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Research Article

The Regularized Trace Formula of the Spectrum of a Dirichlet Boundary Value Problem with Turning Point

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We calculate the regularized trace formula of the infinite sequence of eigenvalues for some version of a Dirichlet boundary value problem with turning points.

1. Introduction

The study of regularized traces of ordinary differential operators has a long history and there are a large number of papers and books studying this issue. The trace formulae for the scalar differential operators have been found by Gelfand and Levitan [1]. The formula obtained there gave rise to a large and very important theory, which started from the investigation of specific operators and further embraced the analysis of regularized traces of discrete operators in general form. In a short time, a number of authors turned their attention to trace theory and obtained interesting results. Dikiĭ [2] demonstrated a technique of using the trace of a resolvent for finding traces. Dikiĭ provided a proof of the Gelfand-Levitan formula in [2] on the basis of direct methods of perturbation theory, and in [3], he derived trace formulas of all orders for the Sturm-Liouville operator by constructing the fractional powers of the operator in closed form and by computing an analytic extension for its zeta function. Later, Levitan [4] suggested one more method for computing the traces of the Sturm-Liouville operator: by matching the expressions for the characteristic determinant via the solution of an appropriate Cauchy problem and via the corresponding infinite product, he found and compared the coefficients of the asymptotic expansions of these expressions thus obtaining trace formulas. The investigation carried out in 1957 by Faddeev [5] linked the trace theory to a substantially new class of problems, singular differential operators. Gasymov's paper [6] was the first paper in which a singular differential operator with discrete spectrum was considered. Afterwards these investigations were continued in many directions, such as Dirac operators, differential operators with abstract operator-valued coefficients, and the case of matrix-valued Sturm-Liouville operators (see, [7]).

Beyond their aesthetic appeal, trace formulas play an important role in the inverse spectral theory [8, 9]. Equation (1.1) is called differential equation with turning points if the weight function $\rho(x)$, which is given by (1.3), changes sign. The turning points appear in elasticity, optics, geophysics, and other branches of natural sciences. The inverse problems for equations with turning points and singularities help to study blow-up solutions for some nonlinear integrable evolution equations of mathematical physics. The turning points cause analytical difficulties, not only in calculating the eigenvalues asymptotes, but also in calculating the trace formula. In [10–12], the authors studied the spectral analysis of problem (1.1)-(1.2). They investigated the asymptotic relations of both eigenvalues and eigenfunctions also studied the eigenfunction expansion formula and proved the equiconvergence formula of that eigenfunctions. To complement the picture of spectral analysis of problem (1.1)-(1.2), we crown the series of papers [10–12] with the present work. In the present work, we evaluate the regularized trace formula for the problem (1.1)-(1.2) by using contour integration method as in [13]. It should be noted here that in [13] the author studied such formula for continuous spectrum in the whole line, while the present work contains point spectrum in a finite interval.

Consider the following Dirichlet problem

$$-y'' + q(x)y = \lambda\rho(x)y, \quad 0 \le x \le \pi, \tag{1.1}$$

$$y(0) = 0, \qquad y(\pi) = 0, \tag{1.2}$$

where $q(x)$ is a nonnegative real function, which has a second piecewise integrable derivatives of the second order in $(0,\pi)$, λ is a spectral parameter and the weight function or the explosive factor $\rho(x)$ is of the form

$$\rho(x) = \begin{cases} 1; & 0 \le x \le a < \pi, \\ -1; & a < x \le \pi. \end{cases} \tag{1.3}$$

Following [10], we state the basic notations and results that are needed in the subsequent calculation. In [10], the author proved that the Dirichlet problem (1.1)-(1.2) has a countable number of eigenvalues $\lambda_n^{\pm}$, $n = 0, 1, 2, \ldots$ where λ_n^{+} are the nonnegative eigenvalues and λ_n^{-} are the negative eigenvalues which admit the asymptotic formulas

$$\begin{aligned} \lambda_n^{+} &= \frac{\pi^2}{a^2}\left(n - \frac{1}{4}\right)^2 + \frac{2k_o\pi}{a} + \left(\frac{2k_1\pi}{a}\right)\frac{1}{n} + O\left(\frac{1}{n^2}\right), \\ \lambda_n^{-} &= -\frac{\pi^2}{(\pi-a)^2}\left(n - \frac{1}{4}\right)^2 - \frac{2k_o\pi}{\pi - a} - \left(\frac{2h_1\pi}{\pi - a}\right)\frac{1}{n} + O\left(\frac{1}{n^2}\right), \end{aligned} \tag{1.4}$$

where

$$
\begin{gathered}
k_o = -\frac{1}{8\pi}\int_0^a q(t)dt, \\
k_1(x) = \frac{a}{2\pi^2}\left\{ \left[\beta_1(a) - \alpha_1(a)\right] \left[2\gamma_1(a) + \frac{1}{2}\beta_1(a)\right] - \frac{1}{2}\alpha_1(a) + \alpha_2(a) - \beta_2(a) + \gamma_2(a) + \delta_2(a) \right\}, \\
h_1(x) = \frac{\pi - a}{a}k_1
\end{gathered} \tag{1.5}
$$

and the constants $\alpha_1(a),\ \alpha_2(a),\ \beta_1(a),\ \beta_2(a),\ \gamma_1(a),\ \gamma_2(a),\ \delta_1(a)$, and $\delta_2(a)$ are given by

$$
\begin{gathered}
\alpha_1(x) = \frac{1}{2}\int_0^x q(t)dt, \\
\alpha_2(x) = \frac{1}{4}\left(\int_0^x q(t)dt\right)^2 + \frac{1}{4}\left[q(0) - q(x)\right], \\
\beta_1(x) = \frac{1}{4}\int_0^x q(t)dt, \\
\beta_2(x) = \frac{1}{8}\left(\int_0^x q(t)dt\right)^2 + \frac{1}{4}\left[q(0) - q(x)\right], \\
\beta_3(x) = \frac{1}{24}\left(\int_0^x q(t)dt\right)^3 + \frac{1}{8}\left[q(0)\int_0^x q(t)dt + q'(0) + q'(x)\right], \\
\gamma_1(x) = \frac{1}{2}\int_x^\pi q(t)dt, \\
\gamma_2(x) = -\frac{1}{8}\left(\int_x^\pi q(t)dt\right)^2 + \frac{1}{4}\left[q(x) + q(\pi)\right], \\
\delta_1(x) = -\frac{1}{8}\left(\int_x^\pi q(t)dt\right)^2 + \frac{1}{4}\left[q(x) - q(\pi)\right], \\
\delta_2(x) = \frac{1}{48}\left(\int_x^\pi q(t)dt\right)^3 + \frac{1}{8}\int_x^\pi q(t)dt\left[3q(x) - q(\pi)\right] + \frac{1}{8}\left[q'(\pi) + q'(x)\right].
\end{gathered} \tag{1.6}
$$

We introduced in [10] the function $W(\lambda)$ as the Wronskian of the two solutions $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ of (1.1)-(1.2). We denote by $\Psi(s)$ the function $W(\lambda)$, for $\lambda = s^2$, which has the following asymptotic formula

$$
\Psi(s) = \frac{Z_o(s)}{s} + \frac{Z_1(s)}{s^2} + \frac{Z_2(s)}{s^3} + O\left(\frac{e^{|\operatorname{Im} s|a + |\operatorname{Re} s|(\pi - a)}}{|s|^4}\right), \tag{1.7}
$$

where

$$Z_o(s) = -\sin sa \cosh s(\pi - a) - \cos sa \sinh s(\pi - a),$$
$$Z_1(s) = -P_1 \sin sa \sinh s(\pi - a) - P_2 \cos sa \cosh s(\pi - a), \tag{1.8}$$
$$Z_2(s) = -Q_1 \sin sa \cosh s(\pi - a) - Q_2 \cos sa \sinh s(\pi - a),$$

$$P_1 = \beta_1(a) + \gamma_1(a), \qquad P_2 = \alpha_1(a) + \gamma_1(a),$$
$$Q_1 = \alpha_2(a) + \beta_1(a)\gamma_1(a) + \delta_2(a), \tag{1.9}$$
$$Q_2 = \alpha_1(a)\ \gamma_1(a) + \beta_2(a) + \gamma_2(a),$$

where α_1, α_2, β_1, β_2, γ_1, γ_2, and δ_2 are expressed, in (1.6), in terms of $q(x)$ and a. We also prove that the roots of $\Psi(s) = 0$ coincide with the eigenvalues of the Dirichlet problem (1.1)-(1.2) and these eigenvalues are simple. Our aim is to calculate the summation of these eigenvalues which we call the trace formula or more precisely the regularized trace formula. During the calculation of the eigenvalues and the eigenfunctions, the condition (1.2) forced us to evaluate up to the term containing $O(1/|\lambda|^4)$. We must also notice that, the formula obtained in the present work, due to the Dirichlet condition (1.2), contained $q(x)$ together with its first derivatives on $[0, \pi]$. We used the methodology as in [4, 14], by Levitan, but our problem contains more difficulties because of the presence of the $\rho(x)$ as $\pm$ sign. In the following theorem we calculate the summation of these eigenvalues in a certain form called the regularized trace formula for the Dirichlet problem (1.1)-(1.2).

Theorem 1.1. *Suppose that $q(x)$ has a second-order piecewise integrable derivatives on $[0, \pi]$ then, in view of the introduced notations,* (1.4) *and* (1.7), *the following regularized trace formula takes place*

$$\sum_{k=0}^{\infty}\left(\lambda_k^{+} - \lambda_k^{o+} - m^{+}\right) + \sum_{k=0}^{\infty}\left(\lambda_k^{-} - \lambda_k^{o-} - m^{-}\right)$$
$$= M_1 + M_2 \ln\left[\frac{a}{\pi - a}\right] + M_3 \ln\left[2 - \frac{\pi(\pi - 2a)}{a(\pi - a)}\right] + M_4 \tan^{-1}\left[\frac{\pi - a}{\pi}\right], \tag{1.10}$$

where the constants m^{+}, m^{-}, M_1, M_2, M_3, and M_4 are given by

$$m^{+} = \frac{-1}{4a} = \int_0^a q(t)dt, \qquad m^{-} = \frac{-1}{4(\pi - a)} = \int_a^{\pi} q(t)dt,$$

$$
\begin{aligned}
M_1 &= \frac{9}{16}\left(\int_0^a q(t)dt\right)\left(\int_0^\pi q(t)dt\right) + \frac{1}{96}\left(\int_a^\pi q(t)dt\right)^3 - \frac{1}{16}\left(\int_a^\pi q(t)dt\right) \\
&\quad - \frac{1}{16a}\int_0^a q(t)dt - \frac{1}{16(\pi-a)}\int_a^\pi q(t)dt + \frac{1}{16}\left[3q\,(a+0) - q(\pi)\right]\int_a^\pi q(t)dt \\
&\quad + \frac{1}{16}\left[4q(0) - 4q(a-0) + 2q(a+0) + 2q(\pi) + q'(\pi) + q'(a+0)\right], \\
M_2 &= \frac{13}{32\pi}\left(\int_0^a q(t)dt\right)^2 + \frac{1}{8\pi}\left(\int_0^a q(t)dt\right)\left(\int_a^\pi q(t)dt\right) \\
&\quad - \frac{1}{48\pi}\left(\int_a^\pi q(t)dt\right)^3 + \frac{1}{8\pi}\left[3q\,(a+0) - q(\pi)\right]\int_a^\pi q(t)dt \\
&\quad + \frac{1}{8\pi}\left[4q\,(0) - 4q\,(a-0) + 2q\,(a+0) + 2q\,(\pi) + q'\,(\pi) + q'\,(a+0)\right], \\
M_3 &= \frac{1}{16\pi}\left(\int_0^\pi q(t)dt\right)^2, \qquad M_4 = \frac{3}{4\pi}\left(\int_0^a q(t)dt\right)\left(\int_0^\pi q(t)dt\right).
\end{aligned}
\tag{1.11}
$$

Proof. We use the well-known formula from the theory of functions of a complex variable

$$
2\sum_{k=0}^{n}\left[\lambda_k^+ + \lambda_k^-\right] = \frac{1}{2\pi i}\oint_{\Gamma_n} s^2 d\,\ln\Psi(s), \tag{1.12}
$$

where $\Psi(s)$ is given by (1.7) and the contour Γ_n is a quadratic contour on the s-domain as defined in [10]

$$
\Gamma_n = \left\{|\mathrm{Re}\,s| \le \frac{\pi}{a}\left(n - \frac{1}{4}\right) + \frac{\pi}{2a}, |\mathrm{Im}\,s| \le \frac{\pi}{\pi - a}\left(n - \frac{1}{4}\right) + \frac{\pi}{2(\pi - a)}\right\}. \tag{1.13}
$$

From (1.7) we have

$$
\Psi(s) = \frac{Z_o(s)}{s}\left[1 + r(s)\right], \quad \text{where } r(s) = \frac{Z_1(s)}{sZ_o(s)} + \frac{Z_2(s)}{s^2 Z_o(s)} + O\left(\frac{e^{|\,\mathrm{Im}\,s|a + |\,\mathrm{Re}\,s|(\pi-a)}}{s^3 Z_o(s)}\right), \tag{1.14}
$$

where, $Z_o(s)$, $Z_1(s)$, and $Z_2(s)$ are given by (1.8). It is clear that, on the contour Γ_n, the term $Z_o(s)$ satisfies the following inequality from below

$$
|Z_o(s)| \ge C\,e^{|\,\mathrm{Im}\,s|a + |\,\mathrm{Re}\,s|(\pi-a)} \quad C \text{ is constant.} \tag{1.15}
$$

which can be shown by expressing $\cos sa, \sin sa, \cosh sa$, and $\sinh sa$ in terms of the exponential functions. So that

$$r(s) = \frac{Z_1(s)}{sZ_o(s)} + \frac{Z_2(s)}{s^2 Z_o(s)} + O\left(\frac{1}{|s|^3}\right). \tag{1.16}$$

From (1.14) the Equation (1.12) can be put in the form

$$2\sum_{k=0}^{n}\left[\lambda_k^+ + \lambda_k^-\right] = \frac{1}{2\pi i}\oint_{\Gamma_n} s^2 d\,\ln\left(\frac{Z_o(s)}{s}\right) + \frac{1}{2\pi i}\oint_{\Gamma_n} s^2 d\,\ln[1 + r(s)], \tag{1.17}$$

further,

$$\frac{1}{2\pi i}\oint_{\Gamma_n} s^2 d\,\ln\left(\frac{Z_o(s)}{s}\right) = 2\sum_{k=0}^{n}\left[\lambda_k^{o+} + \lambda_k^{o-}\right], \tag{1.18}$$

where $\lambda_k^{o\pm}$ are the eigenvalues of (1.1)-(1.2) for $q(x) = 0$. Integrating by part the last term of (1.17) we obtain

$$\frac{1}{2\pi i}\oint_{\Gamma_n} s^2 d\,\ln[1 + r(s)] = \frac{-1}{2\pi i}\oint_{\Gamma_n} 2s\left[r(s) - \frac{1}{2}r^2(s) + O\left(\frac{1}{s^3}\right)\right]ds. \tag{1.19}$$

Substituting from (1.16) into (1.19) we have

$$\frac{1}{2\pi i}\oint_{\Gamma_n} s^2 d\,\ln[1 + r(s)] = \frac{-1}{2\pi i}\oint_{\Gamma_n} 2s\left[\frac{Z_1(s)}{sZ_o(s)} + \frac{Z_2(s)}{s^2 Z_o(s)} - \frac{Z_1^2(s)}{2s^2 Z_o^2(s)} + O\left(\frac{1}{|s|^3}\right)\right]ds. \tag{1.20}$$

From (1.18) and (1.20), (1.17) takes the form

$$\begin{aligned}
&2\sum_{k=0}^{n}\left[\left(\lambda_k^+ - \lambda_k^{o+}\right)\right] - 2\sum_{k=0}^{n}\left[\lambda_k^- - \lambda_k^{o-}\right] \\
&\quad = \frac{-1}{\pi i}\oint_{\Gamma_n} \frac{Z_1(s)}{Z_o(s)}ds - \frac{1}{\pi i}\oint_{\Gamma_n} \frac{Z_2(s)}{sZ_o(s)}ds + \frac{1}{2\pi i}\oint_{\Gamma_n} \frac{Z_1^2(s)}{sZ_o^2(s)}ds + \oint_{\Gamma_n} O\left(\frac{1}{|s|^2}\right)ds.
\end{aligned} \tag{1.21}$$

We evaluate each term of (1.21), first of all, by direct calculation it can be easily shown that

$$\oint_{\Gamma_n} O\left(\frac{1}{|s|^2}\right)ds \longrightarrow 0 \quad \text{as}\, n \longrightarrow \infty. \tag{1.22}$$

To evaluate the integration $(-1/\pi i)\oint_{\Gamma_n}(Z_1(s)/Z_o(s))\,ds$, notice that the function $Z_1(s)/Z_o(s)$ is odd, so that from (1.8) we have

$$\begin{aligned}
&\frac{-1}{\pi i}\oint_{\Gamma_n}\frac{Z_1(s)}{Z_o(s)}ds\\
&\quad=\frac{-1}{\pi i}\oint_{\Gamma_n}\frac{P_1\sin sa\sinh\, s(\pi-a)+P_2\cos sa\cosh\, s(\pi-a)}{\sin sa\cosh\, s(\pi-a)+\cos sa\sinh\, s(\pi-a)}ds\\
&\quad=\frac{-2}{\pi}\int_{-\xi_n}^{\xi_n}\frac{P_1\sin(\sigma_n+i\xi)a\ \sinh(\sigma_n+i\xi)(\pi-a)+P_2\cos(\sigma_n+i\xi)a\ \cosh(\sigma_n+i\xi)(\pi-a)}{\sin(\sigma_n+i\xi)a\ \cosh(\sigma_n+i\xi)(\pi-a)+\cos(\sigma_n+i\xi)a\ \sinh(\sigma_n+i\xi)(\pi-a)}d\xi\\
&\qquad+\frac{2}{\pi i}\int_{-\sigma_n}^{\sigma_n}\frac{P_1\sin(\sigma+i\xi_n)a\ \sinh(\sigma+i\xi_n)(\pi-a)+P_2\cos(\sigma+i\xi_n)a\ \cosh(\sigma+i\xi_n)(\pi-a)}{\sin(\sigma+i\xi_n)a\ \cosh(\sigma+i\xi_n)(\pi-a)+\cos(\sigma+i\xi_n)a\ \sinh(\sigma+i\xi_n)(\pi-a)}d\sigma.
\end{aligned}\tag{1.23}$$

From which (1.23) can be written in the form

$$\begin{aligned}
&\frac{-1}{\pi i}\oint_{\Gamma_n}\frac{Z_1(s)}{Z_o(s)}ds=I_1+I_2,\\
&\text{where } I_1=\frac{-2}{\pi}\int_{-\xi_n}^{\xi_n}\frac{P_1\tan(\sigma_n+i\xi)a\ \tanh(\sigma_n+i\xi)(\pi-a)+P_2}{\tan(\sigma_n+i\xi)a+\tanh(\sigma_n+i\xi)(\pi-a)}d\xi,\\
&I_2=\frac{2}{\pi i}\int_{-\sigma_n}^{\sigma_n}\frac{P_1\tan(\sigma+i\xi_n)a\ \tanh(\sigma+i\xi_n)(\pi-a)+P_2}{\tan(\sigma+i\xi_n)a+\tanh(\sigma+i\xi_n)(\pi-a)}d\sigma.
\end{aligned}\tag{1.24}$$

We notice that in the integration I_1 the function $\tan(\sigma_n+i\xi)$ is bounded while

$$\tanh(\sigma_n+i\xi)(\pi-a)=1+O\left(e^{-2\sigma_n(\pi-a)}\right),\tag{1.25}$$

so that I_1 becomes of the form

$$I_1=\frac{-2}{\pi}\int_{-\xi_n}^{\xi_n}\frac{P_2+P_1\tan(\sigma_n+i\xi)a}{\tan(\sigma_n+i\xi)a-1}d\xi+o(1).\tag{1.26}$$

By the help of the relations

$$\begin{aligned}
\cos(\sigma_n+i\xi)a&=\frac{(-1)^n}{\sqrt{2}}[\cosh\xi a-i\sinh\xi a]\\
\sin(\sigma_n+i\xi)a&=\frac{(-1)^n}{\sqrt{2}}[\cosh\xi a+i\sinh\xi a]
\end{aligned}\tag{1.27}$$

(1.26) becomes

$$I_1 = \frac{-2}{\pi - a}(P_1 - P_2)\left(n + \frac{1}{4}\right) + o(1). \tag{1.28}$$

To evaluate I_2 we have $\tanh(\sigma + i\xi_n)(\pi - a)$ is bounded and

$$\tan(\sigma + i\xi_n)a = i + O\left(e^{-2\xi_n a}\right) \tag{1.29}$$

and hence I_2 becomes of the form

$$I_2 = \frac{2}{\pi i}\int_{-\sigma_n}^{\sigma_n} \frac{iP_1 \sinh(\sigma + i\xi_n)(\pi - a) + P_2 \cosh(\sigma + i\xi_n)(\pi - a)}{i\cosh(\sigma + i\xi_n)(\pi - a) + \sinh(\sigma + i\xi_n)(\pi - a)} d\sigma + o(1). \tag{1.30}$$

As before, by applying the relations

$$\begin{aligned}\cosh(\sigma + i\xi_n)(\pi - a) &= \frac{(-1)^n}{\sqrt{2}}[\cosh\, \sigma\, (\pi - a) + i \sinh\, \sigma\, (\pi - a)],\\ \sinh(\sigma + i\xi_n)(\pi - a) &= \frac{(-1)^n}{\sqrt{2}}[\sinh \sigma(\pi - a) + i \cosh \sigma(\pi - a)]\end{aligned} \tag{1.31}$$

to (1.30), we have

$$I_2 = \frac{2}{a}(P_1 - P_2)\left(n + \frac{1}{4}\right) + o(1). \tag{1.32}$$

From (1.28) and (1.32) by substitution into (1.24) we have

$$\frac{-1}{\pi i}\oint_{\Gamma_n} \frac{Z_1(s)}{Z_o(s)} ds = \left[\frac{2(P_2 - P_1)}{\pi - a} + \frac{2(P_1 - P_2)}{a}\right]\left(n + \frac{1}{4}\right) + o(1). \tag{1.33}$$

To evaluate $(-1/\pi i)\oint_{\Gamma_n}(Z_2(s)/sZ_o(s))ds$ we notice that $Z_2(s)/sZ_o(s)$ is odd function, then

$$\begin{aligned}&\frac{-1}{\pi i}\oint_{\Gamma_n} \frac{Z_2(s)}{sZ_o(s)} ds\\ &= \frac{-1}{\pi i}\oint_{\Gamma_n} \frac{Q_1 \sin sa \cosh s(\pi - a) + Q_2 \cos sa \sinh s(\pi - a)}{s[\sin sa \cosh s(\pi - a) + \cos sa \sinh s(\pi - a)]} ds\\ &= \frac{-2}{\pi}\int_{-\xi_n}^{\xi_n} \frac{Q_1 \sin(\sigma_n + i\xi)a\, \cosh(\sigma_n + i\xi)(\pi - a) + Q_2 \cos(\sigma_n + i\xi)a \sinh(\sigma_n + i\xi)(\pi - a)}{(\sigma_n + i\xi)[\sin(\sigma_n + i\xi)a \cosh(\sigma_n + i\xi)(\pi - a) + \cos(\sigma_n + i\xi)a \sinh(\sigma_n + i\xi)(\pi - a)]} d\xi\\ &\quad + \frac{2}{\pi i}\int_{-\sigma_n}^{\sigma_n} \frac{A_2 \sin(\sigma + i\xi_n)a \cosh(\sigma + i\xi_n)(\pi - a) + B_2 \cos(\sigma + i\xi_n)a \sinh(\sigma + i\xi_n)(\pi - a)}{(\sigma + i\xi_n)[\sin(\sigma + i\xi_n)a \cosh(\sigma + i\xi_n)(\pi - a) + \cos(\sigma + i\xi_n)a \sinh(\sigma + i\xi_n)(\pi - a)]} d\sigma.\end{aligned} \tag{1.34}$$

From (1.34) we have

$$
\begin{aligned}
&\frac{-1}{\pi i}\oint_{\Gamma_n}\frac{Z_2(s)}{sZ_o(s)}ds = I_1^* + I_2^*,\\
\text{where } &I_1^* = \frac{-2}{\pi}\int_{-\xi_n}^{\xi_n}\frac{Q_1\tan(\sigma_n + i\xi)a + Q_2\tanh(\sigma_n + i\xi)(\pi - a)}{(\sigma_n + i\xi)[\tan(\sigma_n + i\xi)a + \tanh(\sigma_n + i\xi)(\pi - a)]}d\xi,\\
&I_2^* = \frac{2}{\pi i}\int_{-\sigma_n}^{\sigma_n}\frac{Q_1\tan(\sigma + i\xi_n)a + Q_2\tanh(\sigma + i\xi_n)(\pi - a)}{(\sigma + i\xi_n)[\tan(\sigma + i\xi_n)a + \tanh(\sigma + i\xi_n)(\pi - a)]}d\sigma.
\end{aligned}
\tag{1.35}
$$

By using (1.25) and keeping in mind that $\tan(\sigma + i\xi_n)a$ is bounded we have

$$
I_1^* = \frac{-2}{\pi}\int_{-\xi_n}^{\xi_n}\frac{Q_1\,\sin(\sigma_n + i\xi)a + Q_2\,\cos(\sigma_n + i\xi)a}{(\sigma_n + i\xi)[\cos(\sigma_n + i\xi)a + \sin(\sigma_n + i\xi)a]}d\xi + o(1).
\tag{1.36}
$$

By the help of (1.27), after some elementary calculation, I_1^* takes the form

$$
I_1^* = \frac{2(Q_1 + Q_2)}{\pi}\tan^{-1}\left(\frac{a}{\pi - a}\right) + \frac{(Q_2 - Q_1)}{\pi}\ln\left[1 + \frac{a^2}{(\pi - a)^2}\right] + o(1).
\tag{1.37}
$$

Now we evaluate I_2^*, by using (1.29) and the roundedness of $\tanh(\sigma + i\xi_n)(\pi - a)$ we have from (1.35)

$$
I_2^* = \frac{-2}{\pi i}\int_{-\sigma_n}^{\sigma_n}\frac{Q_2\,\sinh(\sigma + i\xi_n)(\pi - a) + Q_1\,\cosh(\sigma + i\xi_n)(\pi - a)}{(\sigma + i\xi_n)[i\,\cosh(\sigma + i\xi_n)(\pi - a) + \sinh(\sigma + i\xi_n)(\pi - a)]}d\sigma + o(1).
\tag{1.38}
$$

From (1.38) and (1.31) after calculation we have

$$
I_2^* = \frac{2(Q_1 + Q_2)}{\pi}\tan^{-1}\left(\frac{\pi - a}{a}\right) - \frac{(Q_2 - Q_1)}{\pi}\ln\left[1 + \frac{(\pi - a)^2}{a^2}\right] + o(1).
\tag{1.39}
$$

By substitution from (1.39) and (1.36) into (1.35), we have

$$
\frac{-1}{\pi i}\oint_{\Gamma_n}\frac{Z_2(s)}{sZ_o(s)}ds = Q_1 + Q_2 + \frac{Q_2 - Q_1}{\pi}\ln\frac{a^2}{(\pi - a)^2}.
\tag{1.40}
$$

We evaluate the third integral of (1.21)

$$
\begin{aligned}
&\frac{-1}{2\pi i}\oint_{\Gamma_n}\frac{Z_1^2(s)}{sZ_o^2(s)}ds\\
&=\frac{-1}{2\pi i}\oint_{\Gamma_n}\frac{[P_1\sin sa\sinh s(\pi-a)+P_2\cos sa\cosh s(\pi-a)]^2}{s[\sin sa\cosh s(\pi-a)+\cos sa\sinh s(\pi-a)]^2}ds\\
&=\frac{1}{\pi}\int_{-\xi_n}^{\xi_n}\frac{[P_1\sin(\sigma_n+i\xi)a\sinh(\sigma_n+i\xi)(\pi-a)+P_2\cos(\sigma_n+i\xi)a\cosh(\sigma_n+i\xi)(\pi-a)]^2}{(\sigma_n+i\xi)[\sin(\sigma_n+i\xi)a\cosh(\sigma_n+i\xi)(\pi-a)+\cos(\sigma_n+i\xi)a\sinh(\sigma_n+i\xi)(\pi-a)]^2}d\xi\\
&\quad+\frac{1}{\pi i}\int_{-\sigma_n}^{\sigma_n}\frac{[P_1\sin(\sigma+i\xi_n)a\sinh(\sigma+i\xi_n)(\pi-a)+Q_2\cos(\sigma+i\xi_n)a\cosh(\sigma+i\xi_n)(\pi-a)]^2}{(\sigma+i\xi_n)[\sin(\sigma+i\xi_n)a\cosh(\sigma+i\xi_n)(\pi-a)+\cos(\sigma+i\xi_n)a\sinh(\sigma+i\xi_n)(\pi-a)]^2}d\sigma.
\end{aligned}
\tag{1.41}
$$

After simplification (1.41) becomes

$$
\begin{aligned}
&\frac{1}{2\pi i}\oint_{\Gamma_n}\frac{Z_1^2(s)}{sZ_o^2(s)}ds=I_1^{**}+I_2^{**},\\
&\text{where } I_1^{**}=\frac{1}{\pi}\int_{-\xi_n}^{\xi_n}\left[\frac{P_1-P_2}{2}+\frac{P_1+P_2}{2i}\frac{\cosh\,\xi a}{\sinh\,\xi a}\right]^2\frac{1}{\sigma_n+i\xi}d\xi,\\
&I_2^{**}=\frac{1}{\pi i}\int_{-\sigma_n}^{\sigma_n}\left[\frac{i(P_1+P_2)}{-2}+\frac{P_2-P_1}{-2}\frac{\cosh\sigma(\pi-a)}{\sinh\sigma(\pi-a)}\right]^2\frac{1}{\sigma+i\xi_n}d\sigma.
\end{aligned}
\tag{1.42}
$$

.

For I_1^{**} we have

$$
\begin{aligned}
I_1^{**}&=\frac{1}{4\pi}\int_{-\xi_n}^{\xi_n}\frac{(P_1-P_2)^2}{\sigma_n+i\xi}d\xi-\frac{1}{2\pi i}\int_{-\xi_n}^{\xi_n}\frac{\left(P_1^2-P_2^2\right)}{\sigma_n+i\xi}\frac{\cosh\xi a}{\sinh\xi a}d\xi\\
&\quad-\frac{1}{8\pi}\int_{-\xi_n}^{\xi_n}\frac{(P_1-P_2)^2}{\sigma_n+i\xi}\frac{\cosh^2\xi a}{\sinh^2\xi a}d\xi
\end{aligned}
\tag{1.43}
$$

by calculating the integrations in (1.43) we have

$$
I_1^{**}=\frac{-2P_1P_2}{\pi}\tan^{-1}\frac{a}{\pi-a}+\frac{\left(P_1^2+P_2^2\right)}{2\pi}\ln\left(1+\frac{a^2}{(\pi-a)^2}\right)+o(1).
\tag{1.44}
$$

On the Other hand,

$$I_2^{**} = \frac{-1}{4\pi i}\int_{-\sigma_n}^{\sigma_n} \frac{(P_1+P_2)^2}{\sigma+i\xi_n}d\sigma - \frac{-1}{2\pi i}\int_{-\sigma_n}^{\sigma_n} \frac{\left(P_2^2-P_1^2\right)}{\sigma+i\xi_n}\frac{\cosh\sigma(\pi-a)}{\sinh\sigma(\pi-a)}d\sigma - \frac{1}{4\pi i}\int_{-\sigma_n}^{\sigma_n}\frac{(P_2-P_1)^2}{\sigma+i\xi_n}\frac{\cosh^2\sigma(\pi-a)}{\sinh^2\sigma(\pi-a)}d\sigma \tag{1.45}$$

from which after calculation we have

$$I_2^{**} = \frac{2P_1P_2}{\pi}\tan^{-1}\frac{\pi-a}{a} + \frac{\left(P_2^2-P1^2\right)}{2\pi}\ln\left(1+\frac{(\pi-a)^2}{a^2}\right) + o(1). \tag{1.46}$$

Substituting from (1.44) and (1.46) into (1.42), we have

$$\frac{1}{2\pi i}\oint_{\Gamma_n}\frac{Z_1^2(s)}{sZ_o^2(s)}ds = \frac{P_1^2}{\pi}\ln\frac{a}{\pi-a} + \frac{P_2^2}{2\pi}\ln\left(2-\frac{\pi(\pi-2a)}{a(\pi-a)}\right) + \frac{4P_1P_2}{\pi}\tan^{-1}\frac{\pi-a}{a} - P_1P_2. \tag{1.47}$$

Substituting from (1.47), (1.40), (1.33), and (1.22) into (1.21) we have

$$\begin{aligned}
2\sum_{k=0}^{n}&\left[\left(\lambda_k^+ - \lambda_k^{o+}\right)\right] + 2\sum_{k=0}^{n}\left[\left(\lambda_k^- - \lambda_k^{o-}\right)\right] \\
&= \frac{2(P_2-P_1)}{(\pi-a)}\left(n+\frac{1}{4}\right) \\
&\quad + \frac{2(P_1-P_2)}{a}\left(n+\frac{1}{4}\right) + Q_1 + Q_2 - P_1P_2 \\
&\quad + \left[\frac{2}{\pi}(Q_2-Q_1) + \frac{P_1^2}{\pi}\right]\ln\frac{a}{\pi-a} \\
&\quad + \frac{P_2^2}{2\pi}\ln\left[2-\frac{\pi(\pi-2a)}{a(\pi-a)}\right] + \frac{4P_1P_2}{\pi}\tan^{-1}\left[\frac{\pi-a}{\pi}\right] + o(1).
\end{aligned} \tag{1.48}$$

Passing to the limit as $n \to \infty$ and by using (1.9) and (1.6) we reach to the required formula (1.10). □

2. Conclusion and Comment

It should be noted here that, we cannot expect that the trace formula must become the Gelfand-Levitan trace formula as the point $x = a$ approaches the point "π", because we do not put the condition $y(a) = 0$ at the point $x = a$.

Due to the presence of the turning point (1.3), as it is well known for all unbounded operator, the convergent series $\sum_n(\lambda_n - \lambda_n^{(o)})$ is actually the sum of two infinite siriases, each of which cannot be summed separately.

Moreover the presence of the turning point helps in the solution of inverse problem for different densities medium [15]. The present work belongs to the school of Gasymov [6].

Conflict of Interests

The authors declare that they have no competing interests.

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References

[1] I. M. Gelfand and B. M. Levitan, "On a simple identity for the characteristic values of a differential operator of the second order," *Doklady Akademii Nauk SSSR*, vol. 88, pp. 593–596, 1953.

[2] L. A. Dikiĭ, "On a formula of Gelfand-Levitan," *Uspekhi Matematicheskikh Nauk*, vol. 8, no. 2, pp. 119–123, 1953.

[3] L. A. Dikiĭ, "The zeta function of an ordinary differential equation on a finite interval," *Izvestiya Akademii Nauk SSSR*, vol. 19, pp. 187–200, 1955.

[4] B. M. Levitan, "Calculation of the regularized trace for the Sturm-Liouville operator," *Uspekhi Matematicheskikh Nauk*, vol. 19, no. 1, pp. 161–165, 1964.

[5] L. D. Faddeev, "On the expression for the trace of the difference of two singular differential operators of the Sturm-Liouville type," *Doklady Akademii Nauk SSSR*, vol. 115, no. 5, pp. 878–881, 1957.

[6] M. G. Gasymov, "On the sum of the differences of the eigenvalues of two self-adjoint operators," *Doklady Akademii Nauk SSSR*, vol. 150, no. 6, pp. 1202–1205, 1963.

[7] V. A. Sadovnichii and V. E. Podolski, "Traces of differential operators," *Differential Equations*, vol. 45, no. 4, pp. 477–493, 2009.

[8] S. Clark and F. Gesztesy, "Weyl-Titchmarsh M-function asymptotics, local uniqueness results, trace formulas, and Borg-type theorems for Dirac operators," *Transactions of the American Mathematical Society*, vol. 354, no. 9, pp. 3475–3534, 2002.

[9] R. Carlson, "Large eigenvalues and trace formulas for matrix Sturm-Liouville problems," *SIAM Journal on Mathematical Analysis*, vol. 30, no. 5, pp. 949–962, 1999.

[10] Z. F. A. El-Raheem and A. H. Nasser, "On the spectral property of a Dirichlet problem with explosive factor," *Applied Mathematics and Computation*, vol. 138, no. 2-3, pp. 355–374, 2003.

[11] Z. F. A. El-Raheem and A. H. Nasser, "The equiconvergence of the eigenfunction expansion for a singular version of one-dimensional Schrödinger operator with explosive factor," *Boundary Value Problems*, vol. 2011, article 45, 2011.

[12] Z. F. A. El-Raheem and A. H. Nasser, "The eigen function expansion for a Dirichlet problem with explosive factor," *Abstract and Applied Analysis*, vol. 2011, Article ID 828176, 16 pages, 2011.

[13] Z. F. A. El-Reheem, "On some trace formula for the Sturm-Liouville operator," *Pure Mathematics and Applications*, vol. 7, no. 1-2, pp. 61–68, 1996.

[14] V. A. Marchenko, *Sturm-Liouville Operators and Applications, Revised Edition*, American Mathematical Society, 2011.

[15] M. G. Gasymov and Z. F. Rekheem, "On the theory of inverse Sturm-Liouville problems with discontinuous sign-alternating weight," *Doklady. Akademiya Nauk Azerbaĭdzhana*, vol. 48–50, no. 1–12, pp. 13–16, 1996.