Research Article

Synchronization between Bidirectional Coupled Nonautonomous Delayed Cohen–Grossberg Neural Networks

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Based on using suitable Lyapunov function and the properties of M-matrix, sufficient conditions for complete synchronization of bidirectional coupled nonautonomous Cohen-Grossberg neural networks are obtained. The methods for discussing synchronization avoid complicated error system of Cohen-Grossberg neural networks. Two numerical examples are given to show the effectiveness of the proposed synchronization method.

1. Introduction

It is well known that chaos synchronization has gained considerable attention due to the truth that many benefits of chaos synchronization exist in various engineering fields such as secure communication, image processing, and harmless oscillation generation. Since the pioneering works of Pecora and Carroll [1], chaos synchronization has been widely investigated, and many effective research methods such as feedback control, impulsive control, adaptive control, and important results have been presented in [2–10] and references cited therein. Synchronization of coupled chaotic systems also has received considerable attention [11–13].

Since Cohen and Grossberg proposed the Cohen-Grossberg neural networks (CGNNs) in 1983 [14] and soon the networks have been the subject of active research due to their many applications in various engineering and scientific areas such as patter recognition, associative memory, and combination, and many dynamic analyses of equilibria or periodic solutions of the networks with delays are investigated [15–20]. Unfortunately, so far, only a few works have been done on synchronization of CGNNs, which remains challenging. In 2010, Li et al. investigated the synchronization of discrete-time CGNNs with delays [6] and Chen discussed

the synchronization of impulsive delayed CGNNs under noise perturbation [7], in 2010, Zhu and Cao discussed adaptive synchronization of delayed CGNNs [21] and in 2012, Gan also discussed adaptive synchronization of delayed CGNNs [22], and in 2011, Yu et al. discuss synchronization of delayed CGNNs via periodically intermittent control [23].

Note that the previous methods in [6, 7, 21–23] have obtained error dynamical system first, and then discuss the error system, but, the error dynamical system of Cohen-Grossberg neural network which comparing the Hopfield neural networks becomes very complex because of existence of the amplification functions of the neural network; hence the results above turn to be quite complex, too. Comparing with foregoing works, the objective of this paper is to study the complete synchronization of nonautonomous CGNNs with mixed time delays by using suitable Lyapunov function and the properties of M-matrix, and our methods can avoid writing explicit complex error system which leads to complicated conditions for synchronization.

The rest of this paper is organized as follows. The model, some definitions, and assumptions are presented in Section 2. The sufficient conditions for complete synchronization of bidirectional coupled CGNNs are obtained in Section 3. Two examples are given in Section 4 to demonstrate the main results.

2. Model Description and Preliminaries

Consider the following bidirectional coupled nonautonomous CGNNs with time delays:

$$
\dot{x}_i(t) = -a_i(x_i(t)) \left[b_i(t, x_i(t)) - \sum_{j=1}^{n} u_{ij}(t) f_j(x_j(t)) - \sum_{j=1}^{n} v_{ij}(t) f_j(x_j(t - \tau_{ij}(t))) \right.
$$
$$
\left. - \sum_{j=1}^{n} w_{ij}(t) \int_{0}^{+\infty} k_{ij}(s) f_j(x_j(t-s)) ds - I_i(t) \right] \tag{2.1}
$$
$$
+ \sum_{j=1}^{n} \alpha_{ij}(t) [y_j(t) - x_j(t)],
$$

$$
\dot{y}_i(t) = -a_i(y_i(t)) \left[b_i(t, y_i(t)) - \sum_{j=1}^{n} u_{ij}(t) f_j(y_j(t)) - \sum_{j=1}^{n} v_{ij}(t) f_j(y_j(t - \tau_{ij}(t))) \right.
$$
$$
\left. - \sum_{j=1}^{n} w_{ij}(t) \int_{0}^{+\infty} k_{ij}(s) f_j(y_j(t-s)) ds - I_i(t) \right] \tag{2.2}
$$
$$
+ \sum_{j=1}^{n} \beta_{ij}(t) [x_j(t) - y_j(t)]
$$

for $1 \leq i \leq n$, $t > 0$. $x_i(t)$ and $y_i(t)$ denote the state variable of the ith neuron in (2.1) and (2.2), $f_j(\cdot)$ denote the signal functions of the jth neuron at time t; $I_i(t)$ denote inputs of the ith neuron at time t; $a_i(\cdot)$ represent amplification functions; $b_i(t, \cdot)$ are appropriately behaved functions; $u_{ij}(t)$, $v_{ij}(t)$, and $w_{ij}(t)$ are bounded connection weights of the neural networks, respectively; $\tau_{ij}(t)$ correspond to the finite speed of the axonal signal transmission, there exist

positive τ_{ij} and $\dot{\tau}_{ij}^+$ such that $0 \leq \tau_{ij}(t) \leq \tau_{ij}$ and $\dot{\tau}_{ij}(t) \leq \dot{\tau}_{ij}^+ < 1$; $k_{ij}(\cdot)$ correspond to the delay kernel functions, coupled matrix are $(\alpha_{ij}(t))_{n \times n}$ and $(\beta_{ij}(t))_{n \times n}$ in which $\alpha_{ii}(t) \geq 0$ and $\beta_{ii}(t) \geq 0$ and $\alpha_{ij}(t)$, $\beta_{ij}(t)$ are bounded on $[0, +\infty)$.

Throughout this paper, we assume for system (2.1) and (2.2) that

(H_1) amplification functions $a_i(\cdot)$ are continuous and there exist constants $\underline{a}_i$, $\overline{a}_i$ such that $0 < \underline{a}_i \leq a_i(\cdot) \leq \overline{a}_i$ for $1 \leq i \leq n$;

(H_2) there exist positive continuous and bounded functions $b_i(t)$ such that

$$\frac{b_i(t, x) - b_i(t, y)}{x - y} \geq b_i(t) > 0 \tag{2.3}$$

for all $x \neq y$, $1 \leq i \leq n$;

(H_3) for activation functions $f_j(\cdot)$, there exist positive constants L_j such that

$$L_j = \sup_{x \neq y} \left| \frac{f_j(x) - f_j(y)}{x - y} \right| \tag{2.4}$$

for all $x \neq y$, $1 \leq j \leq n$;

(H_4) the kernel functions $k_{ij}(s)$ are nonnegative continuous function on $[0, +\infty)$ and satisfy

$$\int_0^{+\infty} s e^{\lambda s} k_{ij}(s) ds < +\infty,$$

$$K_{ij}(\lambda) = \int_0^{+\infty} e^{\lambda s} k_{ij}(s) ds \tag{2.5}$$

are differentiable functions for $\lambda \in [0, r_{ij})$, $0 < r_{ij} < +\infty$, $K_{ij}(0) = 1$ and $\lim_{\lambda \to r_{ij}^-} K_{ij}(\lambda) = +\infty$.

Remark 2.1. A typical example of kernel function is given by $k_{ij}(s) = (s^r / r!) r_{ij}^{r+1} e^{-r_{ij} s}$ for $s \in [0, +\infty)$, where $r_{ij} \in (0, +\infty)$, $r \in \{0, 1, \ldots, n\}$. These kernel functions are called as the gamma memory filter [24] and satisfy condition (H_4).

For any bounded function $s(t)$ on $[0, +\infty)$, $\underline{s}$ and $\overline{s}$ denote $\inf_{t \in [0, +\infty)} \{|s(t)|\}$ and $\sup_{t \in [0, +\infty)} \{|s(t)|\}$, respectively.

For any $x(t) = (x_i(t), x_2(t), \ldots, x_k(t))^T \in R^k$, $t > 0$, define $\|x(t)\|_1 = \sum_{i=1}^k |x_i(t)|$, and for any $\varphi(s) = (\varphi_1(s), \varphi_2(s), \ldots, \varphi_k(s))^T \in R^k$, $s \in (-\infty, 0]$, define $\|\varphi\| = \sup_{s \in (-\infty, 0]} \sum_{i=1}^k |\varphi_i(s)|$.

Denote

$$C\left((-\infty, 0], R^k\right) = \left\{ \psi : (-\infty, 0] \to R^k \mid \psi(s) \text{ is bounded and continuous for } s \in (-\infty, 0] \right\}, \tag{2.6}$$

Then $C((-\infty, 0], R^k)$ is a Banach space with respect to $\| \cdot \|$.

The initial conditions of system (2.1) and (2.2) are given by

$$x_i(s) = \varphi_i(s), \quad -\infty < s \le 0, \ 1 \le i \le n,$$
$$y_i(s) = \varsigma_i(s), \quad -\infty < s \le 0, \ 1 \le i \le n. \tag{2.7}$$

Definition 2.2. System (2.1) and system (2.2) are said to achieve global exponential complete synchronization, if for any solution $x(t, \psi_1)$ of system (2.1) and any solution $y(t, \psi_2)$ of system (2.2), there exist positive constants λ and M such that

$$\|y(t, \psi_2) - x(t, \psi_1)\|_1 \le M \|\psi_2 - \psi_1\| e^{-\lambda t}, \quad t \ge 0, \tag{2.8}$$

where $\psi_2(s) = (\varsigma_1(s), \varsigma_2(s), \dots, \varsigma_n(s))^T$, $\psi_1(s) = (\varphi_1(s), \varphi_2(s), \dots, \varphi_n(s))^T \in C((-\infty, 0], R^n)$.

Definition 2.3. A real matrix $A = (a_{ij})_{n \times n}$ is said to be a nonsingular M-matrix if $a_{ij} \le 0$ ($i, j = 1, 2, \dots, n, \ i \ne j$), and all successive principle minors of A are positive.

Lemma 2.4 (see [25]). *A matrix with nonpositive offdiagonal elements $A = (a_{ij})_{n \times n}$ is a nonsingular M-matrix if and only if there exists a vector $p = (p_i)_{1 \times n} > 0$ such that $pA > 0$ or $Ap^T > 0$ holds.*

Lemma 2.5 (see [26]). *Let $a < b \le +\infty$. Suppose that $v(t) = (v_1(t), v_2(t), \dots, v_n(t))^T$ satisfies the following differential equality:*

$$D^+ v(t) \le P v(t) + (Q \otimes \tilde{v}(t)) e_n + \int_0^{+\infty} (R \otimes K(s)) v(t - s) ds, \quad t \in [a, b),$$
$$v(a + s) \in C((-\infty, 0], R^n), \quad s \in (-\infty, 0], \tag{2.9}$$

where

$$P = (p_{ij})_{n \times n}, \quad p_{ij} \ge 0 \ \ for \ \ i \ne j,$$
$$Q = (q_{ij})_{n \times n}, \quad q_{ij} \ge 0, \quad R = (r_{ij})_{n \times n}, \quad r_{ij} \ge 0, \tag{2.10}$$
$$\tilde{v}(t) = (v_j(t - \tau_{ij}(t)))_{n \times n}, \quad e_n = (1, 1, \dots, 1)^T \in R^n, \quad K(s) = (|k_{ij}(s)|)_{n \times n},$$

and $\otimes$ means Hadamard product. If initial conditions satisfy

$$v(t) \le \kappa \xi e^{-\lambda(t-a)}, \quad \kappa \ge 0, \ t \in (-\infty, a], \tag{2.11}$$

where $\xi = (\xi_1, \xi_2, \dots, \xi_n)^T > 0$ and the positive number λ is determined by the following inequality:

$$[\lambda E + P + Q \otimes \varepsilon(\lambda) + (R \otimes \mathcal{K}(\lambda))]\xi < 0 \tag{2.12}$$

in which

$$\varepsilon(\lambda) = \left(e^{\lambda \tau_{ij}}\right)_{n \times n}, \qquad \mathcal{K}(\lambda) = \left(\mathcal{K}_{ij}(\lambda)\right)_{n \times n}, \qquad \mathcal{K}_{ij}(\lambda) = \int_0^{+\infty} e^{\lambda s} |k_{ij}(s)| ds, \qquad (2.13)$$

and E is an identity matrix. Then $v(t) \leq \kappa \xi e^{-\lambda(t-a)}$ for $t \in [a, b]$.

3. Main Results

Theorem 3.1. *Under assumptions (H_1)–(H_4), system (2.1) and system (2.2) will achieve global exponential synchronization, if the following conditions hold:*

(H_5) $\mathcal{M}_1 = A_1 - C_1$ *is a nonsingular M-matrix, where*

$$A_1 = \mathrm{diag}\left(\underline{b}_1 + \frac{\underline{\gamma}_{11}}{\overline{a}_1}, \underline{b}_2 + \frac{\underline{\gamma}_{22}}{\overline{a}_2}, \ldots, \underline{b}_n + \frac{\underline{\gamma}_{nn}}{\overline{a}_n}\right),$$

$$C_1 = \left(c_{ij}\right)_{n \times n}, \quad c_{ij} = \left(\overline{u}_{ij} + \overline{v}_{ij}\frac{1}{1 - \dot{\tau}_{ij}^+} + \overline{w}_{ij}\right)L_j + q_{ij}, \tag{3.1}$$

where $q_{ij} = \overline{\gamma}_{ij}/\underline{a}_i$, $j \neq i$ and $q_{ii} = 0$, $\gamma_{ij}(t) = \alpha_{ij}(t) + \beta_{ij}(t)$, $i, j = 1, 2, \ldots, n$.

Proof. Let $x(t)$, $y(t)$ be two solutions of system (2.1) with initial value $\psi_1 = (\varphi_1, \varphi_2, \ldots, \varphi_n)$ and system (2.2) with $\psi_2 = (\varsigma_1, \varsigma_2, \ldots, \varsigma_n) \in C((-\infty, 0], R^n)$, respectively.

Denote $e_i(t) = y_i(t) - x_i(t)$.

Note that conditions (H_5), $\mathcal{M}_1$ is a nonsingular M-matrix implies that $\mathcal{M}_1^T$ is a nonsingular M-matrix. From Lemma 2.4, we know that there exists a vector $p = (p_1, p_2, \ldots, p_n)^T$ such that $\mathcal{M}_1^T p > 0$, that is

$$p_i\left(\underline{b}_i + \frac{\underline{\gamma}_{ii}}{\overline{a}_i}\right) - \sum_{j=1}^n p_j\left(\overline{u}_{ji} + \frac{\overline{v}_{ji}}{1 - \dot{\tau}_{ji}^+} + \overline{w}_{ji}\right)L_i - \sum_{j=1, j \neq i}^n p_j\frac{\overline{\gamma}_{ji}}{\underline{a}_j} > 0 \tag{3.2}$$

for $1 \leq i \leq n$.

Denote

$$F_i(\theta) = p_i\left(-\frac{\theta}{\underline{a}_i} + \underline{b}_i + \frac{\underline{\gamma}_{ii}}{\overline{a}_i}\right) - \sum_{j=1}^n p_j\left(\overline{u}_{ji}L_i + \overline{v}_{ji}\frac{e^{\theta \tau_{ji}}}{1 - \dot{\tau}_{ji}^+}L_i + \overline{w}_{ji}L_i\int_0^{+\infty} k_{ji}(s)e^{\theta s}ds\right)$$

$$- \sum_{j=1, j \neq i}^n p_j\frac{\overline{\gamma}_{ji}}{\underline{a}_j}, \tag{3.3}$$

for $1 \leq i \leq n$, which indicates $F_i(0) > 0$. Since $F_i(\theta)$ are continuous and differential on $[0, r_{ji})$ on $[0, r_{ji})$ in which r_{ji} are some positive constants, and $\lim_{\theta \to r_{ji}^-} F_i(\theta) = -\infty$ according

to condition (H$_4$), furthermore, $F_i'(\theta) < 0$ for $\theta \in [0, r_{ji})$. There exist constants θ_i such that $F_i(\theta_i) = 0$ for $i = 1, 2, \ldots, n$. So we can choose

$$0 < \lambda \leq \min_{1 \leq i \leq n} \{\theta_i\} \tag{3.4}$$

such that

$$F_i(\lambda) \geq 0. \tag{3.5}$$

Let

$$V_i(t) = e^{\lambda t} \operatorname{sign}(e_i(t)) \int_{x_i(t)}^{y_i(t)} \frac{1}{a_i(s)} ds \quad \text{for } i = 1, 2, \ldots, n, \tag{3.6}$$

where $e_i(t) = y_i(t) - x_i(t)$.

Calculating the upper right derivative of $V_i(t)$ along solutions of (2.1) and (2.2), we get

$$
\begin{aligned}
D^+ V_i(t) \leq e^{\lambda t} \Bigg\{ &\frac{\lambda}{\underline{a_i}} |e_i(t)| - \operatorname{sign}(e_i(t)) (b_i(t, y_i(t)) - b_i(t, x_i(t))) \\
&+ \left| \sum_{j=1}^{n} u_{ij}(t)(f_j(y_j(t)) - f_j(x_j(t))) \right| \\
&+ \left| \sum_{j=1}^{n} v_{ij}(t)(f_j(y_j(t - \tau_{ij}(t))) - f_j(x_j(t - \tau_{ij}(t)))) \right| \\
&+ \left| \sum_{j=1}^{n} w_{ij}(t) \left(\int_0^{+\infty} k_{ij}(s) f_j(y_j(t-s)) ds - \int_0^{+\infty} k_{ij}(s) f_j(x_j(t-s)) ds \right) \right| \\
&- \frac{\underline{\gamma_{ii}}}{\overline{a_i}} |e_i(t)| + \sum_{j=1, j \neq i}^{n} \frac{\overline{\gamma_{ij}}}{\underline{a_i}} |e_j(t)| \Bigg\} \\
\leq e^{\lambda t} \Bigg\{ &\left(\frac{\lambda}{\underline{a_i}} - \underline{b_i} - \frac{\underline{\gamma_{ii}}}{\overline{a_i}} \right) |e_i(t)| + \sum_{j=1}^{n} \overline{u_{ij}} L_j |e_j(t)| + \sum_{j=1}^{n} \overline{v_{ij}} L_j |e_j(t - \tau_{ij}(t))| \\
&+ \sum_{j=1}^{n} \overline{w_{ij}} L_j \int_0^{+\infty} k_{ij}(s) |e_j(t-s)| ds + \sum_{j=1, j \neq i}^{n} \frac{\overline{\gamma_{ij}}}{\underline{a_i}} |e_j(t)| \Bigg\},
\end{aligned}
\tag{3.7}
$$

where $\gamma_{ij}(t) = \alpha_{ij}(t) + \beta_{ij}(t)$.

Now we define a Lyapunov function $V(t)$ by

$$V(t) = \sum_{i=1}^{n} p_i \left\{ V_i(t) + \sum_{j=1}^{n} \overline{v}_{ij} L_j \frac{e^{\lambda \tau_{ij}}}{1 - \dot{\tau}_{ij}^+} \int_{t-\tau_{ij}(t)}^{t} |e_j(s)| e^{\lambda s} ds \right. $$
$$\left. + \sum_{j=1}^{n} \overline{w}_{ij} L_j \int_{0}^{+\infty} k_{ij}(s) \int_{t-s}^{t} |e_j(\mu)| e^{\lambda(s+\mu)} d\mu\, ds \right\}. \tag{3.8}$$

We can obtain that

$$D^+ V(t) \leq e^{\lambda t} \sum_{i=1}^{n} p_i \left\{ \left(\frac{\lambda}{\underline{a}_i} - \underline{b}_i - \frac{\underline{\gamma}_{ii}}{\overline{a}_i} \right) |e_i(t)| \right. $$
$$+ \sum_{j=1}^{n} \left(\overline{u}_{ij} + \overline{v}_{ij} \frac{e^{\lambda \tau_{ij}}}{1 - \dot{\tau}_{ij}^+} + \overline{w}_{ij} \int_{0}^{+\infty} k_{ij}(s) e^{\lambda s} ds \right) $$
$$\times L_j |e_j(t)| + \sum_{j=1, j \neq i}^{n} \frac{\overline{\gamma}_{ij}}{\overline{a}_i} |e_j(t)| \right\} \tag{3.9}$$
$$= - e^{\lambda t} \sum_{i=1}^{n} F_i(\lambda) |e_i(t)| \leq 0,$$

which together with (3.6) and (3.8) leads to

$$m_0 e^{\lambda t} \sum_{i=1}^{n} |e_i(t)| \leq V(t) \leq V(0) \leq M_0 \|\psi_2(l) - \psi_1(l)\|, \tag{3.10}$$

where

$$m_0 = \min_{1 \leq i \leq n} \left\{ \frac{p_i}{\overline{a}_i} \right\}, \qquad M_0 = \max\{M_1, M_2, M_3\},$$

$$M_1 = \max_{1 \leq i \leq n} \left(\frac{p_i}{\underline{a}_i} \right),$$

$$M_2 = \max_{1 \leq i \leq n} \left(\frac{(e^{\lambda \tau} - 1)}{\lambda} \sum_{j=1}^{n} p_j \overline{v}_{ji} \frac{L_i}{1 - \dot{\tau}_{ji}^+} \right), \tag{3.11}$$

$$M_3 = \sum_{j=1}^{n} p_j \max_{1 \leq i \leq n} (\overline{w}_{ji} L_i) \int_{0}^{+\infty} s e^{\lambda s} \max_{1 \leq i \leq n} k_{ji}(s) ds.$$

Hence, we obtain that the following inequality holds:

$$\sum_{i=1}^{n} |e_i(t)| \leq \frac{M_0}{m_0} \|\psi_2(l) - \psi_1(l)\| e^{-\lambda t}, \quad t \geq 0, \tag{3.12}$$

that is,

$$\|y(t) - x(t)\|_1 \leq \frac{M_0}{m_0} \|\psi_2(l) - \psi_1(l)\| e^{-\lambda t}, \quad t \geq 0. \tag{3.13}$$

This completes the proof. $\qquad\square$

Theorem 3.2. *Under assumptions (H_1)–(H_4), system (2.2) and system (2.1) will achieve global exponential synchronization, if the following conditions hold:*
   (H_6) $\mathcal{M}_2 = A_2 - C_2$ *is a nonsingular M-matrix, where*

$$A_2 = \text{diag}\left(\underline{a}_1\left(\underline{b}_1 + \frac{\underline{\gamma}_{11}}{\overline{a}_1} \right), \underline{a}_2\left(\underline{b}_2 + \frac{\underline{\gamma}_{22}}{\overline{a}_2} \right), \dots, \underline{a}_n\left(\underline{b}_n + \frac{\underline{\gamma}_{nn}}{\overline{a}_n} \right) \right),$$

$$C_2 = (c_{ij})_{n \times n}, \quad c_{ij} = \overline{a}_j\left[(\overline{u}_{ij} + \overline{v}_{ij} + \overline{w}_{ij}) L_j + q_{ij} \right], \tag{3.14}$$

in which $q_{ij} = \overline{\gamma}_{ij}/\underline{a}_i$, $j \neq i$, $q_{ii} = 0$ and $\gamma_{ij}(t) = \alpha_{ij}(t) + \beta_{ij}(t)$, $i, j = 1, 2, \dots, n$.

 Proof. Let $x(t), y(t)$ be two solutions of system (2.1) with initial value $\psi_1 = (\varphi_1, \varphi_2, \dots, \varphi_n)$ and system (2.2) with $\psi_2 = (\varsigma_1, \varsigma_2, \dots, \varsigma_n) \in C((-\infty, 0], R^n)$, respectively.
   Denote $e_i(t) = y_i(t) - x_i(t)$.
   Let

$$V_i(t) = \text{sign}(e_i(t)) \int_{x_i(t)}^{y_i(t)} \frac{1}{a_i(s)} ds \quad \text{for } i = 1, 2, \dots, n \tag{3.15}$$

and note that

$$\frac{1}{\overline{a}_i} |e_i(t)| \leq V_i(t) \leq \frac{1}{\underline{a}_i} |e_i(t)|. \tag{3.16}$$

Calculating the upper right derivative of $V_i(t)$ along solutions of (2.1) and (2.2), we can get

$$
\begin{aligned}
D^+V_i(t) \leq\ & -\underline{b}_i|e_i(t)| + \sum_{j=1}^{n}\overline{u}_{ij}L_j|e_j(t)| + \sum_{j=1}^{n}\overline{v}_{ij}L_j|e_j(t-\tau_{ij}(t))| \\
& + \sum_{j=1}^{n}\overline{w}_{ij}L_j\int_0^{+\infty}k_{ij}(s)|e_j(t-s)|ds - \frac{\gamma_{ii}(t)}{\overline{a}_i}|e_i(t)| + \sum_{j=1,j\neq i}^{n}\frac{\overline{\gamma}_{ij}}{\underline{a}_i}|e_j(t)| \\
\leq\ & -\underline{b}_i\underline{a}_iV_i(t) + \sum_{j=1}^{n}\overline{u}_{ij}L_j\overline{a}_jV_j(t) + \sum_{j=1}^{n}\overline{v}_{ij}L_j\overline{a}_jV_j(t-\tau_{ij}(t)) \\
& + \sum_{j=1}^{n}\overline{w}_{ij}L_j\overline{a}_j\int_0^{+\infty}k_{ij}(s)V_j(t-s)ds - \frac{\gamma_{ii}\underline{a}_i}{\overline{a}_i}V_i(t) + \sum_{j=1,j\neq i}^{n}\frac{\overline{\gamma}_{ij}\overline{a}_j}{\underline{a}_i}V_j(t),
\end{aligned}
\tag{3.17}
$$

that is,

$$
D^+V(t) \leq AV(t) + \left(C\otimes\widetilde{V}(t)\right)E_n + \int_0^{+\infty}(D\otimes K(s))V(t-s)ds,
\tag{3.18}
$$

where

$$
\begin{gathered}
V(t) = (V_1(t), V_2(t), \ldots, V_n(t))^T, \qquad \widetilde{V}(t) = \left(V_j(t-\tau_{ij}(t))\right)_{n\times n}, \\
A = \mathrm{diag}\left(-\underline{a}_1\left(\underline{b}_1 + \frac{\gamma_{11}}{\overline{a}_1}\right), -\underline{a}_2\left(\underline{b}_2 + \frac{\gamma_{22}}{\overline{a}_2}\right), \ldots, -\underline{a}_n\left(\underline{b}_n + \frac{\gamma_{nn}}{\overline{a}_n}\right)\right) + H, \\
H = (h_{ij})_{n\times n}, \quad h_{ij} = \overline{a}_j\overline{u}_{ij}L_j + \widetilde{q}_{ij}, \quad \widetilde{q}_{ij} = \frac{\overline{a}_j\overline{\gamma}_{ij}}{\underline{a}_i}, \quad j\neq i, \quad \widetilde{q}_{ii} = 0, \quad i,j = 1,2,\ldots,n, \\
C = (\widetilde{c}_{ij})_{n\times n}, \quad \widetilde{c}_{ij} = \overline{a}_j\overline{v}_{ij}L_j, \qquad D = (d_{ij})_{n\times n}, \quad d_{ij} = \overline{a}_j\overline{w}_{ij}L_j, \\
E_n = (1, 1, \ldots, 1)^T \in R^n, \qquad K(s) = (k_{ij}(s))_{n\times n}.
\end{gathered}
\tag{3.19}
$$

We know from (H_6) that M_2 is nonsingular M-matrix, which implies $-(A+D+C)$ is also a nonsingular M-matrix. From Lemma 2.4, there exists a vector $\xi = (\xi_1, \xi_2, \ldots, \xi_n)^T > 0$ such that $-(A+D+C)\xi > 0$, consequently,

$$
-\xi_i\underline{a}_i\left(\underline{b}_i + \frac{\gamma_{ii}\underline{a}_i}{\overline{a}_i}\right) + \sum_{j=1}^{n}\xi_j\overline{a}_j(\overline{u}_{ij} + \overline{v}_{ij} + \overline{w}_{ij})L_j + \sum_{j=1,j\neq i}^{n}\xi_j\frac{\overline{a}_j\overline{\gamma}_{ij}}{\underline{a}_i} > 0.
\tag{3.20}
$$

Consider the function

$$F_i(\theta) = \xi_i\left(\theta - \underline{a}_i\left(\underline{b}_i + \frac{\Upsilon_{ii}\underline{a}_i}{\overline{a}_i}\right)\right) + \sum_{j=1}^{n}\xi_j\overline{a}_j\left(\overline{u}_{ij} + \overline{v}_{ij}e^{\theta\tau_{ij}} + \overline{w}_{ij}K_{ij}(\theta)\right)L_j$$
$$+ \sum_{j=1,j\neq i}^{n}\xi_j\frac{\overline{a}_j\overline{\Upsilon}_{ij}}{\underline{a}_i}, \tag{3.21}$$

in which $K_{ij}(\theta) = \int_0^{+\infty}e^{\theta\tau_{ij}}k_{ij}(s)ds$.

We obtain from condition (H$_4$) that $K_{ij}(0) = 1$, which, together with (3.20), leads to $F_i(0) < 0$, and similar to proof of Theorem 3.1 above, there exist

$$0 < \lambda < \min_{1\leq i\leq n}\{\theta_i\} \tag{3.22}$$

such that

$$F_i(\lambda) < 0, \tag{3.23}$$

that is,

$$\left[\lambda E + A + C \otimes \left(e^{\lambda\tau_{ij}}\right)_{n\times n} + D \otimes \mathcal{K}(\lambda)\right]\xi < 0, \tag{3.24}$$

where $K(\lambda) = (K_{ij}(\lambda))_{n\times n}$ in which $K_{ij}(\lambda)$ shown in (3.21).

Note that $V_i(s) \leq 1/\underline{a}_i\|\psi_2 - \psi_1\|$ and denote $\kappa = \|\psi_2 - \psi_1\|/\min_{1\leq i\leq n}\{\underline{a}_i\xi_i\}$, we have

$$V(t) \leq \kappa\xi e^{-\lambda t}, \quad t \in (-\infty, 0]. \tag{3.25}$$

We have from Lemma 2.5, (3.18), (3.24), and (3.25) that

$$V(t) \leq \kappa\xi e^{-\lambda t}, \quad t \geq 0. \tag{3.26}$$

It follows from (3.16) that

$$|e_i(t)| \leq \overline{a}_i V_i(t) \leq \overline{a}_i\widetilde{\kappa}\xi_i\|\psi_2 - \psi_1\|e^{-\lambda t} \tag{3.27}$$

in which $\widetilde{\kappa} = 1/\min_{1\leq i\leq n}\{\underline{a}_i\xi_i\}$.

Hence

$$\|y(t) - x(t)\|_1 \leq M\|\psi_2 - \psi_1\|e^{-\lambda t}, \quad t \geq 0, \tag{3.28}$$

where $M = (\sum_{i=1}^{n}\overline{a}_i\xi_i)/\min_{1\leq i\leq n}\{\underline{a}_i\xi_i\}$. This completes the proof. $\qquad\square$

Corollary 3.3. *Under assumptions (H_1)–(H_4), response system (2.2) will be globally exponentially synchronized with master system (2.1) with $\alpha_{ij} = 0$, $i, j = 1, 2, \ldots, n$, if M_1 in Theorem 3.1 and M_2 in Theorem 3.2 are nonsingular M-matrices with $\gamma_{ij}(t) = \beta_{ij}(t)$.*

Remark 3.4. If $a_i(\cdot) = 1$, $b_i(t, x_i(t)) = b_i(t)x_i(t)$ and $b_i(t, y_i(t)) = b_i(t)y_i(t)$, system (2.1) and (2.2) reduce to coupled Hopfield neural networks. Both Theorems 3.1 and 3.2 reduce to the simpler cases in which $\overline{a}_i = \underline{a}_i = 1$.

Remark 3.5. If $\alpha_{ij}(t) = \beta_{ij}(t) = 0$, M_1 in Theorem 3.1 and M_2 in Theorem 3.2 still be nonsingular M-matrix, both theorems imply the global asymptotical stability of solutions of system (2.1), consequently, system (2.2) and (2.1) achieve complete synchronization for sure. In addition, if system (2.1) and (2.2) reduce to autonomous systems, the results in both theorems above still hold.

4. Two Simple Examples

Example 4.1. Consider the following coupled CGNNs:

$$\dot{x}(t) = -a(x(t))\left[Ax(t) - B\int_0^{+\infty} K(s)\tanh(x(t-s))ds - I(t)\right]$$
$$+ K_1(t)(y(t) - x(t)), \tag{4.1}$$

$$\dot{y}(t) = -a(y(t))\left[Ay(t) - B\int_0^{+\infty} K(s)\tanh(y(t-s))ds - I(t)\right]$$
$$+ K_2(t)(x(t) - y(t)), \tag{4.2}$$

where we write system (4.1) and (4.2) in the vector-matrix form and

$$x(t) = (x_1(t), x_2(t))^T, \qquad y(t) = (y_1(t), y_2(t))^T, \qquad I(t) = (2, 2)^T,$$

$$\tanh(x(t)) = (\tanh(x_1(t)), \tanh(x_2(t)))^T,$$

$$\tanh(y(t)) = (\tanh(y_1(t)), \tanh(y_2(t)))^T,$$

$$a(x(t)) = \text{diag}(2 + \sin(x_1(t)), 2 + \sin(x_2(t))),$$

$$a(y(t)) = \text{diag}(2 + \sin(y_1(t)), 2 + \sin(y_2(t))),$$

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & -2\cos(t) \\ 2 & -0 \end{pmatrix}, \qquad K(s) = \begin{pmatrix} 0 & e^{-t} \\ e^{-t}, & 0 \end{pmatrix}, \tag{4.3}$$

$$K_1(t) = \begin{pmatrix} \alpha_1(t) & 0 \\ 0 & \alpha_2(t) \end{pmatrix},$$

$$K_2(t) = \begin{pmatrix} \beta_1(t) & 0 \\ 0 & \beta_2(t) \end{pmatrix}.$$

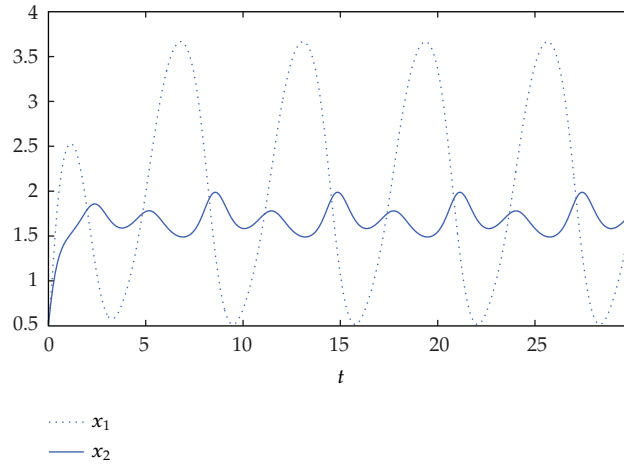Let $K_1(t) = 0$. Figure 1 shows the dynamical behaviors of (4.1) with $K_1(t) = 0$.

Figure 1: Time response of state x_1, x_2 of system (4.1) with initial value $(0.1, 0.1)$ and $K_1 = 0$.

Let $K_1(t) = \operatorname{diag}(1, 1)$ and $K_2(t) = \operatorname{diag}(5, 0)$, it is easy to know $M_1 = \left(\begin{smallmatrix} 5 & -2 \\ -2 & 1 \end{smallmatrix}\right)$ is still a M-matrix. From Theorem 3.1, we know coupled system (4.1) and system (4.2) achieve complete synchronization, and Figure 2 shows the dynamical behaviors of complete synchronization with initial conditions $(y_1(s), y_2(s)) = (2, 2)$ and $(x_1(s), x_2(s)) = (0.1, 0.1)$.

Example 4.2. Consider the following Cohen-Grossberg neural network with discrete delay:

$$\dot{x}_1(t) = -3[x_1(t) - 3.5\sin(x_2(t - 0.4))],$$
$$\dot{x}_2(t) = -2[2x_2(t) - 3\sin(x_1(t - 0.4))]. \tag{4.4}$$

System (4.4) is chaotic system. Figure 3 shows the chaotic behaviors of system (4.4) with initial condition $(0.2, 0.2)$.

For driving system (4.4), we construct the response system as follows:

$$\dot{y}_1(t) = -3[y_1(t) - 3.5\sin(y_2(t - 0.4))] + \alpha_{11}(x_1(t) - y_1(t)),$$
$$\dot{y}_2(t) = -2[2y_2(t) - 3\sin(y_1(t - 0.4))] + \alpha_{22}(x_2(t) - y_2(t)), \tag{4.5}$$

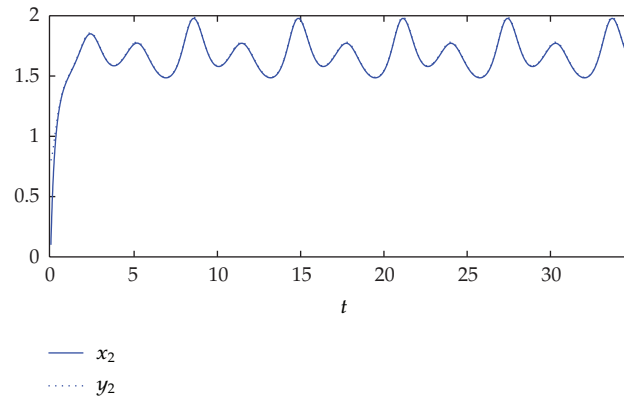where $\alpha_{11} > 0$, $\alpha_{22} > 0$.

Let $\alpha_{11} = 9$, $\alpha_{22} = 0$. It is easy to know

$$M_1 = \begin{pmatrix} 3 & -3.5 \\ -3 & 4 \end{pmatrix} \tag{4.6}$$

is a M-matrix. From Corollary 3.3, we know the response system (4.5) achieves complete synchronization with system (4.4), and Figure 4 shows the dynamical behaviors of complete synchronization with initial conditions $(y_1(s), y_2(s)) = (2, 2)$ and $(x_1(s), x_2(s)) = (0.2, 0.2)$.
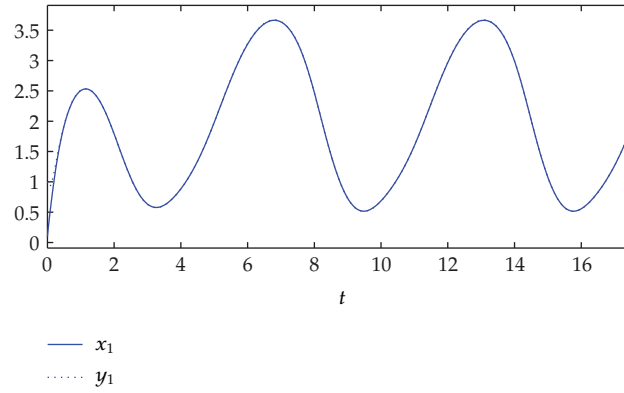
(a)

(b)

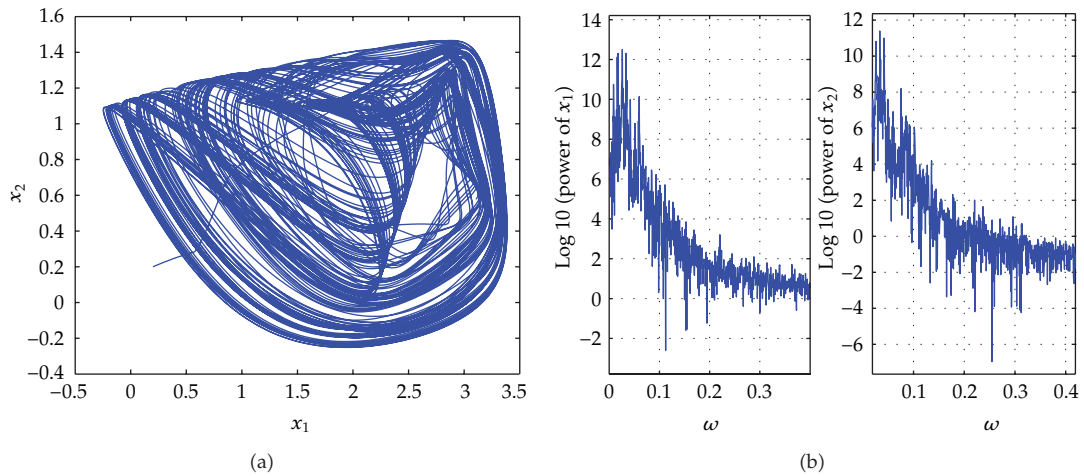**Figure 2:** Complete synchronization of coupled systems (4.1) and (4.2).

(a)

(b)

Figure 3: (a) Phase plot of system (4.4) with initial condition $(0.2, 0.2)$. (b) Power spectrums of time series of x_1 and x_2 for system (4.4).
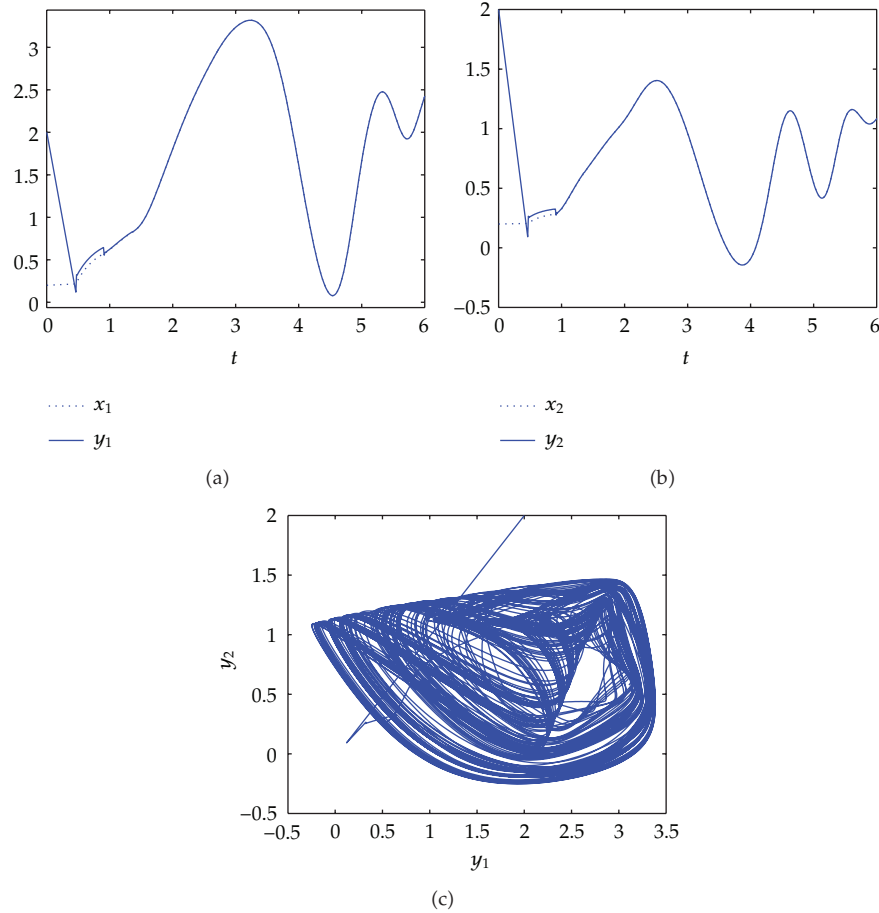
(a)

(b)

(c)

Figure 4: Complete synchronization of the response system (4.5) and the driving system (4.4).

5. Conclusions

Based on using suitable Lyapunov function and the properties of M-matrix, sufficient conditions for complete synchronization of bidirectional coupled CGNNs are directly obtained without writing the explicit error system. Two examples show the effectiveness of the proposed method. Note that M_1 and M_2 must be M-matrix if we let $\alpha_{ii}(t) + \beta_{ii}(t)$, that is, $\gamma_{ii}(t)$ be big enough to a certain extent such that M_1 and M_2 are strongly diagonally dominant matrix, this shows that our criteria are easy to verify and are useful for synchronization of CGNNs.

Acknowledgments

References

[1] L. M. Pecora and T. L. Carroll, "Synchronization in chaotic systems," *Physical Review Letters*, vol. 64, no. 8, pp. 821–824, 1990.

[2] Y. Sun and J. Cao, "Adaptive lag synchronization of unknown chaotic delayed neural networks with noise perturbation," *Physics Letters A*, vol. 364, no. 3-4, pp. 277–285, 2007.

[3] K. Wang, Z. Teng, and H. Jiang, "Adaptive synchronization of neural networks with time-varying delay and distributed delay," *Physica A*, vol. 387, no. 2-3, pp. 631–642, 2008.

[4] B. Cui and X. Lou, "Synchronization of chaotic recurrent neural networks with time-varying delays using nonlinear feedback control," *Chaos, Solitons and Fractals*, vol. 39, no. 1, pp. 288–294, 2009.

[5] H. Huang and G. Feng, "Synchronization of nonidentical chaotic neural networks with time delays," *Neural Networks*, vol. 22, no. 7, pp. 869–874, 2009.

[6] T. Li, A. Song, and S. Fei, "Synchronization control for arrays of coupled discrete-time delayed Cohen-Grossberg neural networks," *Neurocomputing*, vol. 74, no. 1–3, pp. 197–204, 2010.

[7] Z. Chen, "Complete synchronization for impulsive Cohen-Grossberg neural networks with delay under noise perturbation," *Chaos, Solitons and Fractals*, vol. 42, no. 3, pp. 1664–1669, 2009.

[8] J. Cao, G. Chen, and P. Li, "Global synchronization in an array of delayed neural networks with hybrid coupling," *IEEE Transactions on Systems, Man, and Cybernetics B*, vol. 38, no. 2, pp. 488–498, 2008.

[9] J. Cao, Z. Wang, and Y. Sun, "Synchronization in an array of linearly stochastically coupled networks with time delays," *Physica A*, vol. 385, no. 2, pp. 718–728, 2007.

[10] X. Yang, J. Cao, and J. Lu, "Stochastic synchronization of complex networks with nonidentical nodes via hybrid adaptive and impulsive control," *IEEE Transactions on Circuits and Systems. I*, vol. 59, no. 2, pp. 371–384, 2012.

[11] C. H. Chiu, W. W. Lin, and C. C. Peng, "Asymptotic synchronization in lattices of coupled nonidentical Lorenz equations," *International Journal of Bifurcation and Chaos in Applied Sciences and Engineering*, vol. 10, no. 12, pp. 2717–2728, 2000.

[12] J. Lu, T. Zhou, and S. Zhang, "Chaos synchronization between linearly coupled chaotic systems," *Chaos, Solitons and Fractals*, vol. 14, no. 4, pp. 529–541, 2002.

[13] Z. M. Gu, M. Zhao, T. Zhou, C. P. Zhu, and B. H. Wang, "Phase synchronization of non-Abelian oscillators on small-world networks," *Physics Letters A*, vol. 362, no. 2-3, pp. 115–119, 2007.

[14] M. A. Cohen and S. Grossberg, "Absolute stability of global pattern formation and parallel memory storage by competitive neural networks," *IEEE Transactions on Systems, Man, and Cybernetics*, vol. 13, no. 5, pp. 815–826, 1983.

[15] H. Zhao and L. Wang, "Hopf bifurcation in Cohen-Grossberg neural network with distributed delays," *Nonlinear Analysis*, vol. 8, no. 1, pp. 73–89, 2007.

[16] C. H. Li and S. Y. Yang, "Existence and attractivity of periodic solutions to non-autonomous Cohen-Grossberg neural networks with time delays," *Chaos, Solitons and Fractals*, vol. 41, no. 3, pp. 1235–1244, 2009.

[17] Q. Zhou, L. Wan, and J. Sun, "Exponential stability of reaction-diffusion generalized Cohen-Grossberg neural networks with time-varying delays," *Chaos, Solitons and Fractals*, vol. 32, no. 5, pp. 1713–1719, 2007.

[18] Z. Li and K. Li, "Stability analysis of impulsive Cohen-Grossberg neural networks with distributed delays and reaction-diffusion terms," *Applied Mathematical Modelling*, vol. 33, no. 3, pp. 1337–1348, 2009.

[19] J. Cao and Q. Song, "Stability in Cohen-Grossberg-type bidirectional associative memory neural networks with time-varying delays," *Nonlinearity*, vol. 19, no. 7, pp. 1601–1617, 2006.

[20] J. Cao and X. Li, "Stability in delayed Cohen-Grossberg neural networks: LMI optimization approach," *Physica D*, vol. 212, no. 1-2, pp. 54–65, 2005.

[21] Q. Zhu and J. Cao, "Adaptive synchronization of chaotic Cohen-Crossberg neural networks with mixed time delays," *Nonlinear Dynamics*, vol. 61, no. 3, pp. 517–534, 2010.

[22] Q. Gan, "Adaptive synchronization of Cohen-Grossberg neural networks with unknown parameters and mixed time-varying delays," *Communications in Nonlinear Science and Numerical Simulation*, vol. 17, no. 7, pp. 3040–3049, 2012.

[23] J. Yu, C. Hu, H. Jiang, and Z. Teng, "Exponential synchronization of Cohen-Grossberg neural networks via periodically intermittent control," *Neurocomputing*, vol. 74, no. 10, pp. 1776–1782, 2011.

[24] J. C. Principe, J. M. Kuo, and S. Celebi, "An analysis of the gamma memory in dynamic neural networks," *IEEE Transactions on Neural Networks*, vol. 5, no. 2, pp. 331–337, 1994.

[25] R. S. Varga, *Matrix Iterative Analysis*, vol. 27 of *Springer Series in Computational Mathematics*, Springer, Berlin, Germany, 2000.

[26] K. Li, "Stability analysis for impulsive Cohen-Grossberg neural networks with time-varying delays and distributed delays," *Nonlinear Analysis*, vol. 10, no. 5, pp. 2784–2798, 2009.