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Research Article

Asymptotic Properties of Derivatives of the Stieltjes Polynomials

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Let $w_\lambda(x) := (1-x^2)^{\lambda-1/2}$ and $P_{\lambda,n}(x)$ be the ultraspherical polynomials with respect to $w_\lambda(x)$. Then, we denote the Stieltjes polynomials with respect to $w_\lambda(x)$ by $E_{\lambda,n+1}(x)$ satisfying $\int_{-1}^{1} w_\lambda(x)P_{\lambda,n}(x)E_{\lambda,n+1}(x)x^m dx = 0$, $0 \le m < n+1$, $\int_{-1}^{1} w_\lambda(x)P_{\lambda,n}(x)E_{\lambda,n+1}(x)x^m dx \neq 0$, $m = n+1$. In this paper, we investigate asymptotic properties of derivatives of the Stieltjes polynomials $E_{\lambda,n+1}(x)$ and the product $E_{\lambda,n+1}(x)P_{\lambda,n}(x)$. Especially, we estimate the even-order derivative values of $E_{\lambda,n+1}(x)$ and $E_{\lambda,n+1}(x)P_{\lambda,n}(x)$ at the zeros of $E_{\lambda,n+1}(x)$ and the product $E_{\lambda,n+1}(x)P_{\lambda,n}(x)$, respectively. Moreover, we estimate asymptotic representations for the odd derivatives values of $E_{\lambda,n+1}(x)$ and $E_{\lambda,n+1}(x)P_{\lambda,n}(x)$ at the zeros of $E_{\lambda,n+1}(x)$ and $E_{\lambda,n+1}(x)P_{\lambda,n}(x)$ on a closed subset of $(-1,1)$, respectively. These estimates will play important roles in investigating convergence and divergence of the higher-order Hermite-Fejér interpolation polynomials.

1. Introduction

Consider the generalized Stieltjes polynomials $E_{\lambda,n+1}(x)$ defined (up to a multiplicative constant) by

$$\int_{-1}^{1} w_\lambda(x)P_{\lambda,n}(x)E_{\lambda,n+1}(x)x^k dx = 0, \quad k = 0,1,2,\ldots,n, \ n \geqslant 1, \tag{1.1}$$

where $w_\lambda(x) = (1-x^2)^{\lambda-1/2}$, $\lambda > -1/2$, and $P_{\lambda,n}(x)$ is the nth ultraspherical polynomial for the weight function $w_\lambda(x)$.

The polynomials $E_{\lambda,n+1}(x)$, introduced by Stieltjes and studied by Szegö, have been used in numerical integration, whereas the polynomials $P_{\lambda,n}(x)E_{\lambda,n+1}(x)$ have been used in extended Lagrange interpolation. In this paper, we will prove pointwise and asymptotic estimates for the higher-order derivatives of $E_{\lambda,n+1}(x)$ and $P_{\lambda,n}(x)E_{\lambda,n+1}(x)$. It is well known that these kind of estimates are useful for studying interpolation processes with multiple nodes.

In 1934, G. Szegö [1] showed that the zeros of the generalized Stieltjes polynomials $E_{\lambda,n+1}(x)$ are real and inside $[-1,1]$ and interlace with the zeros of $P_{\lambda,n}(x)$ whenever $0 \leqslant \lambda \leqslant 2$. Recently, several authors [2–8] studied further interesting properties for these Stieltjes polynomials. Ehrich and Mastroianni [3, 4] gave accurate pointwise bounds of $E_{\lambda,n+1}(x)$ $(0 \leqslant \lambda \leqslant 1)$ and the product $F_{\lambda,2n+1} := E_{\lambda,n+1}(x)P_{\lambda,n}(x)$ $(0 \leqslant \lambda \leqslant 1)$ on $[-1,1]$, and they estimated asymptotic representations for $E'_{\lambda,n+1}(x)$ and $F'_{\lambda,2n+1}(x)$ at the zeros of $E_{\lambda,n+1}(x)$ and $F_{\lambda,2n+1}(x)$, respectively. In [6], pointwise upper bounds of $E'_{\lambda,n+1}(x)$, $E''_{\lambda,n+1}(x)$, $F'_{\lambda,2n+1}(x)$, and $F''_{\lambda,2n+1}(x)$ are obtained using the asymptotic differential relations of the first and the second order for the Stieltjes polynomials $E_{\lambda,n+1}(x)$ $(0 \leqslant \lambda \leqslant 1)$ and $F_{\lambda,2n+1}(x)$ $(0 \leqslant \lambda \leqslant 1)$. Also the values of $E''_{\lambda,n+1}(x)$ and $F''_{\lambda,2n+1}(x)$ at the zeros of $E_{\lambda,n+1}(x)$ and $F_{\lambda,2n+1}(x)$ are estimated in [6]. Moreover, using the results of [6], the Lebesgue constants of Hermite-Fejér interpolatory process are estimated in [7].

In this paper, we find pointwise upper bounds of $E^{(r)}_{\lambda,n+1}(x)$ and $F^{(r)}_{\lambda,2n+1}(x)$ for two cases of an odd order and of even order. Using these relations, we investigate asymptotic properties of derivatives of the Stieltjes polynomials $E_{\lambda,n+1}(x)$ and $F_{\lambda,2n+1}(x)$ and we also estimate the values of $E^{(2\ell)}_{\lambda,n+1}(x)$ and $F^{(2\ell)}_{\lambda,2n+1}(x)$ at the zeros of $E_{\lambda,n+1}(x)$ and $F_{\lambda,2n+1}(x)$, respectively. Especially, for the value of $F^{(2\ell)}_{\lambda,2n+1}(x)$ at the zeros of $F_{\lambda,2n+1}(x)$, we will estimate $P^{(r)}_{\lambda,n}(x)$ and $E^{(r)}_{\lambda,n+1}(x)$ for an odd r at the zeros of $E_{\lambda,n+1}(x)$ and $P_{\lambda,n}(x)$, respectively. Finally, we investigate asymptotic representations for the values of $E^{(2\ell+1)}_{\lambda,n+1}$ and $F^{(2\ell+1)}_{\lambda,2n+1}(x)$ at the zeros of $E_{\lambda,n+1}(x)$ and $F_{\lambda,2n+1}(x)$ on a closed subset of $(-1,1)$, respectively. These estimates will play important roles in investigating convergence and divergence of the higher-order Hermite-Fejér interpolation polynomials.

This paper is organized as follows. In Section 2, we will introduce the main results. In Section 3, we will introduce the known results in order to prove the main results. Finally, we will prove the results in Section 4.

2. Main Results

We first introduce some notations, which we use in the following. For the ultraspherical polynomials $P_{\lambda,n}$, $\lambda \neq 0$, we use the normalization $P_{\lambda,n}(1) = \binom{n+2\lambda-1}{n}$ and then we know that $P_{\lambda,n}(1) \sim n^{2\lambda-1}$. We denote the zeros of $P_{\lambda,n}$ by $x^{(\lambda)}_{\nu,n}$, $\nu = 1,\dots,n$, and the zeros of Stieltjes polynomials $E_{\lambda,n+1}$ by $\xi^{(\lambda)}_{\mu,n+1}$, $\mu = 1,\dots,n+1$. We denote the zeros of $F_{\lambda,2n+1} := P_{\lambda,n}E_{\lambda,n+1}$ by $y^{(\lambda)}_{\nu,2n+1}$, $\nu = 1,\dots,2n+1$. All nodes are ordered by increasing magnitude. We set $\varphi(x) := \sqrt{1-x^2}$, and, for any two sequences $\{b_n\}_n$ and $\{c_n\}_n$ of nonzero real numbers (or functions), we write $b_n \lesssim c_n$, if there exists a constant $C > 0$, independent of n (and x) such that $b_n \leqslant Cc_n$ for n large enough and write $b_n \sim c_n$ if $b_n \lesssim c_n$ and $c_n \lesssim b_n$. We denote by $\mathcal{P}_n$ the space of polynomials of degree at most n.

For the Chebyshev polynomial $T_n(x)$, note that for $\lambda = 0$ and $\lambda = 1$

$$
\begin{aligned}
E_{0,n+1}(x) &= \frac{2n}{\pi}(T_{n+1}(x) - T_{n-1}(x)) \\
E_{1,n+1}(x) &= \frac{2}{\pi}T_{n+1}(x).
\end{aligned}
\tag{2.1}
$$

Therefore, we will consider $E_{\lambda,n+1}(x)$ for $0 < \lambda < 1$.

Theorem 2.1. *Let $0<\lambda<1$ and $r\geqslant 1$ be a positive integer. Then, for all $x\in[\xi_{1,n+1}^{(\lambda)},\xi_{n+1,n+1}^{(\lambda)}]$,*

$$\left|E_{\lambda,n+1}^{(r)}(x)\right|\lesssim n^{r+1-\lambda}\varphi^{1-r-\lambda}(x). \tag{2.2}$$

Moreover, one has

$$\max_{x\in[-1,1]}\left|E_{\lambda,n+1}^{(r)}(x)\right|\sim n^{2r}, \tag{2.3}$$

and especially one has, for $x\in[-1,\xi_{1,n+1}^{(\lambda)}]\cup[\xi_{n+1,n+1}^{(\lambda)},1]$,

$$\left|E_{\lambda,n+1}^{(r)}(x)\right|\sim n^{2r}. \tag{2.4}$$

Theorem 2.2. *Let $0<\lambda<1$ and $r\geqslant 1$ be a positive integer. Then, for all $x\in[\xi_{1,n+1}^{(\lambda)},\xi_{n+1,n+1}^{(\lambda)}]$,*

$$\left|F_{\lambda,2n+1}^{(r)}(x)\right|\lesssim n^{r}\varphi^{1-2\lambda-r}(x). \tag{2.5}$$

Moreover, one has

$$\max_{x\in[-1,1]}\left|F_{\lambda,2n+1}^{(r)}(x)\right|\lesssim n^{2\lambda+2r-1}, \tag{2.6}$$

and especially, for $x\in[-1,\xi_{1,n+1}^{(\lambda)}]\cup[\xi_{n+1,n+1}^{(\lambda)},1]$,

$$\left|F_{\lambda,2n+1}^{(r)}(x)\right|\sim n^{2\lambda+2r-1}. \tag{2.7}$$

In the following, we also estimate the values of $E_{\lambda,n+1}^{(2\ell)}(x)$ and $F_{\lambda,2n+1}^{(2\ell)}(x)$, $\ell\geqslant 1$ at the zeros $\{\xi_{\mu,n+1}^{(\lambda)}\}$ of $E_{\lambda,n+1}(x)$ and the zeros $\{y_{\nu,2n+1}^{(\lambda)}\}$ of $F_{\lambda,2n+1}(x)$, respectively.

Theorem 2.3. *Let $0<\lambda<1$ and $r\geqslant 2$ be an even integer. For $1\leqslant\mu\leqslant n+1$, one has*

$$\left|E_{\lambda,n+1}^{(r)}\left(\xi_{\mu,n+1}^{(\lambda)}\right)\right|\lesssim n^{r}\varphi^{-r}\left(\xi_{\mu,n+1}^{(\lambda)}\right). \tag{2.8}$$

Theorem 2.4. *Let $0<\lambda<1$ and $r\geqslant 2$ be an even integer. For $1\leqslant\nu\leqslant 2n+1$, one has*

$$\left|F_{\lambda,2n+1}^{(r)}\left(y_{\nu,2n+1}^{(\lambda)}\right)\right|\lesssim n^{r-1+\lambda}\varphi^{-r-\lambda}\left(y_{\nu,2n+1}^{(\lambda)}\right). \tag{2.9}$$

Finally, we obtain the asymptotic representations for the values of $E_{\lambda,n+1}^{(2\ell+1)}(x)$ and $F_{\lambda,2n+1}^{(2\ell+1)}(x)$ at the zeros of $E_{\lambda,n+1}(x)$ and $F_{\lambda,2n+1}(x)$ on a closed subset of $(-1,1)$, respectively.

Theorem 2.5. *Let* $0 < \lambda < 1$ *and* $0 < \varepsilon < 1$. *Suppose* $|\xi_{\mu,n+1}^{(\lambda)}| \leqslant 1 - \varepsilon$. *Then,*

(a)

$$E_{\lambda,n+1}^{(2\ell+1)}\left(\xi_{\mu,n+1}^{(\lambda)}\right) = (-1)^{\ell}(n+1)^{2\ell}\varphi^{-2\ell}\left(\xi_{\mu,n+1}^{(\lambda)}\right)E'_{\lambda,n+1}\left(\xi_{\mu,n+1}^{(\lambda)}\right) + O\left(n^{2\ell+1}\right), \tag{2.10}$$

(b)

$$P_{\lambda,n}^{(2\ell)}\left(\xi_{\mu,n+1}^{(\lambda)}\right) = (-1)^{\ell}n^{\ell}(n+2\lambda)^{\ell}\varphi^{-2\ell}\left(\xi_{\mu,n+1}^{(\lambda)}\right)P_{\lambda,n}\left(\xi_{\mu,n+1}^{(\lambda)}\right) + O\left(n^{2\lambda+2\ell-3}\right). \tag{2.11}$$

In addition,

$$P_{\lambda,n}^{(2\ell)}\left(\xi_{\mu,n+1}^{(\lambda)}\right) = (-1)^{\ell}(n+1)^{2\ell}\varphi^{-2\ell}\left(\xi_{\mu,n+1}^{(\lambda)}\right)P_{\lambda,n}\left(\xi_{\mu,n+1}^{(\lambda)}\right) + O\left(n^{2\ell+\lambda-2}\right). \tag{2.12}$$

Theorem 2.6. *Let* $0 < \lambda < 1$ *and* $0 < \varepsilon < 1$. *Suppose* $|x_{\nu,n}^{(\lambda)}| \leqslant 1 - \varepsilon$. *Then,*

(a)

$$E_{\lambda,n+1}^{(2\ell)}\left(x_{\nu,n}^{(\lambda)}\right) = (-1)^{\ell}(n+1)^{2\ell}\varphi^{-2\ell}\left(x_{\nu,n}^{(\lambda)}\right)E_{\lambda,n+1}\left(x_{\nu,n}^{(\lambda)}\right) + O\left(n^{2\ell}\right), \tag{2.13}$$

(b)

$$P_{\lambda,n}^{(2\ell+1)}\left(x_{\nu,n}^{(\lambda)}\right) = (-1)^{\ell}n^{\ell}(n+2\lambda)^{\ell}\varphi^{-2\ell}\left(x_{\nu,n}^{(\lambda)}\right)P'_{\lambda,n}\left(x_{\nu,n}^{(\lambda)}\right) + O\left(n^{\lambda+2\ell-2}\right). \tag{2.14}$$

In addition,

$$P_{\lambda,n}^{(2\ell+1)}\left(x_{\nu,n}^{(\lambda)}\right) = (-1)^{\ell}(n+1)^{2\ell}\varphi^{-2\ell}\left(x_{\nu,n}^{(\lambda)}\right)P'_{\lambda,n}\left(x_{\nu,n}^{(\lambda)}\right) + O\left(n^{\lambda+2\ell-1}\right). \tag{2.15}$$

Theorem 2.7. *Let* $0 < \lambda < 1$ *and* $0 < \varepsilon < 1$. *Suppose* $|y_{\nu,2n+1}^{(\lambda)}| \leqslant 1 - \varepsilon$. *Then, one has, for a positive integer* $\ell \geqslant 1$,

$$F_{\lambda,2n+1}^{(2\ell+1)}\left(y_{\nu,2n+1}^{(\lambda)}\right) = c_{\ell}(-1)^{\ell}(n+1)^{2\ell}\varphi^{-2\ell}\left(y_{\nu,2n+1}^{(\lambda)}\right)F'_{\lambda,2n+1}\left(y_{\nu,2n+1}^{(\lambda)}\right) + O\left(n^{\lambda+2\ell}\right), \tag{2.16}$$

where $c_{\ell} = 4^{\ell} - 2^{\ell-1}$.

3. The Known Results

In this section, we will introduce the known results in [4, 6, 9] to prove main results.

Proposition 3.1. (a) Let $\lambda > -1/2$. Then, $P_{\lambda,n}(x)$ satisfies the second-order differential equation as follows:

$$\left(1-x^2\right)P''_{\lambda,n}(x) - (2\lambda+1)xP'_{\lambda,n}(x) + n(n+2\lambda)P_{\lambda,n}(x) = 0. \tag{3.1}$$

(b) Let $\lambda > -1/2$. Then,

$$P'_{\lambda,n}(x) = 2\lambda P_{\lambda+1,n-1}(x). \tag{3.2}$$

(c) Let $\lambda > -1/2$. Then, for $1 \leqslant \mu \leqslant n+1$,

$$\left|P'_{\lambda,n}\left(\xi^{(\lambda)}_{\mu,n+1}\right)\right| \lesssim n^{2\lambda-1}\varphi^{-2}\left(\xi^{(\lambda)}_{\mu,n+1}\right). \tag{3.3}$$

(d) Let $\lambda > -1/2$. Then, for $1 \leqslant \nu \leqslant n$,

$$\left|P'_{\lambda,n}\left(x^{(\lambda)}_{\nu,n}\right)\right| \sim n^{\lambda}\varphi^{-\lambda-1}\left(x^{(\lambda)}_{\nu,n}\right). \tag{3.4}$$

(e) Let $\lambda > -1/2$. Then, for $x \in (-1,1)$ and $r \geqslant 0$,

$$\left|P^{(r)}_{\lambda,n}(x)\right| \lesssim n^{\lambda+r-1}\varphi^{-\lambda-r}(x). \tag{3.5}$$

(f) Let $0 \leqslant \lambda \leqslant 1$. Then, for $1 \leqslant \nu \leqslant n$,

$$\left|E'_{\lambda,n+1}\left(x^{(\lambda)}_{\nu,n}\right)\right| \lesssim n\varphi^{-1}\left(x^{(\lambda)}_{\nu,n}\right). \tag{3.6}$$

(g) Let $\lambda > -1/2$ and $r \geqslant 0$. Then, $P_{\lambda,n}(x)$ satisfies the higher-order differential equation as follows:

$$\left(1-x^2\right)P^{(r+2)}_{\lambda,n}(x) - (2\lambda+2r+1)xP^{(r+1)}_{\lambda,n}(x) + \left(n^2+2\lambda n - r(2\lambda-r+2)\right)P^{(r)}_{\lambda,n}(x) = 0. \tag{3.7}$$

Proof. (a) It is from [9, (4.2.1)]. (b) It is from [9, (4.7.14)]. (c) It is from [6, Lemma 3.4]. (d) It is from [9, (8.9.7)]. (e) For $r = 0$, it follows from [9, (7.33.5)], and, for $r \geqslant 1$, it comes from (b) and the case of $r = 0$. (f) It is from [6, Lemma 3.3 (3.23)]. (g) Equation (3.7) comes from (a). □

Proposition 3.2 (see [4]). *Let $0 < \lambda < 1$. Let $\xi_{\mu,n+1}^{(\lambda)} := \cos\theta_{\mu,n+1}^{(\lambda)}$, $\mu = 1,\dots,n+1$ and $y_{\nu,2n+1}^{(\lambda)} := \cos\psi_{\nu,2n+1}^{(\lambda)}$, $\nu = 1,\dots,2n+1$. Then, for $\mu = 0,1,\dots,n+2$ and $\nu = 0,1,\dots,2n+2$,*

$$\left|\theta_{\mu,n+1}^{(\lambda)} - \theta_{\mu+1,n+1}^{(\lambda)}\right| \sim \left|\psi_{\nu,2n+1}^{(\lambda)} - \psi_{\nu+1,2n+1}^{(\lambda)}\right| \sim n^{-1}, \tag{3.8}$$

where $\psi_{0,2n+1}^{(\lambda)} := \theta_{0,n+1}^{(\lambda)} := \pi$ and $\psi_{2n+2,2n+1}^{(\lambda)} := \theta_{n+2,n+1}^{(\lambda)} := 0$.

Proposition 3.3 ([6, Proposition 2.3]). *Let $0 < \lambda < 1$. Then, for all $x \in [-1,1]$,*

$$\left(1 - x^2\right)E_{\lambda,n+1}''(x) - xE_{\lambda,n+1}'(x) + (n+1)^2E_{\lambda,n+1}(x) = I_{\lambda,n}(x), \tag{3.9}$$

where

$$I_{\lambda,n}(x) = \frac{8}{\gamma_n^{(\lambda)}}\sum_{\nu=1}^{[(n+1)/2]}(n+1-\nu)\nu\alpha_{\nu,n}^{(\lambda)}T_{n+1-2\nu}(x). \tag{3.10}$$

Then $I_{\lambda,n}(x)$ is a polynomial of degree $n-1$ satisfying

$$\max_{x\in[-1,1]}|I_{\lambda,n}(x)| \lesssim n^2. \tag{3.11}$$

Proposition 3.4 ([4, Theorem 2.1]). *Let $0 < \lambda < 1$. Then, for $n \geqslant 0$,*

$$|E_{\lambda,n+1}(x)| \lesssim n^{1-\lambda}\varphi^{1-\lambda}(x) + 1 \quad -1 \leqslant x \leqslant 1. \tag{3.12}$$

Furthermore, $E_{\lambda,n+1}(1) \gtrsim 1$.

Proposition 3.5 ([6, Theorem 2.5]). *Let $0 < \lambda < 1$.*

(a) *For all $x \in [\xi_{1,n+1}^{(\lambda)}, \xi_{n+1,n+1}^{(\lambda)}]$,*

$$\left|E_{\lambda,n+1}'(x)\right| \lesssim n^{2-\lambda}\varphi^{-\lambda}(x). \tag{3.13}$$

Moreover, one has, for $x \in [-1, \xi_{1,n+1}^{(\lambda)}] \cup [\xi_{n+1,n+1}^{(\lambda)}, 1]$,

$$\left|E_{\lambda,n+1}'(x)\right| \sim n^2. \tag{3.14}$$

(b) *For all $x \in [\xi_{1,n+1}^{(\lambda)}, \xi_{n+1,n+1}^{(\lambda)}]$,*

$$\left|E_{\lambda,n+1}''(x)\right| \lesssim n^{3-\lambda}\varphi^{-1-\lambda}(x). \tag{3.15}$$

Moreover, one has, for $x \in [-1, \xi_{1,n+1}^{(\lambda)}] \cup [\xi_{n+1,n+1}^{(\lambda)}, 1]$,

$$\left|E''_{\lambda,n+1}(x)\right| \sim n^4. \tag{3.16}$$

Proposition 3.6 ([6, Corollary 2.6]). *Let* $0 < \lambda < 1$. *Then, for all* $x \in [-1, 1]$,

$$\left(1 - x^2\right)F''_{\lambda,2n+1}(x) - xF'_{\lambda,2n+1}(x) + \left(2n^2 + 2(1+\lambda)n + 1\right)F_{\lambda,2n+1}(x) = J_{\lambda,n}(x). \tag{3.17}$$

Here, $J_{\lambda,n}(x)$ *is a polynomial of degree of* $2n+1$ *defined in* (4.37) *such that, for* $x \in [\xi_{1,n+1}^{(\lambda)}, \xi_{n+1,n+1}^{(\lambda)}]$,

$$|J_{\lambda,n}(x)| \lesssim n^2 \varphi^{1-2\lambda}(x) \tag{3.18}$$

and, for $x \in [-1, \xi_{1,n+1}^{(\lambda)}] \cup [\xi_{n+1,n+1}^{(\lambda)}, 1]$,

$$|J_{\lambda,n}(x)| \lesssim n^{1+2\lambda}. \tag{3.19}$$

Proposition 3.7 ([6, Corollary 2.7]). *Let* $0 < \lambda < 1$.

(a) *For all* $x \in [\xi_{1,n+1}^{(\lambda)}, \xi_{n+1,n+1}^{(\lambda)}]$,

$$\left|F'_{\lambda,2n+1}(x)\right| \lesssim n\varphi^{-2\lambda}(x). \tag{3.20}$$

Moreover, one has, for $x \in [-1, \xi_{1,n+1}^{(\lambda)}] \cup [\xi_{n+1,n+1}^{(\lambda)}, 1]$,

$$\left|F'_{\lambda,2n+1}(x)\right| \sim n^{1+2\lambda}. \tag{3.21}$$

(b) *For all* $x \in [\xi_{1,n+1}^{(\lambda)}, \xi_{n+1,n+1}^{(\lambda)}]$,

$$\left|F''_{\lambda,2n+1}(x)\right| \lesssim n^2\varphi^{-1-2\lambda}(x). \tag{3.22}$$

Moreover, one has, for $x \in [-1, \xi_{1,n+1}^{(\lambda)}] \cup [\xi_{n+1,n+1}^{(\lambda)}, 1]$,

$$\left|F''_{\lambda,2n+1}(x)\right| \sim n^{3+2\lambda}. \tag{3.23}$$

Proposition 3.8 ([4, Lemma 5.5]). *Let* $0 < \lambda < 1$. *Then, for* $\mu = 1, 2, \ldots, n+1$,

$$\left|E'_{\lambda,n+1}\left(\xi_{\mu,n+1}^{(\lambda)}\right)\right| \sim n^{2-\lambda}\varphi^{-\lambda}\left(\xi_{\mu,n+1}^{(\lambda)}\right) \tag{3.24}$$

and, for $\nu = 1, 2, \ldots, 2n+1$,

$$\left|F'_{\lambda,2n+1}\left(y^{(\lambda)}_{\nu,2n+1}\right)\right| \sim n\varphi^{-2\lambda}\left(y^{(\lambda)}_{\nu,2n+1}\right). \tag{3.25}$$

We now estimate the second derivatives at the zeros of $E_{\lambda,n+1}$ and $F_{\lambda,2n+1}$.

Proposition 3.9 ([6, Theorem 2.9]). *Let* $0 < \lambda < 1$. *Then, for* $\mu = 1, 2, \ldots, n+1$,

$$\left|E''_{\lambda,n+1}\left(\xi^{(\lambda)}_{\mu,n+1}\right)\right| \lesssim n^2\varphi^{-2}\left(\xi^{(\lambda)}_{\mu,n+1}\right) \tag{3.26}$$

and, for $\nu = 1, 2, \ldots, 2n+1$,

$$\left|F''_{\lambda,2n+1}\left(y^{(\lambda)}_{\nu,2n+1}\right)\right| \lesssim n^{1+\lambda}\varphi^{-2-\lambda}\left(y^{(\lambda)}_{\nu,2n+1}\right). \tag{3.27}$$

4. The Proofs of Main Results

In this section, we let $0 < \lambda < 1$ and $m = \lfloor (n+1/2) \rfloor$. A representation of Stieltjes polynomials $E_{\lambda,n+1}(x)$ is (cf. [1, 10])

$$\frac{\gamma^{(\lambda)}_n}{2}E_{\lambda,n+1}(\cos\theta) = \alpha^{(\lambda)}_{0,n}\cos(n+1)\theta + \alpha^{(\lambda)}_{1,n}\cos(n-1)\theta + \cdots + \begin{cases} \alpha^{(\lambda)}_{n/2,n}\cos\theta, & n \text{ even} \\ \dfrac{1}{2}\alpha^{(\lambda)}_{n+1/2,n}, & n \text{ odd}, \end{cases} \tag{4.1}$$

where

$$\begin{gathered}
\alpha^{(\lambda)}_{0,n} = f^{(\lambda)}_{0,n} = 1, \quad \sum_{\mu=0}^{\nu}\alpha^{(\lambda)}_{\mu,n}f^{(\lambda)}_{\nu-\mu,n} = 0, \quad \nu = 1, 2, \ldots, \\
f^{(\lambda)}_{\nu,n} := \left(1 - \frac{\lambda}{\nu}\right)\left(1 - \frac{\lambda}{n+\nu+\lambda}\right), \quad \nu = 1, 2, \ldots, \\
\gamma^{(\lambda)}_n = \sqrt{\pi}\frac{\Gamma(n+2\lambda)}{\Gamma(n+\lambda+1)} \sim \sqrt{\pi}n^{\lambda-1}.
\end{gathered} \tag{4.2}$$

In the following, we state the asymptotic differential relation of the higher order of $E_{\lambda,n+1}$.

Lemma 4.1. *Let* $0 < \lambda < 1$. *Then, for all* $x \in [-1, 1]$ *and* $r \geqslant 2$,

$$\left(1-x^2\right)E^{(r)}_{\lambda,n+1}(x) = (2r-3)xE^{(r-1)}_{\lambda,n+1}(x) + \left((r-2)^2 - (n+1)^2\right)E^{(r-2)}_{\lambda,n+1}(x) + I^{(r-2)}_{\lambda,n}(x) \tag{4.3}$$

and, for $x \in [\xi^{(\lambda)}_{1,n+1}, \xi^{(\lambda)}_{n+1,n+1}]$,

$$\left|I^{(r-2)}_{\lambda,n}(x)\right| \lesssim n^r\varphi^{2-r}(x). \tag{4.4}$$

Here, $I_{\lambda,n}(x)$ is a polynomial of degree $n-1$ defined in (3.10)*;*

$$I_{\lambda,n}(x) = \frac{8}{\gamma_n^{(\lambda)}} \sum_{\nu=1}^{m} (n+1-\nu)\nu\alpha_{\nu,n}^{(\lambda)} T_{n+1-2\nu}(x), \tag{4.5}$$

such that

$$\max_{x\in[-1,1]} \left| I_{\lambda,n}^{(r-2)}(x) \right| \lesssim n^{2r-2}. \tag{4.6}$$

Proof. For $r \geqslant 2$, (4.3) is obtained by $r-2$ times differentiation of (3.9). Equation (4.6) follows by (3.11) and the use of Markov-Bernstein inequality. Now, we prove (4.4). We know that the Chebyshev polynomial $T_n(x)$ satisfies the second-order differential equation

$$\left(1-x^2\right)T_n''(x) - xT_n'(x) + n^2 T_n(x) = 0, \tag{4.7}$$

so we have, by $r-2$ times differentiation of (4.7),

$$\left(1-x^2\right)T_n^{(r)}(x) - (2r-3)xT_n^{(r-1)}(x) - \left((r-2)^2 - n^2\right)T_n^{(r-2)}(x) = 0. \tag{4.8}$$

Let for a nonnegative integer $j \geqslant 0$,

$$I_{\lambda,n,j}(x) := -\frac{8}{\gamma_n^{(\lambda)}} \sum_{\nu=1}^{m} (n+1-\nu)\nu\alpha_{\nu,n}^{(\lambda)} \left| T_{n+1-2\nu}^{(j)}(x) \right|. \tag{4.9}$$

Observe that in the view of Szegö's result (cf. [1])

$$\alpha_{1,n}^{(\lambda)} < \alpha_{2,n}^{(\lambda)} < \alpha_{3,n}^{(\lambda)} < \cdots < 0, \quad 0 \leqslant \sum_{\nu=0}^{\infty} \alpha_{\nu,n}^{(\lambda)} < 1. \tag{4.10}$$

Then, since $|I_{\lambda,n}^{(j)}(x)| \leqslant I_{\lambda,n,j}(x)$ (note (4.10)), we will prove that, for $x \in [\xi_{1,n+1}^{(\lambda)}, \xi_{n+1,n+1}^{(\lambda)}]$ and $j \geqslant 0$,

$$I_{\lambda,n,j}(x) \lesssim n^{j+2}\varphi^{-j}(x) \tag{4.11}$$

instead of (4.4). Since, from the proof of [6, Proposition 2.3],

$$0 < -\frac{8}{\gamma_n^{(\lambda)}} \sum_{\nu=1}^{m} (n+1-\nu)\nu\alpha_{\nu,n}^{(\lambda)} \lesssim n^2 \tag{4.12}$$

and, for $x = \cos\theta$,

$$\left|T_{n+1-2\nu}'(x)\right| = \left|(n+1-2\nu)\frac{\sin(n+1-2\nu)\theta}{\sin\theta}\right| \lesssim n\varphi^{-1}(x), \tag{4.13}$$

we obtain that $I_{\lambda,n,0}(x) \lesssim n^2$ and $I_{\lambda,n,1}(x) \lesssim n^3\varphi^{-1}(x)$. Using (4.8), we have, for $2 \leqslant j \leqslant n$,

$$\begin{aligned} I_{\lambda,n,j}(x) &= -\frac{8}{\gamma_n^{(\lambda)}}\sum_{\nu=1}^{m}(n+1-\nu)\nu\alpha_{\nu,n}^{(\lambda)}\left|T_{n+1-2\nu}^{(j)}(x)\right| \\ &\leq -\frac{8}{\gamma_n^{(\lambda)}}\sum_{\nu=1}^{m}(n+1-\nu)\nu\alpha_{\nu,n}^{(\lambda)} \\ &\quad\times\left(\frac{(2j-3)|x|}{1-x^2}\left|T_{n+1-2\nu}^{(j-1)}(x)\right| + \frac{\left|(j-2)^2-(n+1-2\nu)^2\right|}{1-x^2}\left|T_{n+1-2\nu}^{(j-2)}(x)\right|\right) \\ &\lesssim \frac{(2j-3)|x|}{1-x^2}I_{\lambda,n,j-1}(x) + \frac{(n+1)^2}{1-x^2}I_{\lambda,n,j-2}(x). \end{aligned} \tag{4.14}$$

Therefore, (4.11) is proved by the mathematical induction on j. Consequently, we have (4.4). □

We obtain pointwise upper bounds of $E_{\lambda,n+1}^{(r)}(x)$ for two cases of an odd order and an even order in the following.

Lemma 4.2. *Let $0 < \lambda < 1$ and $r \geqslant 2$. Let $x \in [\xi_{1,n+1}^{(\lambda)}, \xi_{n+1,n+1}^{(\lambda)}]$. If r is even, then one has*

$$\left|E_{\lambda,n+1}^{(r)}(x)\right| \lesssim n^{r-2}\varphi^{-r}(x)\left|E_{\lambda,n+1}'(x)\right| + n^r\varphi^{-r}(x)|E_{\lambda,n+1}(x)| + n^r\varphi^{-r}(x), \tag{4.15}$$

and, if r is odd, then one has

$$\left|E_{\lambda,n+1}^{(r)}(x)\right| \lesssim n^{r-1}\varphi^{1-r}(x)\left|E_{\lambda,n+1}'(x)\right| + n^{r-1}\varphi^{-1-r}(x)|E_{\lambda,n+1}(x)| + n^r\varphi^{-r}(x). \tag{4.16}$$

Proof. Let $r \geqslant 2$. From (4.3) and (4.4), we have, for $x \in [\xi_{1,n+1}^{(\lambda)}, \xi_{n+1,n+1}^{(\lambda)}]$,

$$\left|E_{\lambda,n+1}^{(r)}(x)\right| \lesssim \varphi^{-2}(x)\left|E_{\lambda,n+1}^{(r-1)}(x)\right| + n^2\varphi^{-2}(x)\left|E_{\lambda,n+1}^{(r-2)}(x)\right| + n^r\varphi^{-r}(x) \tag{4.17}$$

and especially

$$\left|E_{\lambda,n+1}''(x)\right| \lesssim \varphi^{-2}(x)\left|E_{\lambda,n+1}'(x)\right| + n^2\varphi^{-2}(x)|E_{\lambda,n+1}(x)| + n^2\varphi^{-2}(x), \tag{4.18}$$

that is, we have (4.15) for $r = 2$. From Proposition 3.2, we see $1 + \xi_{1,n+1}^{(\lambda)}, 1 - \xi_{n+1,n+1}^{(\lambda)} \gtrsim 1/n$, so we have, for $x \in [\xi_{1,n+1}^{(\lambda)}, \xi_{n+1,n+1}^{(\lambda)}]$,

$$\varphi^{-1}(x) \lesssim n. \tag{4.19}$$

Then, from (4.17) with $r = 3$ and (4.18), we know that

$$\left|E_{\lambda,n+1}^{(3)}(x)\right| \lesssim n^2\varphi^{-2}(x)\left|E'_{\lambda,n+1}(x)\right| + n^2\varphi^{-4}(x)|E_{\lambda,n+1}(x)| + n^3\varphi^{-3}(x), \tag{4.20}$$

that is, we have (4.16) for $r = 3$. Assume that (4.15) and (4.16) hold for $3, 4, \ldots, r-1$ times differentiation. Let r be an even number. Then, we have, from (4.17), (4.19), and the assumptions for $r-1$ and $r-2$,

$$\left|E_{\lambda,n+1}^{(r)}(x)\right| \lesssim n^{r-2}\varphi^{-r}(x)\left|E'_{\lambda,n+1}(x)\right| + n^r\varphi^{-r}(x)|E_{\lambda,n+1}(x)| + n^r\varphi^{-r}(x), \tag{4.21}$$

that is, we have (4.15). Similarly, we also have (4.16) for an odd r. □

Lemma 4.3. *Let $-1 < x_1 < x_2 < \cdots < x_n < 1$ and*

$$P(x) := (x - x_1)(x - x_2)\cdots(x - x_n). \tag{4.22}$$

Then, $P'(x)$ is a polynomial of degree of $n-1$ and has distinct real $n-1$ zeros in $(-1, 1)$. Moreover, if one lets $-1 < y_1 < y_2 < \cdots < y_{n-1} < 1$ be the zeros of $P'(x)$, then $\{y_i\}_{i=1}^{n}$ is interlaced with the zeros of $P(x)$ that is, $x_i < y_i < x_{i+1}$, $i = 1, 2, \ldots, n-1$.

Proof. Since the sign of $P'(x_i)$ is $(-1)^{n-i}$, it is proved. □

Lemma 4.4. *Let r be a nonnegative integer. Then, $E_{\lambda,n+1}^{(r)}(x)$ has distinct $n+1-r$ real zeros on $(-1, 1)$. If one lets $\{x_{n+1}(r, i)\}_{i=1}^{n+1-r}$ be the zeros of the polynomial $E_{\lambda,n+1}^{(r)}(x)$ with*

$$-1 < x_{n+1}(r, 1) < x_{n+1}(r, 2) < \cdots < x_{n+1}(r, n+1-r) < 1, \tag{4.23}$$

then one has, for $1 \leqslant r \leqslant n$ and $k = 1, \ldots, n+1-r$,

$$x_{n+1}(0, k) < x_{n+1}(r, k). \tag{4.24}$$

Proof. From Lemma 4.3, we know that $E_{\lambda,n+1}^{(r)}$ has distinct real $n+1-r$ zeros on $(-1, 1)$. By the interlaced zeros property of Lemma 4.3, we see that, for $k = 1, \ldots, n+1-r$,

$$x_{n+1}(r-1, k) < x_{n+1}(r, k) < x_{n+1}(r-1, k+1). \tag{4.25}$$

Thus, (4.24) is proved. □

Proof of Theorem 2.1. Let $r \geqslant 1$. Equation (2.2) comes from (4.15), (4.16), (3.12), and (3.13). From Propositions 3.4, 3.5, and (4.19), we have

$$\max_{x\in[-1,1]}\left|E_{\lambda,n+1}^{(r)}(x)\right| \sim n^{2r}, \quad r = 1, 2. \tag{4.26}$$

Hence, using the Markov-Bernstein inequality, we have (2.3). To prove (2.4), we will use the mathematical induction. We use (3.14). The formula (2.3) holds for $r = 1$ from (3.14). We suppose that, for $x \in [-1, \xi_{1,n+1}^{(\lambda)}] \cup [\xi_{n+1,n+1}^{(\lambda)}, 1]$ and $r \geqslant 2$,

$$\left| E_{\lambda,n+1}^{(r-1)}(x) \right| \sim n^{2(r-1)}. \tag{4.27}$$

Then, by Lemma 4.4 and (3.8), we have, for $x \in [\xi_{n+1,n+1}^{(\lambda)}, 1]$ and $r \geqslant 2$,

$$\begin{aligned}
&\frac{E_{\lambda,n+1}^{(r)}(x)}{E_{\lambda,n+1}^{(r-1)}(x)} \\
&\quad = \sum_{k=1}^{n+2-r} \frac{1}{x - x_{n+1}(r-1,k)} \geqslant \frac{1}{\xi_{n+1,n+1}^{(\lambda)} - x_{n+1}(r-1, n+2-r)} \\
&\quad \geqslant \frac{1}{\xi_{n+1,n+1}^{(\lambda)} - x_{n+1}(0, n+2-r)} = \frac{1}{\xi_{n+1,n+1}^{(\lambda)} - \xi_{n+2-r,n+1}^{(\lambda)}} \\
&\quad \gtrsim \frac{1}{\xi_{n+1,n+1}^{(\lambda)} - \xi_{n,n+1}^{(\lambda)}} \gtrsim n^2.
\end{aligned} \tag{4.28}$$

Here, the last inequality is obtained by Proposition 3.2, that is,

$$\xi_{n+1,n+1}^{(\lambda)} - \xi_{n,n+1}^{(\lambda)} = \cos\theta_{n+1,n+1}^{(\lambda)} - \cos\theta_{n,n+1}^{(\lambda)} \sim n^{-2}. \tag{4.29}$$

Therefore, we have, for $x \in [\xi_{n+1,n+1}^{(\lambda)}, 1]$ and $r \geqslant 2$,

$$E_{\lambda,n+1}^{(r)}(x) \gtrsim E_{\lambda,n+1}^{(r-1)}(x) n^2 \sim n^{2r}. \tag{4.30}$$

Hence, from (2.3), we have (2.4). For $x \in [-1, \xi_{1,n+1}^{(\lambda)}]$, the proof is similar. □

Lemma 4.5. *Let ℓ be a nonnegative integer and $x \in [\xi_{1,n+1}^{(\lambda)}, \xi_{n+1,n+1}^{(\lambda)}]$. Then,*

$$\left| \left(x E_{\lambda,n+1}(x) P_{\lambda,n}'(x) \right)^{(\ell)} \right| \lesssim n^{\ell+1} \varphi^{-\ell-2\lambda}(x), \tag{4.31a}$$

$$\left| \left(I_{\lambda,n}(x) P_{\lambda,n}(x) \right)^{(\ell)} \right| \lesssim n^{\ell+1+\lambda} \varphi^{-\ell-\lambda}(x), \tag{4.31b}$$

$$\left| \left(E_{\lambda,n+1}'(x) P_{\lambda,n}'(x) \right)^{(\ell)} \right| \lesssim n^{\ell+2} \varphi^{-2\lambda-\ell-1}(x). \tag{4.31c}$$

Proof. (a) When $\ell = 0$, it is obvious from (3.5) and (3.12). Now, suppose $\ell \geqslant 1$. From (3.12), (2.2), and (3.5), we have

$$\begin{aligned}\left|\left(xE_{\lambda,n+1}(x)P'_{\lambda,n}(x)\right)^{(\ell)}\right| &\lesssim \sum_{\ell-1\leqslant q+r\leqslant \ell}\left|E^{(q)}_{\lambda,n+1}(x)P^{(r+1)}_{\lambda,n}(x)\right| \\ &\lesssim \sum_{\ell-1\leqslant q+r\leqslant \ell} n^{q+r+1}\varphi^{-(q+r)-2\lambda}(x) \lesssim n^{\ell+1}\varphi^{-\ell-2\lambda}(x).\end{aligned} \tag{4.32}$$

(b) From (4.4) and (3.5), we have

$$\begin{aligned}\left|\left(I_{\lambda,n}(x)P_{\lambda,n}(x)\right)^{(\ell)}\right| &\lesssim \sum_{q+r=\ell}\left|I^{(q)}_{\lambda,n}(x)P^{(r)}_{\lambda,n}(x)\right| \\ &\lesssim n^{\ell+1+\lambda}\varphi^{-\ell-\lambda}(x).\end{aligned} \tag{4.33}$$

(c) Similarly to the proof of (a), we have, from (2.2) and (3.5),

$$\begin{aligned}\left|\left(E'_{\lambda,n+1}(x)P'_{\lambda,n}(x)\right)^{(\ell)}\right| &\lesssim \sum_{q+r=\ell}\left|E^{(q+1)}_{\lambda,n+1}(x)P^{(r+1)}_{\lambda,n}(x)\right| \\ &\lesssim n^{\ell+2}\varphi^{-2\lambda-\ell-1}(x).\end{aligned} \tag{4.34}$$

□

Lemma 4.6. *Let $0 < \lambda < 1$. Then, for all $x \in [-1,1]$ and $r \geqslant 2$,*

$$\begin{aligned}\left(1-x^2\right)F^{(r)}_{\lambda,2n+1}(x) &= (2r-5)xF^{(r-1)}_{\lambda,2n+1}(x) \\ &\quad + \left((r-2)^2-(n+1)^2-n(n+2\lambda)\right)F^{(r-2)}_{\lambda,2n+1}(x) + J^{(r-2)}_{\lambda,n}(x)\end{aligned} \tag{4.35}$$

and, for $x \in [\xi^{(\lambda)}_{1,n+1}, \xi^{(\lambda)}_{n+1,n+1}]$,

$$\left|J^{(r-2)}_{\lambda,n}(x)\right| \lesssim n^r\varphi^{-2\lambda-r+3}(x). \tag{4.36}$$

Here, $J_{\lambda,n}(x)$ is a polynomial of degree of $2n+1$ defined as follows:

$$\begin{aligned}J_{\lambda,n}(x) &= 2\lambda xE_{\lambda,n+1}(x)P'_{\lambda,n}(x) \\ &\quad + 2\left(1-x^2\right)E'_{\lambda,n+1}(x)P'_{\lambda,n}(x) + I_{\lambda,n}(x)P_{\lambda,n}(x).\end{aligned} \tag{4.37}$$

Furthermore, one has

$$\left|J^{(r-2)}_{\lambda,n}(1)\right| \lesssim n^{2\lambda+2r-3}. \tag{4.38}$$

Proof. Similarly to the proof of Lemma 4.1, (4.35) is obtained by $r-2$ times differentiation of the second-order differential relation with respect to $F_{\lambda,2n+1}(x)$, that is, (3.17). So it is sufficient to prove (4.36) and (4.38). From (4.37), we know that

$$\begin{aligned} J_{\lambda,n}^{(r-2)}(x) = {} & \left(2\lambda x E_{\lambda,n+1}(x)P'_{\lambda,n}(x)\right)^{(r-2)} \\ & + \left(2\left(1-x^2\right)E'_{\lambda,n+1}(x)P'_{\lambda,n}(x)\right)^{(r-2)} + \left(I_{\lambda,n}(x)P_{\lambda,n}(x)\right)^{(r-2)}. \end{aligned} \tag{4.39}$$

By Lemma 4.5 (a) and (b), we have, for $x \in [\xi_{1,n+1}^{(\lambda)}, \xi_{n+1,n+1}^{(\lambda)}]$,

$$\begin{aligned} \left|\left(2\lambda x E_{\lambda,n+1}(x)P'_{\lambda,n}(x)\right)^{(r-2)}\right| &\lesssim n^{r-1}\varphi^{-r+2-2\lambda}(x), \\ \left|\left(I_{\lambda,n}(x)P_{\lambda,n}(x)\right)^{(r-2)}\right| &\lesssim n^{r-1+\lambda}\varphi^{-r+2-\lambda}(x). \end{aligned} \tag{4.40}$$

From (4.19) and Lemma 4.5 (c), we have, for $x \in [\xi_{1,n+1}^{(\lambda)}, \xi_{n+1,n+1}^{(\lambda)}]$ and $r \geqslant 4$,

$$\begin{aligned} & \left|\left(2\left(1-x^2\right)E'_{\lambda,n+1}(x)P'_{\lambda,n}(x)\right)^{(r-2)}\right| \\ & \quad \lesssim \left|\left(1-x^2\right)\left(E'_{\lambda,n+1}(x)P'_{\lambda,n}(x)\right)^{(r-2)}\right| + \left|\left(E'_{\lambda,n+1}(x)P'_{\lambda,n}(x)\right)^{(r-3)}\right| \\ & \quad\quad + \left|\left(E'_{\lambda,n+1}(x)P'_{\lambda,n}(x)\right)^{(r-4)}\right| \\ & \quad \lesssim n^r\varphi^{-2\lambda-r+3}(x) + n^{r-1}\varphi^{-2\lambda-r+2}(x) + n^{r-2}\varphi^{-2\lambda-r+3}(x) \\ & \quad \lesssim n^r\varphi^{-2\lambda-r+3}(x). \end{aligned} \tag{4.41}$$

When $r = 2, 3$, we can similarly obtain that

$$\left|\left(2\left(1-x^2\right)E'_{\lambda,n+1}(x)P'_{\lambda,n}(x)\right)^{(r-2)}\right| \lesssim n^r\varphi^{-2\lambda-r+3}(x). \tag{4.42}$$

Therefore, we have (4.36). On the other hand, from (2.4), (4.6), and (3.2), we know that, for a nonnegative integer ℓ,

$$\begin{aligned} E_{\lambda,n+1}^{(\ell)}(1) \sim n^{2\ell}, \qquad & \left|I_{\lambda,n}^{(\ell)}(1)\right| \lesssim n^{2\ell+2}, \\ P_{\lambda,n}^{(\ell)}(1) \sim P_{\lambda+\ell,n-\ell}(1) & \sim n^{2(\lambda+\ell)-1}. \end{aligned} \tag{4.43}$$

Then, similarly to the proof of Lemma 4.5, we obtain that

$$\left|\left(xE_{\lambda,n+1}(x)P'_{\lambda,n}(x)\right)^{(\ell)}\right|_{x=1} \lesssim n^{2\ell+2\lambda+1}, \qquad \left|(I_{\lambda,n}(x)P_{\lambda,n}(x))^{(\ell)}\right|_{x=1} \lesssim n^{2\ell+2\lambda+1},$$
$$\left|\left(E'_{\lambda,n+1}(x)P'_{\lambda,n}(x)\right)^{(\ell)}\right|_{x=1} \lesssim n^{2\ell+2\lambda+3}. \tag{4.44}$$

Therefore, we have (4.38). □

Lemma 4.7. *Let $0 < \lambda < 1$. Then, for $r \geqslant 2$, if r is even, one has, for $x \in [\xi^{(\lambda)}_{1,n+1}, \xi^{(\lambda)}_{n+1,n+1}]$,*

$$\left|F^{(r)}_{\lambda,2n+1}(x)\right| \lesssim n^{r-2}\varphi^{-r}(x)\left|F'_{\lambda,2n+1}(x)\right| + n^r\varphi^{-r}(x)|F_{\lambda,2n+1}(x)| + n^r\varphi^{1-2\lambda-r}(x) \tag{4.45}$$

and, if r is odd, one has

$$\left|F^{(r)}_{\lambda,2n+1}(x)\right| \lesssim n^{r-1}\varphi^{1-r}(x)\left|F'_{\lambda,2n+1}(x)\right| + n^{r-1}\varphi^{-1-r}(x)|F_{\lambda,2n+1}(x)| + n^r\varphi^{1-2\lambda-r}(x). \tag{4.46}$$

Proof. Using (4.35) and (4.36), we obtain the result similarly to the proof of Lemma 4.2. □

Proof of Theorem 2.2. Equation (2.5) comes from Lemma 4.7 and Proposition 3.7. We will show (2.6) and (2.7). From Proposition 3.7, we see

$$\max_{x\in[-1,1]}\left|F'_{\lambda,2n+1}(x)\right| \sim n^{1+2\lambda}. \tag{4.47}$$

Hence, using Markov-Bernstein inequality for $F'_{\lambda,2n+1} \in \mathcal{P}_{2n}$, we have (2.6). Now, we show (2.7). By Proposition 3.7 (a), it is true for $r = 1$. We suppose that, for $r \geqslant 2$,

$$\left|F^{(r-1)}_{\lambda,2n+1}(x)\right| \sim n^{2r+2\lambda-3}, \qquad x \in \left[-1, \xi^{(\lambda)}_{1,n+1}\right] \cup \left[\xi^{(\lambda)}_{n+1,n+1}, 1\right]. \tag{4.48}$$

As the proof of Theorem 2.1, we have, for $r \geqslant 2$ and for $x \in [\xi^{(\lambda)}_{n+1,n+1}, 1]$,

$$\frac{F^{(r)}_{\lambda,2n+1}(x)}{F^{(r-1)}_{\lambda,2n+1}(x)} \gtrsim n^2. \tag{4.49}$$

Therefore, we see that by induction with Proposition 3.7, (2.7) holds for every $r = 1, 2, 3, \ldots$. □

Corollary 4.8. *Let $0 < \lambda < 1$ and $r \geqslant 2$. Then, for $x \in [-1, \xi^{(\lambda)}_{1,n+1}] \cup [\xi^{(\lambda)}_{n+1,n+1}, 1]$,*

$$\left|J^{(r-2)}_{\lambda,n}(x)\right| \lesssim n^{2r+2\lambda-3}. \tag{4.50}$$

Proof. Corollary 4.8 comes from (4.35), (2.7), and (3.8). □

Proof of Theorem 2.3. Equation (2.8) comes from (4.15) and (3.24). □

Lemma 4.9. *For* $1 \leqslant \mu \leqslant n+1$ *and* $1 \leqslant \nu \leqslant 2n+1$,

$$\left|P_{\lambda,n}\left(\xi_{\mu,n+1}^{(\lambda)}\right)\right| \sim n^{\lambda-1}\varphi^{-\lambda}\left(\xi_{\mu,n+1}^{(\lambda)}\right), \tag{4.51}$$

$$\left|E_{\lambda,n+1}\left(x_{\nu,n}^{(\lambda)}\right)\right| \sim n^{1-\lambda}\varphi^{1-\lambda}\left(x_{\nu,n}^{(\lambda)}\right). \tag{4.52}$$

Proof. Since we know from (3.24) and (3.25) that

$$\begin{aligned} n\varphi^{-2\lambda}\left(\xi_{\mu,n+1}^{(\lambda)}\right) \sim \left|F'_{\lambda,2n+1}\left(\xi_{\mu,n+1}^{(\lambda)}\right)\right| &= \left|E'_{\lambda,n+1}\left(\xi_{\mu,n+1}^{(\lambda)}\right)P_{\lambda,n}\left(\xi_{\mu,n+1}^{(\lambda)}\right)\right| \\ &\sim n^{2-\lambda}\varphi^{-\lambda}\left(\xi_{\mu,n+1}^{(\lambda)}\right)\left|P_{\lambda,n}\left(\xi_{\mu,n+1}^{(\lambda)}\right)\right|, \end{aligned} \tag{4.53}$$

(4.51) is obviously proved. Similarly, since we have, from (3.4) and (3.25),

$$\begin{aligned} n\varphi^{-2\lambda}\left(x_{\nu,n}^{(\lambda)}\right) \sim \left|F'_{\lambda,2n+1}\left(x_{\nu,n}^{(\lambda)}\right)\right| &= \left|E_{\lambda,n+1}\left(x_{\nu,n}^{(\lambda)}\right)P'_{\lambda,n}\left(x_{\nu,n}^{(\lambda)}\right)\right| \\ &\sim n^{\lambda}\varphi^{-\lambda-1}\left(x_{\nu,n}^{(\lambda)}\right)\left|E_{\lambda,n+1}\left(x_{\nu,n}^{(\lambda)}\right)\right|, \end{aligned} \tag{4.54}$$

(4.52) is obtained. □

Lemma 4.10. *Let* $0 < \lambda < 1$ *and* $r \geqslant 1$. *Let* r *be an odd integer.*

(a) *For* $1 \leqslant \mu \leqslant n+1$,

$$\left|P_{\lambda,n}^{(r)}\left(\xi_{\mu,n+1}^{(\lambda)}\right)\right| \lesssim n^{2\lambda+r-2}\varphi^{-r-1}\left(\xi_{\mu,n+1}^{(\lambda)}\right). \tag{4.55}$$

(b) *For* $1 \leqslant \nu \leqslant n$,

$$\left|E_{\lambda,n+1}^{(r)}\left(x_{\nu,n}^{(\lambda)}\right)\right| \lesssim n^{r}\varphi^{-r}\left(x_{\nu,n}^{(\lambda)}\right). \tag{4.56}$$

Proof. (a) We know, from (3.3),

$$\left|P'_{\lambda,n}\left(\xi_{\mu,n+1}^{(\lambda)}\right)\right| \lesssim n^{2\lambda-1}\varphi^{-2}\left(\xi_{\mu,n+1}^{(\lambda)}\right). \tag{4.57}$$

So (4.55) holds for $k=1$. Assume that, for $k = 1, 2, \ldots, \ell$,

$$\left|P_{\lambda,n}^{(2k-1)}\left(\xi_{\mu,n+1}^{(\lambda)}\right)\right| \lesssim n^{2\lambda+2k-3}\varphi^{-2k}\left(\xi_{\mu,n+1}^{(\lambda)}\right). \tag{4.58}$$

Then, we have from (3.5), (3.7), and (4.58) that

$$\begin{aligned}
&\left|P_{\lambda,n}^{(2\ell+1)}\left(\xi_{\mu,n+1}^{(\lambda)}\right)\right| \\
&\quad\lesssim \varphi^{-2}\left(\xi_{\mu,n+1}^{(\lambda)}\right)\left|P_{\lambda,n}^{(2\ell)}\left(\xi_{\mu,n+1}^{(\lambda)}\right)\right| + n^2\varphi^{-2}\left(\xi_{\mu,n+1}^{(\lambda)}\right)\left|P_{\lambda,n}^{(2\ell-1)}\left(\xi_{\mu,n+1}^{(\lambda)}\right)\right| \\
&\quad\lesssim n^{2\lambda+2\ell-1}\varphi^{-2\ell-2}\left(\xi_{\mu,n+1}^{(\lambda)}\right).
\end{aligned}\tag{4.59}$$

Therefore, we have the result using the mathematical induction.

(b) For an odd integer $r \geqslant 1$, we have from (3.6), (4.16), and (4.52)

$$\begin{aligned}
&\left|E_{\lambda,n+1}^{(r)}\left(x_{\nu,n}^{(\lambda)}\right)\right| \\
&\quad\lesssim n^{r-1}\varphi^{1-r}(x)\left(\left|E_{\lambda,n+1}'\left(x_{\nu,n}^{(\lambda)}\right)\right| + \left|E_{\lambda,n+1}\left(x_{\nu,n}^{(\lambda)}\right)\right|\right) + n^r\varphi^{-r}\left(x_{\nu,n}^{(\lambda)}\right) \\
&\quad\lesssim n^r\varphi^{-r}\left(x_{\nu,n}^{(\lambda)}\right).
\end{aligned}\tag{4.60}$$

□

Lemma 4.11. *Let $r \geqslant 2$. If r is even, then*

$$\left|P_{\lambda,n}^{(r)}\left(x_{\nu,n}^{(\lambda)}\right)\right| \lesssim n^{\lambda+r-2}\varphi^{-\lambda-r-1}\left(x_{\nu,n}^{(\lambda)}\right).\tag{4.61}$$

Proof. It is easily proved from (3.4) and (3.7). □

Lemma 4.12. *Let k be a positive integer. Then, one has, for $1 \leqslant \nu \leqslant 2n+1$,*

$$\left|J_{\lambda,n}^{(2k)}\left(y_{\nu,2n+1}^{(\lambda)}\right)\right| \lesssim n^{2k+1+\lambda}\varphi^{-2k-\lambda}\left(y_{\nu,2n+1}^{(\lambda)}\right).\tag{4.62}$$

Proof. From (4.37), we know that

$$\begin{aligned}
J_{\lambda,n}(x) &= 2\lambda x E_{\lambda,n+1}(x)P_{\lambda,n}'(x) \\
&\quad + 2\left(1-x^2\right)E_{\lambda,n+1}'(x)P_{\lambda,n}'(x) + I_{\lambda,n}(x)P_{\lambda,n}(x).
\end{aligned}\tag{4.63}$$

From Lemma 4.5 (a) and (b), we know that for $x \in [y_{1,2n+1}^{(\lambda)}, y_{2n+1,2n+1}^{(\lambda)}]$ $(=[\xi_{1,n+1}^{(\lambda)}, \xi_{n+1,n+1}^{(\lambda)}])$

$$\begin{aligned}
\left|\left(xE_{\lambda,n+1}(x)P_{\lambda,n}'(x)\right)^{(2k)}\right| &\lesssim n^{2k+1}\varphi^{-2k-2\lambda}(x) \\
\left|\left(I_{\lambda,n}(x)P_{\lambda,n}(x)\right)^{(2k)}\right| &\lesssim n^{2k+1+\lambda}\varphi^{-2k-\lambda}(x).
\end{aligned}\tag{4.64}$$

On the other hand, we estimate $|((1-x^2)E'_{\lambda,n+1}(x)P'_{\lambda,n}(x))^{(2k)}|$ splitting into three terms as follows:

$$\begin{aligned}
&\left|\left(\left(1-x^2\right)E'_{\lambda,n+1}(x)P'_{\lambda,n}(x)\right)^{(2k)}\right| \\
&\quad\lesssim \sum_{q+r=2k}\left|\left(1-x^2\right)E^{(q+1)}_{\lambda,n+1}(x)P^{(r+1)}_{\lambda,n}(x)\right| \\
&\qquad + O(1)\left[\left|\left(E'_{\lambda,n+1}(x)P'_{\lambda,n}(x)\right)^{(2k-1)}\right| + \left|\left(E'_{\lambda,n+1}(x)P'_{\lambda,n}(x)\right)^{(2k-2)}\right|\right].
\end{aligned} \tag{4.65}$$

Here, from Lemma 4.5 (c), we have for $x \in [y^{(\lambda)}_{1,2n+1}, y^{(\lambda)}_{2n+1,2n+1}]$

$$\begin{aligned}
\left|\left(E'_{\lambda,n+1}(x)P'_{\lambda,n}(x)\right)^{(2k-1)}\right| &\lesssim n^{2k+1}\varphi^{-2\lambda-2k}(x), \\
\left|\left(E'_{\lambda,n+1}(x)P'_{\lambda,n}(x)\right)^{(2k-2)}\right| &\lesssim n^{2k}\varphi^{-2\lambda-2k+1}(x) \lesssim n^{2k+1}\varphi^{-2\lambda-2k}(x).
\end{aligned} \tag{4.66}$$

For the first term, we also split into two terms as follows:

$$\begin{aligned}
&\sum_{q+r=2k}\left|\left(1-x^2\right)E^{(q+1)}_{\lambda,n+1}(x)P^{(r+1)}_{\lambda,n}(x)\right| \\
&\quad= \sum_{q+r=2k,q:\,\text{even},r:\,\text{even}}\left|\left(1-x^2\right)E^{(q+1)}_{\lambda,n+1}(x)P^{(r+1)}_{\lambda,n}(x)\right| \\
&\qquad + \sum_{q+r=2k,q:\,\text{odd},r:\,\text{odd}}\left|\left(1-x^2\right)E^{(q+1)}_{\lambda,n+1}(x)P^{(r+1)}_{\lambda,n}(x)\right| \\
&\quad:= A_1(x) + A_2(x).
\end{aligned} \tag{4.67}$$

From (2.2) and (4.55), we know that for even q and r

$$\begin{aligned}
\left|E^{(q+1)}_{\lambda,n+1}\left(\xi^{(\lambda)}_{\mu,n+1}\right)\right| &\lesssim n^{q+2-\lambda}\varphi^{-q-\lambda}\left(\xi^{(\lambda)}_{\mu,n+1}\right), \\
\left|P^{(r+1)}_{\lambda,n}\left(\xi^{(\lambda)}_{\mu,n+1}\right)\right| &\lesssim n^{2\lambda+r-1}\varphi^{-r-2}\left(\xi^{(\lambda)}_{\mu,n+1}\right).
\end{aligned} \tag{4.68}$$

Also, we know from (4.56) and (3.5) that for even q and r

$$\begin{aligned}
\left|E^{(q+1)}_{\lambda,n+1}\left(x^{(\lambda)}_{\nu,n}\right)\right| &\lesssim n^{q+1}\varphi^{-q-1}\left(x^{(\lambda)}_{\nu,n}\right), \\
\left|P^{(r+1)}_{\lambda,n}\left(x^{(\lambda)}_{\nu,n}\right)\right| &\lesssim n^{\lambda+r}\varphi^{-\lambda-r-1}\left(x^{(\lambda)}_{\nu,n}\right).
\end{aligned} \tag{4.69}$$

Then, we have

$$\begin{aligned}
&\left|A_1\left(\xi_{\mu,n+1}^{(\lambda)}\right)\right| \\
&\quad \lesssim \varphi^2\left(\xi_{\mu,n+1}^{(\lambda)}\right) \sum_{q+r=2k,q:\,\text{even},r:\,\text{even}} \left|E_{\lambda,n+1}^{(q+1)}\left(\xi_{\mu,n+1}^{(\lambda)}\right) P_{\lambda,n}^{(r+1)}\left(\xi_{\mu,n+1}^{(\lambda)}\right)\right| \\
&\quad \lesssim n^{2k+1+\lambda} \varphi^{-2k-\lambda}\left(\xi_{\mu,n+1}^{(\lambda)}\right), \\
&\left|A_1\left(x_{\nu,n}^{(\lambda)}\right)\right| \lesssim \varphi^2\left(x_{\nu,n}^{(\lambda)}\right) \sum_{q+r=2k,q:\,\text{even},r:\,\text{even}} \left|E_{\lambda,n+1}^{(q+1)}\left(x_{\nu,n}^{(\lambda)}\right) P_{\lambda,n}^{(r+1)}\left(x_{\nu,n}^{(\lambda)}\right)\right| \\
&\quad \lesssim n^{2k+1+\lambda} \varphi^{-\lambda-2k}\left(x_{\nu,n}^{(\lambda)}\right).
\end{aligned} \tag{4.70}$$

Similarly, for odd q and r, we have, by (2.8) and (3.5),

$$\left|E_{\lambda,n+1}^{(q+1)}\left(\xi_{\mu,n+1}^{(\lambda)}\right) P_{\lambda,n}^{(r+1)}\left(\xi_{\mu,n+1}^{(\lambda)}\right)\right| \lesssim n^{2k+1+\lambda} \varphi^{-2k-\lambda-2}\left(\xi_{\mu,n+1}^{(\lambda)}\right) \tag{4.71}$$

and, by (2.2) and (4.61),

$$\left|E_{\lambda,n+1}^{(q+1)}\left(x_{\nu,n}^{(\lambda)}\right) P_{\lambda,n}^{(r+1)}\left(x_{\nu,n}^{(\lambda)}\right)\right| \lesssim n^{2k+1+\lambda} \varphi^{-\lambda-2k-2}\left(x_{\nu,n}^{(\lambda)}\right). \tag{4.72}$$

Thus, we have

$$\begin{aligned}
&\left|A_2\left(\xi_{\mu,n+1}^{(\lambda)}\right)\right| \\
&\quad \lesssim \varphi^2\left(\xi_{\mu,n+1}^{(\lambda)}\right) \sum_{q+r=2k,q:\,\text{odd},r:\,\text{odd}} \left|E_{\lambda,n+1}^{(q+1)}\left(\xi_{\mu,n+1}^{(\lambda)}\right) P_{\lambda,n}^{(r+1)}\left(\xi_{\mu,n+1}^{(\lambda)}\right)\right| \\
&\quad \lesssim n^{2k+1+\lambda} \varphi^{-2k-\lambda}\left(\xi_{\mu,n+1}^{(\lambda)}\right), \\
&\left|A_2\left(x_{\nu,n}^{(\lambda)}\right)\right| \lesssim \varphi^2\left(x_{\nu,n}^{(\lambda)}\right) \sum_{q+r=2k,q:\,\text{odd},r:\,\text{odd}} \left|E_{\lambda,n+1}^{(q+1)}\left(x_{\nu,n}^{(\lambda)}\right) P_{\lambda,n}^{(r+1)}\left(x_{\nu,n}^{(\lambda)}\right)\right| \\
&\quad \lesssim n^{2k+1+\lambda} \varphi^{-\lambda-2k}\left(x_{\nu,n}^{(\lambda)}\right).
\end{aligned} \tag{4.73}$$

Therefore, we have the result. □

Proof of Theorem 2.4. When $r = 2$, (2.9) holds from (3.27). Let even $r > 2$, and suppose that (2.9) holds for $r - 2$. Since we know by (4.35), (2.5), (4.62), and (4.19)

$$\begin{aligned}
&\varphi^2\left(y_{\nu,2n+1}^{(\lambda)}\right)\left|F_{\lambda,2n+1}^{(r)}\left(y_{\nu,2n+1}^{(\lambda)}\right)\right| \\
&\quad\lesssim \left|F_{\lambda,2n+1}^{(r-1)}\left(y_{\nu,2n+1}^{(\lambda)}\right)\right| + n^2\left|F_{\lambda,2n+1}^{(r-2)}\left(y_{\nu,2n+1}^{(\lambda)}\right)\right| + \left|J_{\lambda,n}^{(r-2)}\left(y_{\nu,2n+1}^{(\lambda)}\right)\right| \\
&\quad\lesssim n^2\left|F_{\lambda,2n+1}^{(r-2)}\left(y_{\nu,2n+1}^{(\lambda)}\right)\right| + n^{r-1+\lambda}\varphi^{-r+2-\lambda}\left(y_{\nu,2n+1}^{(\lambda)}\right),
\end{aligned} \tag{4.74}$$

we obtain, using mathematical induction,

$$\varphi^2\left(y_{\nu,2n+1}^{(\lambda)}\right)\left|F_{\lambda,2n+1}^{(r)}\left(y_{\nu,2n+1}^{(\lambda)}\right)\right| \lesssim n^{r-1+\lambda}\varphi^{-r+2-\lambda}\left(y_{\nu,2n+1}^{(\lambda)}\right). \tag{4.75}$$

Therefore, (2.9) is proved. □

Proof of Theorem 2.5. (a) From (4.3), we know that

$$\begin{aligned}
\varphi^2\left(\xi_{\mu,n+1}^{(\lambda)}\right)E_{\lambda,n+1}^{(3)}\left(\xi_{\mu,n+1}^{(\lambda)}\right) &= -(n+1)^2E'_{\lambda,n+1}\left(\xi_{\mu,n+1}^{(\lambda)}\right) + O\left(E'_{\lambda,n+1}\left(\xi_{\mu,n+1}^{(\lambda)}\right)\right) \\
&\quad + O\left(E''_{\lambda,n+1}\left(\xi_{\mu,n+1}^{(\lambda)}\right)\right) + O\left(I'_{\lambda,n}\left(\xi_{\mu,n+1}^{(\lambda)}\right)\right).
\end{aligned} \tag{4.76}$$

Therefore, we have, by (2.2), (3.24), and (3.26),

$$E_{\lambda,n+1}^{(3)}\left(\xi_{\mu,n+1}^{(\lambda)}\right) = -(n+1)^2\varphi^{-2}\left(\xi_{\mu,n+1}^{(\lambda)}\right)E'_{\lambda,n+1}\left(\xi_{\mu,n+1}^{(\lambda)}\right) + O\left(n^3\right). \tag{4.77}$$

Suppose that, for an integer $\ell \geqslant 2$,

$$E_{\lambda,n+1}^{(2\ell-1)}\left(\xi_{\mu,n+1}^{(\lambda)}\right) = (-1)^{\ell-1}(n+1)^{2(\ell-1)}\varphi^{-2(\ell-1)}\left(\xi_{\mu,n+1}^{(\lambda)}\right)E'_{\lambda,n+1}\left(\xi_{\mu,n+1}^{(\lambda)}\right) + O\left(n^{2\ell-1}\right). \tag{4.78}$$

Then, from (4.3), we obtain

$$\begin{aligned}
\varphi^2\left(\xi_{\mu,n+1}^{(\lambda)}\right)E_{\lambda,n+1}^{(2\ell+1)}\left(\xi_{\mu,n+1}^{(\lambda)}\right) &= -(n+1)^2E_{\lambda,n+1}^{(2\ell-1)}\left(\xi_{\mu,n+1}^{(\lambda)}\right) + O\left(E_{\lambda,n+1}^{(2\ell)}\left(\xi_{\mu,n+1}^{(\lambda)}\right)\right) \\
&\quad + O\left(E_{\lambda,n+1}^{(2\ell-1)}\left(\xi_{\mu,n+1}^{(\lambda)}\right)\right) + O\left(I_{\lambda,n}^{(2\ell-1)}\left(\xi_{\mu,n+1}^{(\lambda)}\right)\right).
\end{aligned} \tag{4.79}$$

Therefore, by (2.2), (2.8), and (4.4), we have

$$\begin{aligned}
E_{\lambda,n+1}^{(2\ell+1)}\left(\xi_{\mu,n+1}^{(\lambda)}\right) &= -(n+1)^2\varphi^{-2}\left(\xi_{\mu,n+1}^{(\lambda)}\right)E_{\lambda,n+1}^{(2\ell-1)}\left(\xi_{\mu,n+1}^{(\lambda)}\right) + O\left(n^{2\ell+1}\right) \\
&= (-1)^{\ell}(n+1)^{2\ell}\varphi^{-2\ell}\left(\left(\xi_{\mu,n+1}^{(\lambda)}\right)\right)E'_{\lambda,n+1}\left(\xi_{\mu,n+1}^{(\lambda)}\right) + O\left(n^{2\ell+1}\right).
\end{aligned} \tag{4.80}$$

(b) Similarly to the proof of (a), by (4.51), (3.3), and (3.7), we can obtain

$$P_{\lambda,n}^{(2\ell)}\left(\xi_{\mu,n+1}^{(\lambda)}\right) = (-1)^{\ell} n^{\ell}(n+2\lambda)^{\ell}\varphi^{-2\ell}\left(\xi_{\mu,n+1}^{(\lambda)}\right)P_{\lambda,n}\left(\xi_{\mu,n+1}^{(\lambda)}\right) + O\left(n^{2\lambda+2\ell-3}\right). \tag{4.81}$$

In addition, we see that, from (4.51),

$$\begin{aligned} P_{\lambda,n}^{(2\ell)}\left(\xi_{\mu,n+1}^{(\lambda)}\right) &= (-1)^{\ell}(n+1)^{2\ell}\varphi^{-2\ell}\left(\xi_{\mu,n+1}^{(\lambda)}\right)P_{\lambda,n}\left(\xi_{\mu,n+1}^{(\lambda)}\right) \\ &\quad + O\left(n^{2\ell-1}P_{\lambda,n}\left(\xi_{\mu,n+1}^{(\lambda)}\right)\right) + O\left(n^{2\lambda+2\ell-3}\right) \\ &= (-1)^{\ell}(n+1)^{2\ell}\varphi^{-2\ell}\left(\xi_{\mu,n+1}^{(\lambda)}\right)P_{\lambda,n}\left(\xi_{\mu,n+1}^{(\lambda)}\right) + O\left(n^{2\ell+\lambda-2}\right). \end{aligned} \tag{4.82}$$

□

Proof of Theorem 2.6. (a) From (3.9), we know that

$$\varphi^2\left(x_{\nu,n}^{(\lambda)}\right)E''_{\lambda,n+1}\left(x_{\nu,n}^{(\lambda)}\right) = -(n+1)^2E_{\lambda,n+1}\left(x_{\nu,n}^{(\lambda)}\right) + O\left(E'_{\lambda,n+1}\left(x_{\nu,n}^{(\lambda)}\right)\right) + O\left(I_{\lambda,n}\left(x_{\nu,n}^{(\lambda)}\right)\right). \tag{4.83}$$

Therefore, by (4.4) and (4.56), we have

$$E''_{\lambda,n+1}\left(x_{\nu,n}^{(\lambda)}\right) = -(n+1)^2\varphi^{-2}\left(x_{\nu,n}^{(\lambda)}\right)E_{\lambda,n+1}\left(x_{\nu,n}^{(\lambda)}\right) + O\left(n^2\right). \tag{4.84}$$

Then, we obtain from (4.3), (2.2), and (4.56) that

$$\varphi^2\left(x_{\nu,n}^{(\lambda)}\right)E_{\lambda,n+1}^{(2\ell)}\left(x_{\nu,n}^{(\lambda)}\right) = -(n+1)^2E_{\lambda,n+1}^{(2\ell-2)}\left(x_{\nu,n}^{(\lambda)}\right) + O\left(n^{2\ell}\right). \tag{4.85}$$

Therefore, we have the result inductively.

(b) From (3.7), we know that

$$\varphi^2\left(x_{\nu,n}^{(\lambda)}\right)P_{\lambda,n}^{(3)}\left(x_{\nu,n}^{(\lambda)}\right) = -n(n+2\lambda)P'_{\lambda,n}\left(x_{\nu,n}^{(\lambda)}\right) + O\left(P''_{\lambda,n}\left(x_{\nu,n}^{(\lambda)}\right)\right) + O\left(P'_{\lambda,n}\left(x_{\nu,n}^{(\lambda)}\right)\right). \tag{4.86}$$

Therefore, by (3.4) and (4.61), we have

$$P_{\lambda,n}^{(3)}\left(x_{\nu,n}^{(\lambda)}\right) = -n(n+2\lambda)\varphi^{-2}\left(x_{\nu,n}^{(\lambda)}\right)P'_{\lambda,n}\left(x_{\nu,n}^{(\lambda)}\right) + O\left(n^{\lambda}\right). \tag{4.87}$$

Suppose that, for an integer $\ell \geqslant 2$,

$$P_{\lambda,n}^{(2\ell-1)}\left(x_{\nu,n}^{(\lambda)}\right) = (-1)^{\ell-1}n^{\ell-1}(n+2\lambda)^{\ell-1}\varphi^{-2(\ell-1)}\left(x_{\nu,n}^{(\lambda)}\right)P'_{\lambda,n}\left(x_{\nu,n}^{(\lambda)}\right) + O\left(n^{\lambda+2\ell-4}\right). \tag{4.88}$$

Then from (3.5), (3.7), and (4.61)

$$P_{\lambda,n}^{(2\ell+1)}\left(x_{\nu,n}^{(\lambda)}\right) = (-1)^{\ell}n^{\ell}(n+2\lambda)^{\ell}\varphi^{-2\ell}\left(x_{\nu,n}^{(\lambda)}\right)P'_{\lambda,n}\left(x_{\nu,n}^{(\lambda)}\right) + O\left(n^{\lambda+2\ell-2}\right). \tag{4.89}$$

Here, we see that for $\ell \geqslant 2$

$$\{n(n+2\lambda)\}^{\ell} = (n+1)^{2\ell} + O\left(n^{2\ell-1}\right). \tag{4.90}$$

Hence, we obtain, from (3.4),

$$\begin{aligned} P_{\lambda,n}^{(2\ell+1)}\left(x_{\nu,n}^{(\lambda)}\right) &= (-1)^{\ell}(n+1)^{2\ell}\varphi^{-2\ell}\left(x_{\nu,n}^{(\lambda)}\right)P'_{\lambda,n}\left(x_{\nu,n}^{(\lambda)}\right) \\ &\quad + O\left(n^{2\ell-1}P'_{\lambda,n}\left(x_{\nu,n}^{(\lambda)}\right)\right) + O\left(n^{\lambda+2\ell-2}\right) \\ &= (-1)^{\ell}(n+1)^{2\ell}\varphi^{-2\ell}\left(x_{\nu,n}^{(\lambda)}\right)P'_{\lambda,n}\left(x_{\nu,n}^{(\lambda)}\right) + O\left(n^{\lambda+2\ell-1}\right). \end{aligned} \tag{4.91}$$

□

Lemma 4.13. *Let $0 < \varepsilon < 1$ and $|y_{\nu,2n+1}^{(\lambda)}| \leqslant 1-\varepsilon$. Then, for a nonnegative integer $\ell \geqslant 0$,*

$$J_{\lambda,n}^{(2\ell+1)}\left(y_{\nu,2n+1}^{(\lambda)}\right) = 2^{2\ell+1}(-1)^{\ell+1}(n+1)^{2\ell+2}\varphi^{-2\ell}\left(y_{\nu,2n+1}^{(\lambda)}\right)F'\left(y_{\nu,2n+1}^{(\lambda)}\right) + O\left(n^{\lambda+2\ell+2}\right). \tag{4.92}$$

Proof. Let $|y_{\nu,2n+1}^{(\lambda)}| \leqslant 1-\varepsilon$. From (4.37) and Lemma 4.5, we see that

$$\begin{aligned} &J_{\lambda,n}^{(2\ell+1)}\left(y_{\nu,2n+1}^{(\lambda)}\right) \\ &\quad = 2\left(1-x^2\right)\left(E'_{\lambda,n+1}(x)P'_{\lambda,n}(x)\right)^{(2\ell+1)} - 4x(2\ell+1)\left(E'_{\lambda,n+1}(x)P'_{\lambda,n}(x)\right)^{(2\ell)} \\ &\qquad - 2\ell(2\ell+1)\left(E'_{\lambda,n+1}(x)P'_{\lambda,n}(x)\right)^{(2\ell-1)} \\ &\qquad +2\lambda\left(xE_{\lambda,n+1}(x)P'_{\lambda,n}(x)\right)^{(2\ell+1)} + \left(I_{\lambda,n}(x)P_{\lambda,n}(x)\right)^{(2\ell+1)}\Big|_{x=y_{\nu,2n+1}^{(\lambda)}} \\ &\quad = 2\left(1-x^2\right)\left(E'_{\lambda,n+1}(x)P'_{\lambda,n}(x)\right)^{(2\ell+1)}\Big|_{x=y_{\nu,2n+1}^{(\lambda)}} + O\left(n^{2\ell+2+\lambda}\right). \end{aligned} \tag{4.93}$$

Here, we let $2\ell(2\ell+1)(E'_{\lambda,n+1}(x)P'_{\lambda,n}(x))^{(2\ell-1)} = 0$ when $\ell = 0$. To estimate the first term, we split it into two terms as follows:

$$\begin{aligned} &\left(E'_{\lambda,n+1}(x)P'_{\lambda,n}(x)\right)^{(2\ell+1)} \\ &\quad = \sum_{0\leqslant i\leqslant 2\ell+1,\, i:\text{ even}} + \sum_{0\leqslant i\leqslant 2\ell+1,\, i:\text{ odd}} \binom{2\ell+1}{i} E_{\lambda,n+1}^{(i+1)}(x)P_{\lambda,n}^{(2\ell+2-i)}(x). \end{aligned} \tag{4.94}$$

Then, using $|x_{\nu,n}^{(\lambda)}| \leqslant 1-\varepsilon$, from (2.13) and (2.15), we obtain

$$\begin{aligned}
&\sum_{0\leqslant i\leqslant 2\ell+1, i:\,\text{odd}} \binom{2\ell+1}{i} E_{\lambda,n+1}^{(i+1)}\left(x_{\nu,n}^{(\lambda)}\right) P_{\lambda,n}^{(2\ell+2-i)}\left(x_{\nu,n}^{(\lambda)}\right) \\
&\quad = (-1)^{\ell+1}(n+1)^{2\ell+2}\varphi^{-2\ell-2}\left(x_{\nu,n}^{(\lambda)}\right) F'_{\lambda,2n+1}\left(x_{\nu,n}^{(\lambda)}\right) \sum_{0\leqslant i\leqslant 2\ell+1, i:\,\text{odd}} \binom{2\ell+1}{i} + O\left(n^{\lambda+2\ell+2}\right) \\
&\quad = 2^{2\ell}(-1)^{\ell+1}(n+1)^{2\ell+2}\varphi^{-2\ell-2}\left(x_{\nu,n}^{(\lambda)}\right) F'_{\lambda,2n+1}\left(x_{\nu,n}^{(\lambda)}\right) + O\left(n^{\lambda+2\ell+2}\right)
\end{aligned} \tag{4.95}$$

and, from (4.56) and from (4.61),

$$\sum_{0\leqslant i\leqslant 2\ell+1,\ i:\,\text{even}} \binom{2\ell+1}{i} E_{\lambda,n+1}^{(i+1)}\left(x_{\nu,n}^{(\lambda)}\right) P_{\lambda,n}^{(2\ell+2-i)}\left(x_{\nu,n}^{(\lambda)}\right) = O\left(n^{\lambda+2\ell+1}\right). \tag{4.96}$$

Thus, we have, for $|x_{\nu,n}^{(\lambda)}| \leqslant 1-\varepsilon$,

$$\begin{aligned}
&\left(E'_{\lambda,n+1}(x)P'_{\lambda,n}(x)\right)^{(2\ell+1)}\Big|_{x=x_{\nu,n}^{(\lambda)}} \\
&\quad = 2^{2\ell}(-1)^{\ell+1}(n+1)^{2\ell+2}\varphi^{-2\ell-2}\left(x_{\nu,n}^{(\lambda)}\right) F'_{\lambda,2n+1}\left(x_{\nu,n}^{(\lambda)}\right) + O\left(n^{\lambda+2\ell+2}\right).
\end{aligned} \tag{4.97}$$

Similarly, noting $|\xi_{\mu,n}^{(\lambda)}| \leqslant 1-\varepsilon$, from (2.10) and (2.12)

$$\begin{aligned}
&\sum_{0\leqslant i\leqslant 2\ell+1, i:\,\text{even}} \binom{2\ell+1}{i} E_{\lambda,n+1}^{(i+1)}\left(\xi_{\mu,n+1}^{(\lambda)}\right) P_{\lambda,n}^{(2\ell+2-i)}\left(\xi_{\mu,n+1}^{(\lambda)}\right) \\
&\quad = 2^{2\ell}(-1)^{\ell+1}(n+1)^{2\ell+2}\varphi^{-2\ell-2}\left(\xi_{\mu,n+1}^{(\lambda)}\right) F'_{\lambda,2n+1}\left(\xi_{\mu,n+1}^{(\lambda)}\right) + O\left(n^{\lambda+2\ell+2}\right)
\end{aligned} \tag{4.98}$$

and, from (2.8) and (4.55),

$$\sum_{0\leqslant i\leqslant 2\ell+1, i:\,\text{odd}} \binom{2\ell+1}{i} E_{\lambda,n+1}^{(i+1)}\left(\xi_{\mu,n+1}^{(\lambda)}\right) P_{\lambda,n}^{(2\ell+2-i)}\left(\xi_{\mu,n+1}^{(\lambda)}\right) = O\left(n^{2\lambda+2\ell+1}\right). \tag{4.99}$$

Then, we have, for $|\xi_{\mu,n+1}^{(\lambda)}| \leqslant 1-\varepsilon$,

$$\begin{aligned}
&\left(E'_{\lambda,n+1}(x)P'_{\lambda,n}(x)\right)^{(2\ell+1)}\Big|_{x=\xi_{\mu,n+1}^{(\lambda)}} \\
&\quad = 2^{2\ell}(-1)^{\ell+1}(n+1)^{2\ell+2}\varphi^{-2\ell-2}\left(\xi_{\mu,n+1}^{(\lambda)}\right) F'_{\lambda,2n+1}\left(\xi_{\mu,n+1}^{(\lambda)}\right) + O\left(n^{\lambda+2\ell+2}\right).
\end{aligned} \tag{4.100}$$

Therefore, we have, for $|y^{(\lambda)}_{\nu,2n+1}| \leqslant 1-\varepsilon$,

$$\begin{aligned}&\left(E'_{\lambda,n+1}(x)P'_{\lambda,n}(x)\right)^{(2\ell+1)}\Big|_{x=y^{(\lambda)}_{\nu,2n+1}}\\&\quad=2^{2\ell}(-1)^{\ell+1}(n+1)^{2\ell+2}\varphi^{-2\ell-2}\left(y^{(\lambda)}_{\nu,2n+1}\right)F'_{\lambda,2n+1}\left(y^{(\lambda)}_{\nu,2n+1}\right)+O\left(n^{\lambda+2\ell+2}\right).\end{aligned}\tag{4.101}$$

Thus, we have, for $|y^{(\lambda)}_{\nu,2n+1}| \leqslant 1-\varepsilon$,

$$J^{(2\ell+1)}_{\lambda,n}\left(y^{(\lambda)}_{\nu,2n+1}\right)=2^{2\ell+1}(-1)^{\ell+1}(n+1)^{2\ell+2}\varphi^{-2\ell}\left(y^{(\lambda)}_{\nu,2n+1}\right)F'_{\lambda,2n+1}\left(y^{(\lambda)}_{\nu,2n+1}\right)+O\left(n^{\lambda+2\ell+2}\right).\tag{4.102}$$

□

Proof of Theorem 2.7. From (4.35), (3.25), (3.27), and Lemma 4.13, we have

$$\begin{aligned}&\varphi^2\left(y^{(\lambda)}_{\nu,2n+1}\right)F^{(3)}_{\lambda,2n+1}\left(y^{(\lambda)}_{\nu,2n+1}\right)\\&\quad=-2(n+1)^2F'_{\lambda,2n+1}\left(y^{(\lambda)}_{\nu,2n+1}\right)+J'_{\lambda,n}\left(y^{(\lambda)}_{\nu,2n+1}\right)\\&\qquad+O\left(nF'_{\lambda,2n+1}\left(y^{(\lambda)}_{\nu,2n+1}\right)\right)+O\left(F''_{\lambda,2n+1}\left(y^{(\lambda)}_{\nu,2n+1}\right)\right)\\&\quad=-3(n+1)^2F'_{\lambda,2n+1}\left(y^{(\lambda)}_{\nu,2n+1}\right)+O\left(n^{\lambda+2}\right).\end{aligned}\tag{4.103}$$

Suppose that

$$F^{(2\ell-1)}_{\lambda,2n+1}\left(y^{(\lambda)}_{\nu,2n+1}\right)=c_{\ell-1}(-1)^{\ell-1}(n+1)^{2\ell-2}\varphi^{-2\ell+2}\left(y^{(\lambda)}_{\nu,2n+1}\right)F'_{\lambda,2n+1}\left(y^{(\lambda)}_{\nu,2n+1}\right)+O\left(n^{\lambda+2\ell-2}\right).\tag{4.104}$$

Then, we obtain from (4.35)

$$\begin{aligned}&\varphi^2\left(y^{(\lambda)}_{\nu,2n+1}\right)F^{(2\ell+1)}_{\lambda,2n+1}\left(y^{(\lambda)}_{\nu,2n+1}\right)\\&\quad=\left(2c_{\ell-1}+2^{2\ell-1}\right)(-1)^{\ell}(n+1)^{2\ell}\varphi^{-2\ell+2}\left(y^{(\lambda)}_{\nu,2n+1}\right)F'_{\lambda,2n+1}\left(y^{(\lambda)}_{\nu,2n+1}\right)+O\left(n^{\lambda+2\ell}\right)\\&\quad=c_{\ell}(-1)^{\ell}(n+1)^{2\ell}\varphi^{-2\ell+2}\left(y^{(\lambda)}_{\nu,2n+1}\right)F'_{\lambda,2n+1}\left(y^{(\lambda)}_{\nu,2\mathrm{n}+1}\right)+O\left(n^{\lambda+2\ell}\right).\end{aligned}\tag{4.105}$$

Therefore, (2.16) is proved. □

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