurs frequently in experiment design; see for instance [4]. In recent years, the matrix nearness problem has been studied extensively (e.g., [3, 5–19]).

Research on solving the matrix equation pair (1.1) has been actively ongoing for last 40 or more years. For instance, Mitra [20, 21] gave conditions for the existence of a solution and a representation of the general common solution to the matrix equation pair (1.1). Shinozaki and Sibuya [22] and vander Woude [23] discussed conditions for the existence of a common solution to the matrix equation pair (1.1). Navarra et al. [11] derived sufficient and necessary conditions for the existence of a common solution to (1.1). Yuan [18] obtained an analytical expression of the least-squares solutions of (1.1) by using the generalized singular value decomposition (GSVD) of matrices. Recently, some finite iterative algorithms have also been developed to solve matrix equations. Deng et al. [24] studied the consistent conditions and the general expressions about the Hermitian solutions of the matrix equations $(AX, XB) = (C, D)$ and designed an iterative method for its Hermitian minimum norm solutions. Li and Wu [25] gave symmetric and skew-antisymmetric solutions to certain matrix equations $A_1X = C_1$, $XB_3 = C_3$ over the real quaternion algebra H. Dehghan and Hajarian [26] proposed the necessary and sufficient conditions for the solvability of matrix equations $A_1XB_1 = D_1, A_1X = C_1$, $XB_2 = C_2$ and $A_1X = C_1$, $XB_2 = C_2$, $A_3X = C_3$, $XB_4 = C_4$ over the reflexive or antireflexive matrix X and obtained the general expression of the solutions for a solvable case. Wang [27, 28] gave the centrosymmetric solution to the system of quaternion matrix equations $A_1X = C_1$, $A_3XB_3 = C_3$. Wang [29] also solved a system of matrix equations over arbitrary regular rings with identity. For more studies on iterative algorithms on coupled matrix equations, we refer to [6, 7, 15–17, 19, 30–34]. Peng et al. [13] presented iterative methods to obtain the symmetric solutions of (1.1). Sheng and Chen [14] presented a finite iterative method when (1.1) is consistent. Liao and Lei [9] presented an analytical expression of the least-squares solution and an algorithm for (1.1) with the minimum norm. Peng et al. [12] presented an efficient algorithm for the least-squares reflexive solution. Dehghan and Hajarian [3] presented an iterative algorithm for solving a pair of matrix equations (1.1) over generalized centrosymmetric matrices. Cai and Chen [35] presented an iterative algorithm for the least-squares bisymmetric solutions of the matrix equations (1.1). However, the problem of finding the generalized reflexive solutions of matrix equation pair (1.1) has not been solved. In this paper, we construct an iterative algorithm by which the solvability of Problem 1 can be determined automatically, the solution can be obtained within finite iterative steps when Problem 1 is consistent, and the solution of Problem 2 can be obtained by finding the least-norm generalized reflexive solution of a corresponding matrix equation pair.

This paper is organized as follows. In Section 2, we will solve Problem 1 by constructing an iterative algorithm; that is, if Problem 1 is consistent, then for an arbitrary initial matrix $X_1 \in \mathcal{R}_r^{m\times n}(P,Q)$, we can obtain a solution $\overline{X} \in \mathcal{R}_r^{m\times n}(P,Q)$ of Problem 1 within finite iterative steps in the absence of round-off errors. Let $X_1 = A^T HB^T + C^T \widehat{H} D^T + PA^T HB^T Q + PC^T \widehat{H} D^T Q$, where $H \in \mathcal{R}^{p\times q}$, $\widehat{H} \in \mathcal{R}^{s\times t}$ are arbitrary matrices, or more especially, letting $X_1 = \mathbf{0} \in \mathcal{R}_r^{m\times n}(P,Q)$, we can obtain the unique least norm solution of Problem 1. Then in Section 3, we give the optimal approximate solution of Problem 2 by finding the least norm generalized reflexive solution of a corresponding new matrix equation pair. In Section 4, several numerical examples are given to illustrate the application of our iterative algorithm.

2. The Solution of Problem 1

In this section, we will first introduce an iterative algorithm to solve Problem 1 and then prove that it is convergent. The idea of the algorithm and it's proof in this paper are originally inspired by those in [13]. The idea of our algorithm is also inspired by those in [3]. When $P = Q$, $R = S$, $X^T = X$ and $Y^T = Y$, the results in this paper reduce to those in [3].

Algorithm 2.1. Step 1. Input matrices $A \in \mathcal{R}^{p\times m}$, $B \in \mathcal{R}^{n\times q}$, $C \in \mathcal{R}^{s\times m}$, $D \in \mathcal{R}^{n\times t}$, $E \in \mathcal{R}^{p\times q}$, $F \in \mathcal{R}^{s\times t}$, and two generalized reflection matrix $P \in \mathcal{R}^{m\times m}$, $Q \in \mathcal{R}^{n\times n}$.

Step 2. Choose an arbitrary matrix $X_1 \in \mathcal{R}_r^{m\times n}(P,Q)$. Compute

$$\begin{gathered} R_1 = \begin{pmatrix} E - AX_1B & 0 \\ 0 & F - CX_1D \end{pmatrix}, \\ P_1 = \frac{1}{2}\Big(A^T(E - AX_1B)B^T + C^T(F - CX_1D)D^T + PA^T(E - AX_1B)B^TQ \\ + PC^T(F - CX_1D)D^TQ\Big), \end{gathered} \tag{2.1}$$

$k := 1$.

Step 3. If $R_1 = \mathbf{0}$, then stop. Else go to Step 4.

Step 4. Compute

$$X_{k+1} = X_k + \frac{\|R_k\|^2}{\|P_k\|^2} P_k,$$

$$R_{k+1} = \begin{pmatrix} E - AX_{k+1}B & 0 \\ 0 & F - CX_{k+1}D \end{pmatrix} = R_k - \frac{\|R_k\|^2}{\|P_k\|^2} \begin{pmatrix} AP_kB & 0 \\ 0 & CP_kD \end{pmatrix},$$

$$P_{k+1} = \frac{1}{2}\Big(A^T(E - AX_{k+1}B)B^T + C^T(F - CX_{k+1}D)D^T + PA^T(E - AX_{k+1}B)B^TQ + PC^T(F - CX_{k+1}D)D^TQ\Big) + \frac{\|R_{k+1}\|^2}{\|R_k\|^2}P_k. \tag{2.2}$$

Step 5. If $R_{k+1} = \mathbf{0}$, then stop. Else, letting $k := k + 1$, go to Step 4.

Obviously, it can be seen that $P_i \in R_r^{m\times n}(P, Q)$, $X_i \in R_r^{m\times n}(P, Q)$, where $i = 1, 2, \ldots$.

Lemma 2.2. *For the sequences $\{R_i\}$ and $\{P_i\}$ generated in Algorithm 2.1, one has*

$$\begin{aligned}
\operatorname{tr}\left(R_{i+1}^T R_j\right) &= \operatorname{tr}\left(R_i^T R_j\right) - \frac{\|R_i\|^2}{\|P_i\|^2}\operatorname{tr}\left(P_i^T P_j\right) + \frac{\|R_i\|^2\|R_j\|^2}{\|P_i\|^2\|R_{j-1}\|^2}\operatorname{tr}\left(P_i^T P_{j-1}\right),\\
\operatorname{tr}\left(P_{i+1}^T P_j\right) &= \frac{\|P_j\|^2}{\|R_j\|^2}\left(\operatorname{tr}\left(R_{i+1}^T R_j\right) - \operatorname{tr}\left(R_{i+1}^T R_{j+1}\right)\right) + \frac{\|R_{i+1}\|^2}{\|R_i\|^2}\operatorname{tr}\left(P_i^T P_j\right).
\end{aligned} \tag{2.3}$$

Proof. By Algorithm 2.1, we have

$$\begin{aligned}
\operatorname{tr}\left(R_{i+1}^T R_j\right) &= \operatorname{tr}\left(\left(\left(R_i - \frac{\|R_i\|^2}{\|P_i\|^2}\begin{pmatrix} AP_iB & 0 \\ 0 & CP_iD\end{pmatrix}\right)\right)^T R_j\right)\\
&= \operatorname{tr}\left(R_i^T R_j\right) - \frac{\|R_i\|^2}{\|P_i\|^2}\operatorname{tr}\left(\begin{pmatrix} B^TP_i^TA^T & 0 \\ 0 & D^TP_i^TC^T\end{pmatrix}R_j\right)\\
&= \operatorname{tr}\left(R_i^T R_j\right) - \frac{\|R_i\|^2}{\|P_i\|^2}\operatorname{tr}\left(\begin{pmatrix} B^TP_i^TA^T & 0 \\ 0 & D^TP_i^TC^T\end{pmatrix}\begin{pmatrix} E - AX_jB & 0 \\ 0 & F - CX_jD\end{pmatrix}\right)\\
&= \operatorname{tr}\left(R_i^T R_j\right) - \frac{\|R_i\|^2}{\|P_i\|^2}\operatorname{tr}\left(B^TP_i^TA^T(E - AX_jB) + D^TP_i^TC^T(F - CX_jD)\right)\\
&= \operatorname{tr}\left(R_i^T R_j\right) - \frac{\|R_i\|^2}{\|P_i\|^2}\operatorname{tr}\left(P_i^T\left(A^T(E - AX_jB)B^T + C^T(F - CX_jD)D^T\right)\right)\\
&= \operatorname{tr}\left(R_i^T R_j\right) - \frac{\|R_i\|^2}{\|P_i\|^2}
\end{aligned}$$

$$\times \operatorname{tr}\left(P_i^T\left(\frac{A^T(E-AX_jB)B^T+C^T(F-CX_jD)D^T}{2}\right.\right.$$

$$+\frac{PA^T(E-AX_jB)B^TQ+PC^T(F-CX_jD)D^TQ}{2}$$

$$+\frac{A^T(E-AX_jB)B^T+C^T(F-CX_jD)D^T}{2}$$

$$\left.\left.+\frac{-PA^T(E-AX_jB)B^TQ-PC^T(F-CX_jD)D^TQ}{2}\right)\right)$$

$$=\operatorname{tr}\left(R_i^TR_j\right)-\frac{\|R_i\|^2}{\|P_i\|^2}$$

$$\times \operatorname{tr}\left(P_i^T\left\{\frac{A^T(E-AX_jB)B^T+C^T(F-CX_jD)D^T}{2}\right.\right.$$

$$\left.\left.+\frac{PA^T(E-AX_jB)B^TQ+PC^T(F-CX_jD)D^TQ}{2}\right\}\right)$$

$$=\operatorname{tr}\left(R_i^TR_j\right)-\frac{\|R_i\|^2}{\|P_i\|^2}\operatorname{tr}\left(P_i^T\left(P_j-\frac{\|R_j\|^2}{\|R_{j-1}\|^2}P_{j-1}\right)\right)$$

$$=\operatorname{tr}\left(R_i^TR_j\right)-\frac{\|R_i\|^2}{\|P_i\|^2}\operatorname{tr}\left(P_i^TP_j\right)+\frac{\|R_i\|^2\|R_j\|^2}{\|P_i\|^2\|R_{j-1}\|^2}\operatorname{tr}\left(P_i^TP_{j-1}\right),$$

$$\operatorname{tr}\left(P_{i+1}^TP_j\right)=\operatorname{tr}\left(\left(\frac{A^T(E-AX_{i+1}B)B^T+C^T(F-CX_{i+1}D)D^T}{2}\right.\right.$$

$$\left.\left.+\ \frac{PA^T(E-AX_{i+1}B)B^TQ+PC^T(F-CX_{i+1}D)D^TQ}{2}+\frac{\|R_{i+1}\|^2}{\|R_i\|^2}P_i\right)^TP_j\right)$$

$$=\operatorname{tr}\left(\left(\frac{A^T(E-AX_{i+1}B)B^T+C^T(F-CX_{i+1}D)D^T}{2}\right.\right.$$

$$\left.\left.+\ \frac{PA^T(E-AX_{i+1}B)B^TQ+PC^T(F-CX_{i+1}D)D^TQ}{2}\right)^TP_j\right)$$

$$+\frac{\|R_{i+1}\|^2}{\|R_i\|^2}\operatorname{tr}\left(P_i^TP_j\right)$$

$$=\operatorname{tr}\left(P_j^T\left(A^T(E-AX_{i+1}B)B^T+C^T(F-CX_{i+1}D)D^T\right)\right)+\frac{\|R_{i+1}\|^2}{\|R_i\|^2}\operatorname{tr}\left(P_i^TP_j\right)$$

$$
\begin{aligned}
&= \operatorname{tr}\left((E - AX_{i+1}B)^T AP_jB + (F - CX_{i+1}D)^T CP_jD\right) + \frac{\|R_{i+1}\|^2}{\|R_i\|^2}\operatorname{tr}\left(P_i^T P_j\right) \\
&= \operatorname{tr}\left(\begin{pmatrix} (E - AX_{i+1}B)^T & 0 \\ 0 & (F - CX_{i+1}D)^T \end{pmatrix}\begin{pmatrix} AP_jB & 0 \\ 0 & CP_jD \end{pmatrix}\right) \\
&\quad + \frac{\|R_{i+1}\|^2}{\|R_i\|^2}\operatorname{tr}\left(P_i^T P_j\right) \\
&= \frac{\|P_j\|^2}{\|R_j\|^2}\operatorname{tr}\left(R_{i+1}^T (R_j - R_{j+1})\right) + \frac{\|R_{i+1}\|^2}{\|R_i\|^2}\operatorname{tr}\left(P_i^T P_j\right) \\
&= \frac{\|P_j\|^2}{\|R_j\|^2}\left(\operatorname{tr}\left(R_{i+1}^T R_j\right) - \operatorname{tr}\left(R_{i+1}^T R_{j+1}\right)\right) + \frac{\|R_{i+1}\|^2}{\|R_i\|^2}\operatorname{tr}\left(P_i^T P_j\right).
\end{aligned}
\tag{2.4}
$$

This completes the proof. □

Lemma 2.3. *For the sequences* $\{R_i\}$ *and* $\{P_i\}$ *generated by Algorithm 2.1, and* $k \geq 2$, *one has*

$$
\operatorname{tr}\left(R_i^T R_j\right) = 0, \quad \operatorname{tr}\left(P_i^T P_j\right) = 0, \quad i, j = 1, 2, \ldots, k, \; i \neq j. \tag{2.5}
$$

Proof. Since $\operatorname{tr}(R_i^T R_j) = \operatorname{tr}(R_j^T R_i)$ and $\operatorname{tr}(P_i^T P_j) = \operatorname{tr}(P_j^T P_i)$ for all $i, j = 1, 2, \ldots, k$, we only need to prove that $\operatorname{tr}(R_i^T R_j) = 0$, $\operatorname{tr}(P_i^T P_j) = 0$ for all $1 \leq j < i \leq k$. We prove the conclusion by induction, and two steps are required.

Step 1. We will show that

$$
\operatorname{tr}\left(R_{i+1}^T R_i\right) = 0, \quad \operatorname{tr}\left(P_{i+1}^T P_i\right) = 0, \quad i = 1, 2, \ldots, k-1. \tag{2.6}
$$

To prove this conclusion, we also use induction.

For $i = 1$, by Algorithm 2.1 and the proof of Lemma 2.2, we have that

$$
\begin{aligned}
\operatorname{tr}\left(R_2^T R_1\right) &= \operatorname{tr}\left(\left(R_1 - \frac{\|R_1\|^2}{\|P_1\|^2}\begin{pmatrix} AP_1B & 0 \\ 0 & CP_1D \end{pmatrix}\right)^T R_1\right) \\
&= \operatorname{tr}\left(R_1^T R_1\right) - \frac{\|R_1\|^2}{\|P_1\|^2}
\end{aligned}
$$

$$\begin{aligned}
&\times \operatorname{tr}\left(P_1^T\left\{\frac{A^T(E-AX_1B)B^T+C^T(F-CX_1D)D^T}{2}\right.\right.\\
&\qquad\left.\left.+\frac{PA^T(E-AX_1B)B^TQ+PC^T(F-CX_1D)D^TQ}{2}\right\}\right)\\
&=\|R_1\|^2-\frac{\|R_1\|^2}{\|P_1\|^2}\operatorname{tr}\left(P_1^TP_1\right)\\
&=0,\\
\operatorname{tr}\left(P_2^TP_1\right)&=\frac{\|P_1\|^2}{\|R_1\|^2}\left(\operatorname{tr}\left(R_2^TR_1\right)-\operatorname{tr}\left(R_2^TR_2\right)\right)+\frac{\|R_2\|^2}{\|R_1\|^2}\|P_1\|^2\\
&=0.
\end{aligned} \tag{2.7}$$

Assume (2.6) holds for $i=s-1$, that is, $\operatorname{tr}(R_s^TR_{s-1})=0$, $\operatorname{tr}(P_s^TP_{s-1})=0$. When $i=s$, by Lemma 2.2, we have that

$$\begin{aligned}
\operatorname{tr}\left(R_{s+1}^TR_s\right)&=\operatorname{tr}\left(R_s^TR_s\right)-\frac{\|R_s\|^2}{\|P_s\|^2}\operatorname{tr}\left(P_s^TP_s\right)+\frac{\|R_s\|^4}{\|P_s\|^2\|R_{s-1}\|^2}\operatorname{tr}\left(P_s^TP_{s-1}\right)\\
&=\|R_s\|^2-\|R_s\|^2+\frac{\|R_s\|^4}{\|P_s\|^2\|R_{s-1}\|^2}\operatorname{tr}\left(P_s^TP_{s-1}\right)\\
&=0,\\
\operatorname{tr}\left(P_{s+1}^TP_s\right)&=\frac{\|P_s\|^2}{\|R_s\|^2}\left(\operatorname{tr}\left(R_{s+1}^TR_s\right)-\operatorname{tr}\left(R_{s+1}^TR_{s+1}\right)\right)+\frac{\|R_{s+1}\|^2}{\|R_s\|^2}\operatorname{tr}\left(P_s^TP_s\right)\\
&=-\frac{\|P_s\|^2}{\|R_s\|^2}\|R_{s+1}\|^2+\frac{\|R_{s+1}\|^2}{\|R_s\|^2}\|P_s\|^2\\
&=0.
\end{aligned} \tag{2.8}$$

Hence, (2.6) holds for $i=s$. Therefor, (2.6) holds by the principle of induction.

Step 2. Assuming that $\operatorname{tr}(R_s^TR_j)=0$, $\operatorname{tr}(P_s^TP_j)=0$, $j=1,2,\ldots,s-1$, then we show that

$$\operatorname{tr}\left(R_{s+1}^TR_j\right)=0,\quad \operatorname{tr}\left(P_{s+1}^TP_j\right)=0,\quad j=1,2,\ldots,s. \tag{2.9}$$

In fact, by Lemma 2.2 we have

$$\begin{aligned}\operatorname{tr}\left(R_{s+1}^T R_j\right) &= \operatorname{tr}\left(R_s^T R_j\right) - \frac{\|R_s\|^2}{\|P_s\|^2}\operatorname{tr}\left(P_s^T P_j\right) + \frac{\|R_s\|^2\|R_j\|^2}{\|P_s\|^2\|R_{j-1}\|^2}\operatorname{tr}\left(P_s^T P_{j-1}\right)\\ &= 0.\end{aligned} \tag{2.10}$$

From the previous results, we have $\operatorname{tr}(R_{s+1}^T R_{j+1}) = 0$. By Lemma 2.2 we have that

$$\begin{aligned}\operatorname{tr}\left(P_{s+1}^T P_j\right) &= \frac{\|P_j\|^2}{\|R_j\|^2}\left(\operatorname{tr}\left(R_{s+1}^T R_j\right) - \operatorname{tr}\left(R_{s+1}^T R_{j+1}\right)\right) + \frac{\|R_{s+1}\|^2}{\|R_s\|^2}\operatorname{tr}\left(P_s^T P_j\right)\\ &= \frac{\|P_j\|^2}{\|R_j\|^2}\left(\operatorname{tr}\left(R_{s+1}^T R_j\right) - \operatorname{tr}\left(R_{s+1}^T R_{j+1}\right)\right)\\ &= 0.\end{aligned} \tag{2.11}$$

By the principle of induction, (2.9) holds. Note that (2.5) is implied in Steps 1 and 2 by the principle of induction. This completes the proof. □

Lemma 2.4. *Supposing $\overline{X}$ is an arbitrary solution of Problem 1, that is, $A\overline{X}B = E$ and $C\overline{X}D = F$, then*

$$\operatorname{tr}\left(\left(\overline{X} - X_k\right)^T P_k\right) = \|R_k\|^2, \quad k = 1, 2, \ldots, \tag{2.12}$$

where the sequences $\{X_k\}$, $\{R_k\}$, and $\{P_k\}$ are generated by Algorithm 2.1.

Proof. We proof the conclusion by induction.

For $k = 1$, we have that

$$\begin{aligned}\operatorname{tr}\left(\left(\overline{X} - X_1\right)^T P_1\right) = \operatorname{tr}\Big(\left(\overline{X} - X_1\right)^T \frac{1}{2}\Big(&A^T(E - AX_1B)B^T + C^T(F - CX_1D)D^T\\ &+PA^T(E - AX_1B)B^TQ + PC^T(F - CX_1D)D^TQ\Big)\Big)\end{aligned}$$

$$\begin{aligned}
&= \operatorname{tr}\left(\left(\overline{X}-X_1\right)^T\left(A^T(E-AX_1B)B^T+C^T(F-CX_1D)D^T\right)\right)\\
&= \operatorname{tr}\left(\left(\overline{X}-X_1\right)^T A^T(E-AX_1B)B^T+\left(\overline{X}-X_1\right)^T C^T(F-CX_1D)D^T\right)\\
&= \operatorname{tr}\left((E-AX_1B)^T A\left(\overline{X}-X_1\right)B+(F-CX_1D)^T C\left(\overline{X}-X_1\right)D\right)\\
&= \operatorname{tr}\left(\begin{pmatrix}(E-AX_1B)^T & 0\\ 0 & (F-CX_1D)^T\end{pmatrix}\begin{pmatrix}A\left(\overline{X}-X_1\right)B & 0\\ 0 & C\left(\overline{X}-X_1\right)D\end{pmatrix}\right)\\
&= \operatorname{tr}\left(\begin{pmatrix}(E-AX_1B)^T & 0\\ 0 & (F-CX_1D)^T\end{pmatrix}\begin{pmatrix}E-AX_1B & 0\\ 0 & F-CX_1D\end{pmatrix}\right)\\
&= \operatorname{tr}\left(R_1^T R_1\right)=\|R_1\|^2.
\end{aligned} \tag{2.13}$$

Assume (2.12) holds for $k = s$. By **Algorithm 2.1**, we have that

$$\begin{aligned}
&\operatorname{tr}\left(\left(\overline{X}-X_{s+1}\right)^T P_{s+1}\right)\\
&\quad= \operatorname{tr}\left(\left(\overline{X}-X_{s+1}\right)^T\right.\\
&\qquad\times\left(\left(\frac{A^T(E-AX_{s+1}B)B^T+C^T(F-CX_{s+1}D)D^T}{2}\right.\right.\\
&\qquad\qquad\left.\left.\left.+\frac{PA^T(E-AX_{s+1}B)B^TQ+PC^T(F-CX_{s+1}D)D^TQ}{2}\right)+\frac{\|R_{s+1}\|^2}{\|R_s\|^2}P_s\right)\right)\\
&\quad= \operatorname{tr}\left(\left(\overline{X}-X_{s+1}\right)^T\left(A^T(E-AX_{s+1}B)B^T+C^T(F-CX_{s+1}D)D^T+\frac{\|R_{s+1}\|^2}{\|R_s\|^2}P_s\right)\right)\\
&\quad= \operatorname{tr}\left(\begin{pmatrix}(E-AX_{s+1}B)^T & 0\\ 0 & (F-CX_{s+1}D)^T\end{pmatrix}\begin{pmatrix}E-AX_{s+1}B & 0\\ 0 & F-CX_{s+1}D\end{pmatrix}\right)\\
&\qquad+\frac{\|R_{s+1}\|^2}{\|R_s\|^2}\operatorname{tr}\left(\left(\overline{X}-X_{s+1}\right)^T P_s\right)\\
&\quad= \operatorname{tr}\left(R_{s+1}^T R_{s+1}\right)+\frac{\|R_{s+1}\|^2}{\|R_s\|^2}\operatorname{tr}\left(\left(\overline{X}-X_s\right)^T P_s\right)-\frac{\|R_{s+1}\|^2}{\|R_s\|^2}\frac{\|R_s\|^2}{\|P_s\|^2}\operatorname{tr}\left(P_s^T P_s\right)
\end{aligned}$$

$$
\begin{aligned}
&= \|R_{s+1}\|^2 + \frac{\|R_{s+1}\|^2}{\|R_s\|^2}\|R_s\|^2 - \frac{\|R_{s+1}\|^2}{\|R_s\|^2}\frac{\|R_s\|^2}{\|P_s\|^2}\|P_s\|^2 \\
&= \|R_{s+1}\|^2.
\end{aligned} \tag{2.14}
$$

Therefore, (2.12) holds for $k = s+1$. By the principle of induction, (2.12) holds. This completes the proof. □

Theorem 2.5. *Supposing that Problem 1 is consistent, then for an arbitrary initial matrix* $X_1 \in \mathcal{R}_r^{m\times n}(P,Q)$*, a solution of Problem 1 can be obtained with finite iteration steps in the absence of round-off errors.*

Proof. If $R_i \neq \mathbf{0}$, $i = 1,2,\ldots,pq+st$, by Lemma 2.4 we have $P_i \neq \mathbf{0}$, $i = 1,2,\ldots,pq+st$, then we can compute $X_{pq+st+1}, R_{pq+st+1}$ by Algorithm 2.1.

By Lemma 2.3, we have

$$
\begin{aligned}
&\operatorname{tr}\left(R_{pq+st+1}^T R_i\right) = 0, \quad i = 1,2,\ldots,pq+st, \\
&\operatorname{tr}\left(R_i^T R_j\right) = 0, \quad i,j = 1,2,\ldots,pq+st, \ i \neq j.
\end{aligned} \tag{2.15}
$$

Therefore, $R_1, R_2, \ldots, R_{pq+st}$ is an orthogonal basis of the matrix space

$$
S = \left\{ W \mid W = \begin{pmatrix} W_1 & 0 \\ 0 & W_4 \end{pmatrix},\ W_1 \in \mathcal{R}^{p\times q},\ W_4 \in \mathcal{R}^{s\times t} \right\}, \tag{2.16}
$$

which implies that $R_{pq+st+1} = \mathbf{0}$; that is, $X_{pq+st+1}$ is a solution of Problem 1. This completes the proof. □

To show the least norm generalized reflexive solution of Problem 1, we first introduce the following result.

Lemma 2.6 (see [8, Lemma 2.4]). *Supposing that the consistent system of linear equation* $My = b$ *has a solution* $y_0 \in R(M^T)$*, then* y_0 *is the least norm solution of the system of linear equations.*

By Lemma 2.6, the following result can be obtained.

Theorem 2.7. *Suppose that Problem 1 is consistent. If one chooses the initial iterative matrix* $X_1 = A^T H B^T + C^T \widehat{H} D^T + P A^T H B^T Q + P C^T \widehat{H} D^T Q$*, where* $H \in \mathcal{R}^{p\times q}$*,* $\widehat{H} \in \mathcal{R}^{s\times t}$ *are arbitrary matrices, especially, let* $X_1 = \mathbf{0} \in \mathcal{R}_r^{m\times n}$*, one can obtain the unique least norm generalized reflexive solution of Problem 1 within finite iterative steps in the absence of round-off errors by using Algorithm 2.1.*

Proof. By Algorithm 2.1 and Theorem 2.5, if we let $X_1 = A^T H B^T + C^T \widehat{H} D^T + P A^T H B^T Q + P C^T \widehat{H} D^T Q$, where $H \in \mathcal{R}^{p\times q}$, $\widehat{H} \in \mathcal{R}^{s\times t}$ are arbitrary matrices, we can obtain the solution X^* of Problem 1 within finite iterative steps in the absence of round-off errors, and the solution X^* can be represented that $X^* = A^T G B^T + C^T \widehat{G} D^T + P A^T G B^T Q + P C^T \widehat{G} D^T Q$.

In the sequel, we will prove that X^* is just the least norm solution of Problem 1.

Consider the following system of matrix equations:

$$\begin{gathered} AXB = E, \\ CXD = F, \\ APXQB = E, \\ CPXQD = F. \end{gathered} \tag{2.17}$$

If Problem 1 has a solution $X_0 \in \mathcal{R}_r^{m\times n}(P,Q)$, then

$$\begin{gathered} PX_0Q = X_0, \\ AX_0B = E, \qquad CX_0D = F. \end{gathered} \tag{2.18}$$

Thus

$$APX_0QB = E, \qquad CPX_0QD = F. \tag{2.19}$$

Hence, the systems of matrix equations (2.17) also have a solution X_0.

Conversely, if the systems of matrix equations (2.17) have a solution $\overline{X} \in R^{m\times n}$, let $X_0 = (\overline{X} + P\overline{X}Q)/2$, then $X_0 \in R_r^{m\times n}(P,Q)$, and

$$\begin{gathered} AX_0B = \frac{1}{2}A\left(\overline{X} + P\overline{X}Q\right)B = \frac{1}{2}\left(A\overline{X}B + AP\overline{X}QB\right) = \frac{1}{2}(E+E) = E, \\ CX_0D = \frac{1}{2}C\left(\overline{X} + P\overline{X}Q\right)D = \frac{1}{2}\left(C\overline{X}D + CP\overline{X}QD\right) = \frac{1}{2}(F+F) = F. \end{gathered} \tag{2.20}$$

Therefore, X_0 is a solution of Problem 1.

So the solvability of Problem 1 is equivalent to that of the systems of matrix equations (2.17), and the solution of Problem 1 must be the solution of the systems of matrix equations (2.17).

Letting S'_E denote the set of all solutions of the systems of matrix equations (2.17), then we know that $S_E \subset S'_E$, where S_E is the set of all solutions of Problem 1. In order to prove that X^* is the least-norm solution of Problem 1, it is enough to prove that X^* is the least-norm solution of the systems of matrix equations (2.21). Denoting $\mathrm{vec}(X) = x$, $\mathrm{vec}(X^*) = x^*$, $\mathrm{vec}(G) = g_1$, $\mathrm{vec}(\widehat{G}) = g_2$, $\mathrm{vec}(E) = e$, $\mathrm{vec}(F) = f$, then the systems of matrix equations (2.17) are equivalent to the systems of linear equations

$$\begin{pmatrix} B^T \otimes A \\ D^T \otimes C \\ B^TQ \otimes AP \\ D^TQ \otimes CP \end{pmatrix} x = \begin{pmatrix} e \\ f \\ e \\ f \end{pmatrix}. \tag{2.21}$$

Noting that

$$
\begin{aligned}
x^* &= \operatorname{vec}\left(A^T G B^T + C^T \hat{G} D^T + P A^T G B^T Q + P C^T \hat{G} D^T Q\right) \\
&= \left(B \otimes A^T\right) g_1 + \left(D \otimes C^T\right) y_2 + \left(QB \otimes PA^T\right) g_1 + \left(QD \otimes PC^T\right) g_2 \\
&= \left(B \otimes A^T \quad D \otimes C^T \quad QB \otimes PA^T \quad QD \otimes PC^T\right) \begin{pmatrix} g_1 \\ g_2 \\ g_1 \\ g_2 \end{pmatrix} \\
&= \begin{pmatrix} B^T \otimes A \\ D^T \otimes C \\ B^T Q \otimes AP \\ D^T Q \otimes CP \end{pmatrix}^T \begin{pmatrix} g_1 \\ g_2 \\ g_1 \\ g_2 \end{pmatrix} \in \mathcal{R}\left(\begin{pmatrix} B^T \otimes A \\ D^T \otimes C \\ B^T Q \otimes AP \\ D^T Q \otimes CP \end{pmatrix}^T\right),
\end{aligned}
\tag{2.22}
$$

by Lemma 2.6 we know that X^* is the least norm solution of the systems of linear equations (2.21). Since vector operator is isomorphic and X^* is the unique least norm solution of the systems of matrix equations (2.17), then X^* is the unique least norm solution of Problem 1. □

3. The Solution of Problem 2

In this section, we will show that the optimal approximate solution of Problem 2 for a given generalized reflexive matrix can be derived by finding the least norm generalized reflexive solution of a new corresponding matrix equation pair $A\widetilde{X}B = \widetilde{E}$, $C\widetilde{X}D = \widetilde{F}$.

When Problem 1 is consistent, the set of solutions of Problem 1 denoted by S_E is not empty. For a given matrix $X_0 \in \mathcal{R}_r^{m\times n}(P, Q)$ and $X \in S_E$, we have that the matrix equation pair (1.1) is equivalent to the following equation pair:

$$
\begin{aligned}
A\widetilde{X}B &= \widetilde{E}, \\
C\widetilde{X}D &= \widetilde{F},
\end{aligned}
\tag{3.1}
$$

where $\widetilde{X} = X - X_0$, $\widetilde{E} = E - AX_0B$, $\widetilde{F} = F - CX_0D$. Then Problem 2 is equivalent to finding the least norm generalized reflexive solution $\widetilde{X}^*$ of the matrix equation pair (3.1).

By using Algorithm 2.1, let initially iterative matrix $\widetilde{X}_1 = A^T H B^T + C^T \widehat{H} D^T + P A^T H B^T Q + P C^T \widehat{H} D^T Q$, or more especially, letting $\widetilde{X}_1 = \mathbf{0} \in R_r^{m\times n}(P, Q)$, we can obtain the unique least norm generalized reflexive solution $\widetilde{X}^*$ of the matrix equation pair (3.1); then we can obtain the generalized reflexive solution $\widehat{X}$ of Problem 2, and $\widehat{X}$ can be represented that $\widehat{X} = \widetilde{X}^* + X_0$.

4. Examples for the Iterative Methods

In this section, we will show several numerical examples to illustrate our results. All the tests are performed by MATLAB 7.8.

Example 4.1. Consider the generalized reflexive solution of the equation pair (1.1), where

$$A=\begin{pmatrix}1&3&-5&7&-9\\2&0&4&6&-1\\0&-2&9&6&-8\\3&6&2&27&-13\\-5&5&-22&-1&-11\\8&4&-6&-9&-19\end{pmatrix},\quad B=\begin{pmatrix}4&0&8&-5&4\\-1&5&0&-2&3\\4&-1&0&2&5\\0&3&9&2&-6\\-2&7&-8&1&11\end{pmatrix},$$
$$C=\begin{pmatrix}6&32&-5&7&-9\\2&10&4&6&-11\\9&-12&9&3&-8\\13&6&4&27&-15\\-5&15&-22&-13&-11\\2&9&-6&-9&-19\end{pmatrix},\quad D=\begin{pmatrix}7&1&8&-6&14\\-4&5&0&-2&3\\3&-12&0&8&25\\1&6&9&4&-6\\-5&8&-2&9&17\end{pmatrix},$$
$$E=\begin{pmatrix}592&-1191&1216&-244&-1331\\305&431&1234&-518&221\\814&-407&1668&-1176&537\\1434&-179&4083&-1374&-808\\242&-3150&-1362&1104&-2848\\423&-2909&1441&-182&-3326\end{pmatrix},\tag{4.1}$$
$$F=\begin{pmatrix}-2882&2830&299&2291&-4849\\409&670&1090&-783&-793\\3363&-126&2979&-3851&246\\2632&173&4553&-3709&-100\\-1774&-4534&-4548&1256&-6896\\864&-2512&-1136&-1633&-5412\end{pmatrix}.$$

Let

$$P=\begin{pmatrix}0&0&0&1&0\\0&0&0&0&1\\0&0&-1&0&0\\1&0&0&0&0\\0&1&0&0&0\end{pmatrix},\qquad Q=\begin{pmatrix}0&0&0&0&-1\\0&0&0&1&0\\0&0&-1&0&0\\0&1&0&0&0\\-1&0&0&0&0\end{pmatrix}.\tag{4.2}$$

We will find the generalized reflexive solution of the matrix equation pair $AXB = E$, $CXD = F$ by using Algorithm 2.1. It can be verified that the matrix equation pair is consistent over generalized reflexive matrix and has a solution with respect to P, Q as follows:

$$X^* = \begin{pmatrix} 5 & 3 & -6 & 12 & -5 \\ -11 & 8 & -1 & 9 & 7 \\ 13 & -4 & -8 & 4 & 13 \\ 5 & 12 & 6 & 3 & -5 \\ -7 & 9 & 1 & 8 & 11 \end{pmatrix} \in \mathcal{R}_r^{5\times 5}(P, Q). \tag{4.3}$$

Because of the influence of the error of calculation, the residual R_i is usually unequal to zero in the process of the iteration, where $i = 1, 2, \ldots$. For any chosen positive number ε, however small enough, for example, $\varepsilon = 1.0000e - 010$, whenever $\|R_k\| < \varepsilon$, stop the iteration, and X_k is regarded to be a generalized reflexive solution of the matrix equation pair $AXB = E$, $CXD = F$. Choose an initially iterative matrix $X_1 \in \mathcal{R}_r^{5\times 5}(P, Q)$, such as

$$X_1 = \begin{pmatrix} 1 & 10 & -6 & 12 & -5 \\ -6 & 8 & -1 & 14 & 9 \\ 13 & -4 & -8 & 4 & 13 \\ 5 & 12 & 6 & 10 & -1 \\ -9 & 14 & 1 & 8 & 6 \end{pmatrix}. \tag{4.4}$$

By Algorithm 2.1, we have

$$X_{17} = \begin{pmatrix} 5.0000 & 3.0000 & -6.0000 & 12.0000 & -5.0000 \\ -11.0000 & 8.0000 & -1.0000 & 9.0000 & 7.0000 \\ 13.0000 & -4.0000 & -8.0000 & 4.0000 & 13.0000 \\ 5.0000 & 12.0000 & 6.0000 & 3.0000 & -5.0000 \\ -7.0000 & 9.0000 & 1.0000 & 8.0000 & 11.0000 \end{pmatrix},$$
$$\|R_{17}\| = 3.2286e - 011 < \varepsilon. \tag{4.5}$$

So we obtain a generalized reflexive solution of the matrix equation pair $AXB = E$, $CXD = F$ as follows:

$$\overline{X} = \begin{pmatrix} 5.0000 & 3.0000 & -6.0000 & 12.0000 & -5.0000 \\ -11.0000 & 8.0000 & -1.0000 & 9.0000 & 7.0000 \\ 13.0000 & -4.0000 & -8.0000 & 4.0000 & 13.0000 \\ 5.0000 & 12.0000 & 6.0000 & 3.0000 & -5.0000 \\ -7.0000 & 9.0000 & 1.0000 & 8.0000 & 11.0000 \end{pmatrix}. \tag{4.6}$$

The relative error of the solution and the residual are shown in Figure 1, where the relative error $\mathrm{re}_k = \|X_k - X^*\| / \|X^*\|$ and the residual $r_k = \|R_k\|$.

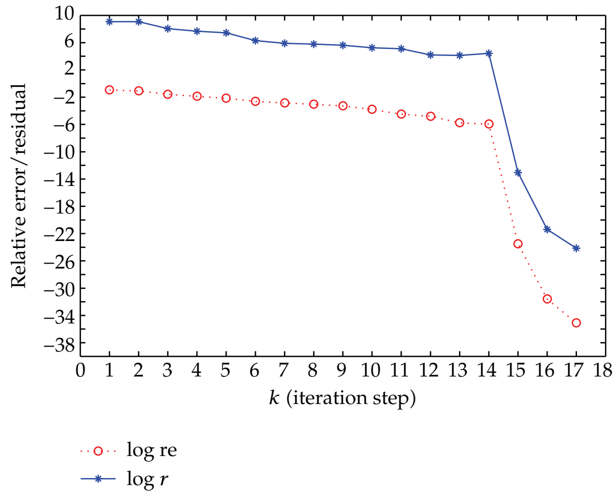

Figure 1: The relative error of the solution and the residual for Example 4.1 with $X_1 \neq \mathbf{0}$.

Letting

$$X_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{4.7}$$

by Algorithm 2.1, we have

$$X_{17} = \begin{pmatrix} 5.0000 & 3.0000 & -6.0000 & 12.0000 & -5.0000 \\ -11.0000 & 8.0000 & -1.0000 & 9.0000 & 7.0000 \\ 13.0000 & -4.0000 & -8.0000 & 4.0000 & 13.0000 \\ 5.0000 & 12.0000 & 6.0000 & 3.0000 & -5.0000 \\ -7.0000 & 9.0000 & 1.0000 & 8.0000 & 11.0000 \end{pmatrix},$$
$$\|R_{17}\| = 3.1999e-011 < \varepsilon. \tag{4.8}$$

So we obtain a generalized reflexive solution of the matrix equation pair $AXB = E$, $CXD = F$ as follows:

$$\overline{X} = \begin{pmatrix} 5.0000 & 3.0000 & -6.0000 & 12.0000 & -5.0000 \\ -11.0000 & 8.0000 & -1.0000 & 9.0000 & 7.0000 \\ 13.0000 & -4.0000 & -8.0000 & 4.0000 & 13.0000 \\ 5.0000 & 12.0000 & 6.0000 & 3.0000 & -5.0000 \\ -7.0000 & 9.0000 & 1.0000 & 8.0000 & 11.0000 \end{pmatrix}. \tag{4.9}$$

The relative error of the solution and the residual are shown in Figure 2.

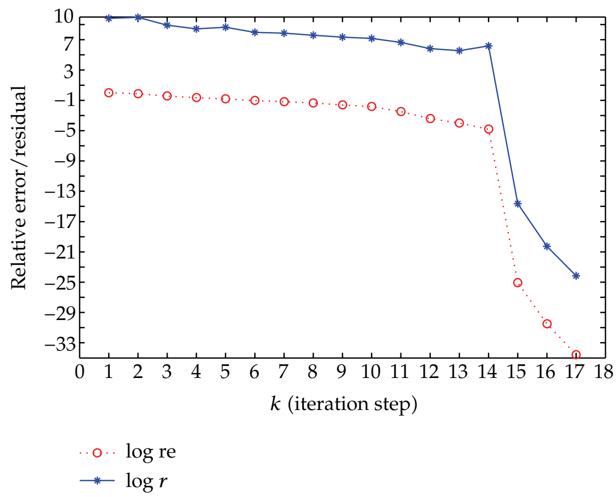

Figure 2: The relative error of the solution and the residual for Example 4.1 with $X_1 = \mathbf{0}$.

Example 4.2. Consider the least norm generalized reflexive solution of the matrix equation pair in Example 4.1. Let

$$H = \begin{pmatrix} 1 & 0 & 1 & 0 & 2 \\ 0 & -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 1 \\ 2 & 0 & 1 & 0 & -3 \\ 0 & 1 & 2 & 1 & 0 \\ -1 & 0 & -2 & -1 & 0 \end{pmatrix}, \qquad \widehat{H} = \begin{pmatrix} -1 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 3 \\ 1 & -1 & 0 & -2 & 0 \\ 2 & 0 & 1 & 0 & -3 \\ 0 & 1 & 2 & 1 & 0 \\ -1 & 0 & -2 & 1 & 2 \end{pmatrix},$$
$$X_1 = A^T H B^T + C^T \widehat{H} D^T + P A^T H B^T Q + P C^T \widehat{H} D^T Q. \tag{4.10}$$

By using Algorithm 2.1, we have

$$X_{19} = \begin{pmatrix} 5.0000 & 3.0000 & -6.0000 & 12.0000 & -5.0000 \\ -11.0000 & 8.0000 & -1.0000 & 9.0000 & 7.0000 \\ 13.0000 & -4.0000 & -8.0000 & 4.0000 & 13.0000 \\ 5.0000 & 12.0000 & 6.0000 & 3.0000 & -5.0000 \\ -7.0000 & 9.0000 & 1.0000 & 8.0000 & 11.0000 \end{pmatrix},$$
$$\|R_{19}\| = 6.3115e - 011 < \varepsilon. \tag{4.11}$$

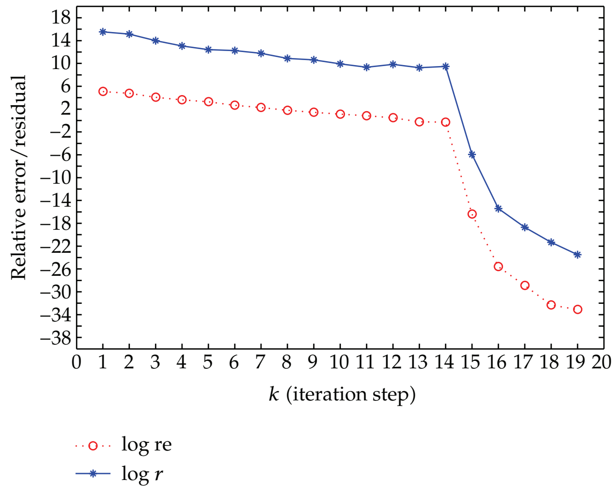

Figure 3: The relative error of the solution and the residual for Example 4.2.

So we obtain the least norm generalized reflexive solution of the matrix equation pair $AXB = E$, $CXD = F$ as follows:

$$X^* = \begin{pmatrix} 5.0000 & 3.0000 & -6.0000 & 12.0000 & -5.0000 \\ -11.0000 & 8.0000 & -1.0000 & 9.0000 & 7.0000 \\ 13.0000 & -4.0000 & -8.0000 & 4.0000 & 13.0000 \\ 5.0000 & 12.0000 & 6.0000 & 3.0000 & -5.0000 \\ -7.0000 & 9.0000 & 1.0000 & 8.0000 & 11.0000 \end{pmatrix}. \tag{4.12}$$

The relative error of the solution and the residual are shown in Figure 3.

Example 4.3. Let S_E denote the set of all generalized reflexive solutions of the matrix equation pair in Example 4.1. For a given matrix,

$$X_0 = \begin{pmatrix} -3 & 3 & 1 & 1 & 1 \\ 0 & -7 & 1 & 6 & 10 \\ 10 & -9 & 0 & 9 & 10 \\ -1 & 1 & -1 & 3 & 3 \\ -10 & 6 & -1 & -7 & 0 \end{pmatrix} \in \mathcal{R}_r^{5\times 5}(P,Q), \tag{4.13}$$

we will find $\widehat{X} \in S_E$, such that

$$\left\| \widehat{X} - X_0 \right\| = \min_{X \in S_E} \|X - X_0\|. \tag{4.14}$$

That is, find the optimal approximate solution to the matrix X_0 in S_E.

Letting $\widetilde{X} = X - X_0$, $\widetilde{E} = E - AX_0B$, $\widetilde{F} = F - CX_0D$, by the method mentioned in Section 3, we can obtain the least norm generalized reflexive solution $\widetilde{X}^*$ of the matrix

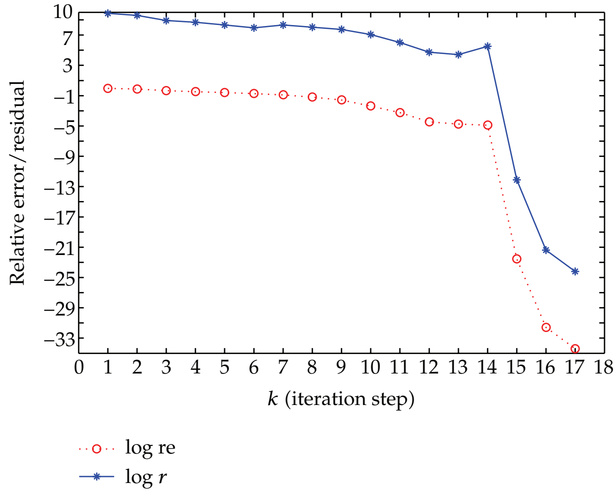

Figure 4: The relative error of the solution and the residual for Example 4.3.

equation pair $A\widetilde{X}B = \widetilde{E}$, $C\widetilde{X}D = \widetilde{F}$ by choosing the initial iteration matrix $\widetilde{X}_1 = \mathbf{0}$, and $\widetilde{X}^*$ is that

$$\widetilde{X}^*_{17} = \begin{pmatrix} 8.0000 & -0.0000 & -7.0000 & 11.0000 & -6.0000 \\ -11.0000 & 15.0000 & -2.0000 & 3.0000 & -3.0000 \\ 3.0000 & 5.0000 & -8.0000 & -5.0000 & 3.0000 \\ 6.0000 & 11.0000 & 7.0000 & -0.0000 & -8.0000 \\ 3.0000 & 3.0000 & 2.0000 & 15.0000 & 11.0000 \end{pmatrix},$$

$$\|R_{17}\| = 3.0690e - 011 < \varepsilon = 1.0000e - 010, \tag{4.15}$$

$$\widehat{X} = \widetilde{X}^*_{17} + X_0 = \begin{pmatrix} 5.0000 & 3.0000 & -6.0000 & 12.0000 & -5.0000 \\ -11.0000 & 8.0000 & -1.0000 & 9.0000 & 7.0000 \\ 13.0000 & -4.0000 & -8.0000 & 4.0000 & 13.0000 \\ 5.0000 & 12.0000 & 6.0000 & 3.0000 & -5.0000 \\ -7.0000 & 9.0000 & 1.0000 & 8.0000 & 11.0000 \end{pmatrix}.$$

The relative error of the solution and the residual are shown in Figure 4, where the relative error $\mathrm{re}_k = \|\widetilde{X}_k + X_0 - X^*\| / \|X^*\|$ and the residual $r_k = \|R_k\|$.

Acknowledgments

The authors are very much indebted to the anonymous referees and our editors for their constructive and valuable comments and suggestions which greatly improved the original manuscript of this paper. This work was partially supported by the Research Fund Project (Natural Science 2010XJKYL018), Opening Fund of Geomathematics Key Laboratory of Sichuan Province (scsxdz2011005), Natural Science Foundation of Sichuan Education Department (12ZB289) and Key Natural Science Foundation of Sichuan Education Department (12ZA008).

References

[1] H.-C. Chen, "Generalized reflexive matrices: special properties and applications," *SIAM Journal on Matrix Analysis and Applications*, vol. 19, no. 1, pp. 140–153, 1998.

[2] J. L. Chen and X. H. Chen, *Special Matrices*, Tsing Hua University Press, 2001.

[3] M. Dehghan and M. Hajarian, "An iterative algorithm for solving a pair of matrix equations $AYB = E$, $CYD = F$ over generalized centro-symmetric matrices," *Computers & Mathematics with Applications*, vol. 56, no. 12, pp. 3246–3260, 2008.

[4] T. Meng, "Experimental design and decision support," in *Expert System, The Technology of Knowledge Management and Decision Making for the 21st Century*, Leondes, Ed., vol. 1, Academic Press, 2001.

[5] A. L. Andrew, "Solution of equations involving centrosymmetric matrices," *Technometrics*, vol. 15, pp. 405–407, 1973.

[6] M. Dehghan and M. Hajarian, "An iterative algorithm for the reflexive solutions of the generalized coupled Sylvester matrix equations and its optimal approximation," *Applied Mathematics and Computation*, vol. 202, no. 2, pp. 571–588, 2008.

[7] M. Dehghan and M. Hajarian, "An iterative method for solving the generalized coupled Sylvester matrix equations over generalized bisymmetric matrices," *Applied Mathematical Modelling. Simulation and Computation for Engineering and Environmental Systems*, vol. 34, no. 3, pp. 639–654, 2010.

[8] G.-X. Huang, F. Yin, and K. Guo, "An iterative method for the skew-symmetric solution and the optimal approximate solution of the matrix equation $AXB = C$," *Journal of Computational and Applied Mathematics*, vol. 212, no. 2, pp. 231–244, 2008.

[9] A.-P. Liao and Y. Lei, "Least-squares solution with the minimum-norm for the matrix equation $(AXB, GXH) = (C, D)$," *Computers & Mathematics with Applications*, vol. 50, no. 3-4, pp. 539–549, 2005.

[10] F. Li, X. Hu, and L. Zhang, "The generalized reflexive solution for a class of matrix equations $(AX = B, XC = D)$," *Acta Mathematica Scientia B*, vol. 28, no. 1, pp. 185–193, 2008.

[11] A. Navarra, P. L. Odell, and D. M. Young, "A representation of the general common solution to the matrix equations $A_1XB_1 = C_1$ and $A_2XB_2 = C_2$ with applications," *Computers & Mathematics with Applications*, vol. 41, no. 7-8, pp. 929–935, 2001.

[12] Z.-h. Peng, X.-y. Hu, and L. Zhang, "An efficient algorithm for the least-squares reflexive solution of the matrix equation $A_1XB_1 = C_1, A_2XB_2 = C_2$," *Applied Mathematics and Computation*, vol. 181, no. 2, pp. 988–999, 2006.

[13] Y.-X. Peng, X.-Y. Hu, and L. Zhang, "An iterative method for symmetric solutions and optimal approximation solution of the system of matrix equations $A_1XB_1 = C_1$, $A_2XB_2 = C_2$," *Applied Mathematics and Computation*, vol. 183, no. 2, pp. 1127–1137, 2006.

[14] X. Sheng and G. Chen, "A finite iterative method for solving a pair of linear matrix equations $(AXB, CXD) = (E, F)$," *Applied Mathematics and Computation*, vol. 189, no. 2, pp. 1350–1358, 2007.

[15] A.-G. Wu, G. Feng, G.-R. Duan, and W.-J. Wu, "Finite iterative solutions to a class of complex matrix equations with conjugate and transpose of the unknowns," *Mathematical and Computer Modelling*, vol. 52, no. 9-10, pp. 1463–1478, 2010.

[16] A.-G. Wu, G. Feng, G.-R. Duan, and W.-J. Wu, "Iterative solutions to coupled Sylvester-conjugate matrix equations," *Computers & Mathematics with Applications*, vol. 60, no. 1, pp. 54–66, 2010.

[17] A.-G. Wu, B. Li, Y. Zhang, and G.-R. Duan, "Finite iterative solutions to coupled Sylvester-conjugate matrix equations," *Applied Mathematical Modelling*, vol. 35, no. 3, pp. 1065–1080, 2011.

[18] Y. X. Yuan, "Least squares solutions of matrix equation $AXB = E$, $CXD = F$," *Journal of East China Shipbuilding Institute*, vol. 18, no. 3, pp. 29–31, 2004.

[19] B. Zhou, Z.-Y. Li, G.-R. Duan, and Y. Wang, "Weighted least squares solutions to general coupled Sylvester matrix equations," *Journal of Computational and Applied Mathematics*, vol. 224, no. 2, pp. 759–776, 2009.

[20] S. K. Mitra, "Common solutions to a pair of linear matrix equations $A_1XB_1 = C_1$ and $A_2XB_2 = C_2$," *Cambridge Philosophical Society*, vol. 74, pp. 213–216, 1973.

[21] S. K. Mitra, "A pair of simultaneous linear matrix equations $A_1XB_1 = C_1, A_2XB_2 = C_2$ and a matrix programming problem," *Linear Algebra and its Applications*, vol. 131, pp. 97–123, 1990.

[22] N. Shinozaki and M. Sibuya, "Consistency of a pair of matrix equations with an application," *Keio Science and Technology Reports*, vol. 27, no. 10, pp. 141–146, 1974.

[23] J. W. vander Woude, *Freeback decoupling and stabilization for linear systems with multiple exogenous variables [Ph.D. thesis]*, 1987.

[24] Y.-B. Deng, Z.-Z. Bai, and Y.-H. Gao, "Iterative orthogonal direction methods for Hermitian minimum norm solutions of two consistent matrix equations," *Numerical Linear Algebra with Applications*, vol. 13, no. 10, pp. 801–823, 2006.

[25] Y.-T. Li and W.-J. Wu, "Symmetric and skew-antisymmetric solutions to systems of real quaternion matrix equations," *Computers & Mathematics with Applications*, vol. 55, no. 6, pp. 1142–1147, 2008.

[26] M. Dehghan and M. Hajarian, "The reflexive and anti-reflexive solutions of a linear matrix equation and systems of matrix equations," *The Rocky Mountain Journal of Mathematics*, vol. 40, no. 3, pp. 825–848, 2010.

[27] Q.-W. Wang, J.-H. Sun, and S.-Z. Li, "Consistency for bi(skew)symmetric solutions to systems of generalized Sylvester equations over a finite central algebra," *Linear Algebra and Its Applications*, vol. 353, pp. 169–182, 2002.

[28] Q.-W. Wang, "Bisymmetric and centrosymmetric solutions to systems of real quaternion matrix equations," *Computers & Mathematics with Applications*, vol. 49, no. 5-6, pp. 641–650, 2005.

[29] Q.-W. Wang, "A system of matrix equations and a linear matrix equation over arbitrary regular rings with identity," *Linear Algebra and Its Applications*, vol. 384, pp. 43–54, 2004.

[30] M. Dehghan and M. Hajarian, "An efficient algorithm for solving general coupled matrix equations and its application," *Mathematical and Computer Modelling*, vol. 51, no. 9-10, pp. 1118–1134, 2010.

[31] M. Dehghan and M. Hajarian, "On the reflexive and anti-reflexive solutions of the generalised coupled Sylvester matrix equations," *International Journal of Systems Science. Principles and Applications of Systems and Integration*, vol. 41, no. 6, pp. 607–625, 2010.

[32] M. Dehghan and M. Hajarian, "The general coupled matrix equations over generalized bisymmetric matrices," *Linear Algebra and Its Applications*, vol. 432, no. 6, pp. 1531–1552, 2010.

[33] I. Jonsson and B. Kågström, "Recursive blocked algorithm for solving triangular systems. I. One-sided and coupled Sylvester-type matrix equations," *ACM Transactions on Mathematical Software*, vol. 28, no. 4, pp. 392–415, 2002.

[34] I. Jonsson and B. Kågström, "Recursive blocked algorithm for solving triangular systems. II. Two-sided and generalized Sylvester and Lyapunov matrix equations," *ACM Transactions on Mathematical Software*, vol. 28, no. 4, pp. 416–435, 2002.

[35] J. Cai and G. Chen, "An iterative algorithm for the least squares bisymmetric solutions of the matrix equations $A_1XB_1 = C_1, A_2XB_2 = C_2$," *Mathematical and Computer Modelling*, vol. 50, no. 7-8, pp. 1237–1244, 2009.