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Research Article

Sharp Bounds by the Generalized Logarithmic Mean for the Geometric Weighted Mean of the Geometric and Harmonic Means

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We present sharp upper and lower generalized logarithmic mean bounds for the geometric weighted mean of the geometric and harmonic means.

1. Introduction

For $p \in \mathbb{R}$ the generalized logarithmic mean $L_p(a,b)$ of two positive numbers a and b is defined by

$$L_p(a,b) = \begin{cases} a, & a = b, \\ \left[\dfrac{a^{p+1} - b^{p+1}}{(p+1)(a-b)}\right]^{1/p}, & p \neq 0,\ p \neq -1,\ a \neq b, \\ \dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{1/(b-a)}, & p = 0,\ a \neq b, \\ \dfrac{b-a}{\log b - \log a}, & p = -1,\ a \neq b. \end{cases} \tag{1.1}$$

It is well-known that $L_p(a,b)$ is continuous and strictly increasing with respect to $p \in \mathbb{R}$ for fixed a and b with $a \neq b$. In the recent past, the generalized logarithmic mean has been the subject of intensive research. In particular, many remarkable inequalities for L_p can be found in the literature [1–23]. The generalized logarithmic mean has applications in convex function, economics, physics, and even in meteorology [24–27]. In [26] the authors study a variant of Jensen's functional equation involving L_p, which appear in a heat conduction problem. Let $A(a,b) = (a+b)/2$, $I(a,b) = (1/e)(b^b/a^a)^{1/(b-a)}$, $L(a,b) = (b-a)/(\log b - \log a)$, $G(a,b) = \sqrt{ab}$, and $H(a,b) = 2ab/(a+b)$ be the arithmetic, identric, logarithmic, geometric, and harmonic means of two positive numbers a and b with $a \neq b$, respectively. Then it is well known that

$$\begin{aligned} \min\{a,b\} &< H(a,b) < G(a,b) = L_{-2}(a,b) < L(a,b) = L_{-1}(a,b) \\ &< I(a,b) = L_0(a,b) < A(a,b) = L_1(a,b) < \max\{a,b\}. \end{aligned} \tag{1.2}$$

In [28–30], the authors present bounds for L and I in terms of G and A.

Proposition 1.1. *For all positive real numbers a and b with $a \neq b$, one has*

$$\begin{gathered} A^{1/3}(a,b)G^{2/3}(a,b) < L(a,b) < \frac{1}{3}A(a,b) + \frac{2}{3}G(a,b), \\ \frac{1}{3}G(a,b) + \frac{2}{3}A(a,b) < I(a,b). \end{gathered} \tag{1.3}$$

The proof of the following Proposition 1.2 can be found in [31].

Proposition 1.2. *For all positive real numbers a and b with $a \neq b$, we have*

$$\sqrt{G(a,b)A(a,b)} < \sqrt{L(a,b)I(a,b)} < \frac{1}{2}(L(a,b) + I(a,b)) < \frac{1}{2}(G(a,b) + A(a,b)). \tag{1.4}$$

For $r \in \mathbb{R}$ the rth power mean $M_r(a,b)$ of two positive numbers a and b is defined by

$$M_r(a,b) = \begin{cases} \left(\dfrac{a^r + b^r}{2}\right)^{1/r}, & r \neq 0, \\ \sqrt{ab}, & r = 0. \end{cases} \tag{1.5}$$

The main properties of these means are given in [32]. Several authors discussed the relationship of certain means to M_r. The following sharp bounds for L, I, $(IL)^{1/2}$, and $(I + L)/2$ in terms of power means are proved in [31, 33–37].

Proposition 1.3. *For all positive real numbers a and b with $a \neq b$ one has*

$$\begin{gathered} M_0(a,b) < L(a,b) < M_{1/3}(a,b), \qquad M_{2/3}(a,b) < I(a,b) < M_{\log 2}(a,b), \\ M_0(a,b) < I^{1/2}(a,b)L^{1/2}(a,b) < M_{1/2}(a,b), \\ \frac{1}{2}[I(a,b) + L(a,b)] < M_{1/2}(a,b). \end{gathered} \tag{1.6}$$

The following three results were established by Alzer and Qiu in [38].

Proposition 1.4. *The inequalities*

$$\alpha A(a,b) + (1-\alpha)G(a,b) < I(a,b) < \beta A(a,b) + (1-\beta)G(a,b) \tag{1.7}$$

hold for all positive real numbers a and b with $a \neq b$ if and only if

$$\alpha \leq \frac{2}{3}, \qquad \beta \geq \frac{2}{e} = 0.73575\cdots. \tag{1.8}$$

Proposition 1.5. *Let a and b be real numbers with $a \neq b$. If $0 < a,\ b \leq e$, then*

$$[G(a,b)]^{A(a,b)} < [L(a,b)]^{I(a,b)} < [A(a,b)]^{G(a,b)}. \tag{1.9}$$

And, if $a, b \geq e$, then

$$[A(a,b)]^{G(a,b)} < [I(a,b)]^{L(a,b)} < [G(a,b)]^{A(a,b)}. \tag{1.10}$$

Proposition 1.6. *For all positive real numbers a and b with $a \neq b$, one has*

$$M_c(a,b) < \frac{1}{2}(L(a,b) + I(a,b)) \tag{1.11}$$

with the best possible parameter $c = \log 2/(1+\log 2) = 0.40938\cdots$

In [39] the authors presented inequalities between the generalized logarithmic mean and the product $A^{\alpha}(a,b)G^{\beta}(a,b)H^{\gamma}(a,b)$ for all $a,b>0$ with $a \neq b$ and $\alpha,\beta>0$ with $\alpha+\beta<1$.

It is the aim of this paper to give a solution to the problem: for $\alpha \in (0,1)$, what are the greatest value p and the least value q, such that the inequality

$$L_p(a,b) \leq G^{\alpha}(a,b)H^{1-\alpha}(a,b) \leq L_q(a,b) \tag{1.12}$$

holds for all $a,b>0$?

2. Main Result

Theorem 2.1. *For $\alpha \in (0,1)$ and all $a,b>0$, one has the following:*

(1) $L_{3\alpha-5}(a,b) = G^{\alpha}(a,b)H^{1-\alpha}(a,b) = L_{-(2/\alpha)}(a,b)$ *for* $\alpha = 2/3$,

(2) $L_{3\alpha-5}(a,b) \geq G^{\alpha}(a,b)H^{1-\alpha}(a,b) \geq L_{-(2/\alpha)}(a,b)$ *for* $0<\alpha<2/3$, *and* $L_{3\alpha-5}(a,b) \leq G^{\alpha}(a,b)H^{1-\alpha}(a,b) \leq L_{-(2/\alpha)}(a,b)$ *for* $2/3<\alpha<1$, *with equality if and only if* $a=b$, *and the parameters* $3\alpha-5$ *and* $-2/\alpha$ *in each inequality cannot be improved.*

Proof. (1) If $\alpha = 2/3$ and $a=b$, then (1.1) implies that $L_{3\alpha-5}(a,b) = G^{\alpha}(a,b)H^{1-\alpha}(a,b) = L_{-(2/\alpha)}(a,b) = a$.

If $\alpha = 2/3$ and $a \neq b$, then (1.1) leads to

$$L_{3\alpha-5}(a,b) = L_{-(2/\alpha)}(a,b) = L_{-3}(a,b) = \left[\frac{a^{-2}-b^{-2}}{2(b-a)}\right]^{-1/3}$$
$$= (ab)^{1/3}\left(\frac{2ab}{a+b}\right)^{1/3} = G^{2/3}(a,b)H^{1/3}(a,b) = G^{\alpha}(a,b)H^{1-\alpha}(a,b). \tag{2.1}$$

(2) If $a = b$, then from (1.1) we clearly see that $L_{3\alpha-5}(a,b) = G^{\alpha}(a,b)H^{1-\alpha}(a,b) = L_{-(2/\alpha)}(a,b) = a$ for any $\alpha \in (0,1)$.

If $a \neq b$, without loss of generality, we assume $a > b$. Let $a/b = t > 1$ and

$$f(t) = \log L_{3\alpha-5}(a,b) - \log\left[G^{\alpha}(a,b)H^{1-\alpha}(a,b)\right]. \tag{2.2}$$

Then (1.1) and simple computations yield

$$f(t) = \frac{1}{3\alpha-5}\log\frac{t^{3\alpha-4}-1}{(3\alpha-4)(t-1)} - \frac{\alpha}{2}\log t - (1-\alpha)\log\frac{2t}{1+t},$$
$$\lim_{t\to 1^{+}} f(t) = 0, \tag{2.3}$$

$$f'(t) = -\frac{t^{4-3\alpha}}{t(t^2-1)\left(t^{4-3\alpha}-1\right)}g(t), \tag{2.4}$$

where $g(t) = (2-\alpha/2)t^{3\alpha-2} - ((2-\alpha)(2-3\alpha)/5-3\alpha)t^{3\alpha-3} + ((1-\alpha)(2-3\alpha)/2(5-3\alpha))t^{3\alpha-4} - ((1-\alpha)(2-3\alpha)/2(5-3\alpha))t^{2} + ((2-\alpha)(2-3\alpha)/(5-3\alpha))t - (2-\alpha)/2$,

$$g(1) = 0,$$
$$g'(t) = \frac{(2-\alpha)(3\alpha-2)}{2}t^{3\alpha-3} - \frac{3(2-\alpha)(2-3\alpha)(\alpha-1)}{5-3\alpha}t^{3\alpha-4} + \frac{(1-\alpha)(2-3\alpha)(3\alpha-4)}{2(5-3\alpha)}t^{3\alpha-5} - \frac{(1-\alpha)(2-3\alpha)}{(5-3\alpha)}t + \frac{(2-\alpha)(2-3\alpha)}{(5-3\alpha)}, \tag{2.5}$$
$$g'(1) = 0,$$
$$g''(t) = \frac{3(2-\alpha)(3\alpha-2)(\alpha-1)}{2}t^{3\alpha-4} - \frac{3(2-\alpha)(2-3\alpha)(\alpha-1)(3\alpha-4)}{5-3\alpha}t^{3\alpha-5} - \frac{(1-\alpha)(2-3\alpha)(3\alpha-4)}{2}t^{3\alpha-6} - \frac{(1-\alpha)(2-3\alpha)}{(5-3\alpha)},$$

$$g''(1) = 0, \tag{2.6}$$

$$g'''(t) = \frac{3}{2}(1-\alpha)(2-\alpha)(4-3\alpha)(3\alpha-2)t^{3\alpha-7}(t-1)^2. \tag{2.7}$$

If $0 < \alpha < 2/3$, then (2.7) implies

$$g'''(t) < 0 \tag{2.8}$$

for $t > 1$.

From (2.3)–(2.6) and (2.8) we know that $f(t) > 0$ for $t > 1$.

If $2/3 < \alpha < 1$, then (2.7) leads to

$$g'''(t) > 0 \tag{2.9}$$

for $t > 1$. Therefore $f(t) < 0$ for $t > 1$ follows from (2.3)–(2.6) and (2.9).

Let

$$h(t) = \log L_{-(2/\alpha)}(a, b) - \log\left[G^{\alpha}(a, b)H^{1-\alpha}(a, b)\right] \tag{2.10}$$

for $t = a/b > 1$; then (1.1) and elementary calculations lead to

$$h(t) = -\frac{\alpha}{2}\log\frac{t^{(\alpha-2)/\alpha} - 1}{((\alpha-2)/\alpha)(t-1)} - \frac{\alpha}{2}\log t - (1-\alpha)\log\frac{2t}{1+t}, \quad \lim_{t\to 1^+} h(t) = 0, \tag{2.11}$$

$$h'(t) = -\frac{t^{(2-\alpha)/\alpha}}{t(t^2-1)\left(t^{(2-\alpha)/\alpha} - 1\right)}v(t), \tag{2.12}$$

where $v(t) = ((2-\alpha)/2)t^{(3\alpha-2)/\alpha} + ((3\alpha-2)/2)t^{(2\alpha-2)/\alpha} - ((3\alpha-2)/2)t - (2-\alpha)/2$,

$$v(1) = 0, \quad v'(t) = \frac{(2-\alpha)(3\alpha-2)}{2\alpha}t^{(2\alpha-2)/\alpha} + \frac{(3\alpha-2)(\alpha-1)}{\alpha}t^{(\alpha-2)/\alpha} - \frac{3\alpha-2}{2}, \tag{2.13}$$

$$v'(1) = 0, \tag{2.14}$$

$$v''(t) = \frac{(2-\alpha)(1-\alpha)(2-3\alpha)}{\alpha^2}t^{-2/\alpha}(t-1). \tag{2.15}$$

If $\alpha \in (0, 2/3)$, then (2.15) implies

$$v''(t) > 0 \tag{2.16}$$

for $t > 1$.

From (2.11)–(2.14) and (2.16) we know that $h(t) < 0$ for $t > 1$.

If $\alpha \in (2/3, 1)$, then (2.15) leads to

$$v''(t) < 0 \tag{2.17}$$

for $t > 1$. Therefore, $h(t) > 0$ for $t > 1$ follows from (2.11)–(2.14) and (2.17).

Next, we prove that the parameters $-(2/\alpha)$ and $3\alpha-5$ in either case cannot be improved. The proof is divided into two cases.

Case 1 ($\alpha \in (0,2/3)$). For any $\epsilon > 0$ and $x \in (0,1)$, from (1.1) one has

$$\begin{aligned}&\left[G^{\alpha}(1,1+x)H^{1-\alpha}(1,1+x)\right]^{5-3\alpha+\epsilon}-\left[L_{3\alpha-5-\epsilon}(1,1+x)\right]^{5-3\alpha+\epsilon}\\&\quad=\frac{f_1(x)}{(1+x/2)^{(1-\alpha)(5-3\alpha+\epsilon)}\left[(1+x)^{4-3\alpha+\epsilon}-1\right]},\end{aligned}\tag{2.18}$$

where $f_1(x) = (1+x)^{(1-\alpha/2)(5-3\alpha+\epsilon)}[(1+x)^{4-3\alpha+\epsilon}-1]-(4-3\alpha+\epsilon)x(1+x)^{4-3\alpha+\epsilon}(1+x/2)^{(1-\alpha)(5-3\alpha+\epsilon)}$.

Let $x \to 0$; making use of the Taylor expansion, we get

$$f_1(x)=\frac{\epsilon(4-3\alpha+\epsilon)(5-3\alpha+\epsilon)}{24}x^3+o\left(x^3\right).\tag{2.19}$$

Equations (2.18) and (2.19) imply that for any $\alpha \in (0,2/3)$ and $\epsilon > 0$ there exists $\delta=\delta(\epsilon,\alpha)\in(0,1)$, such that $L_{3\alpha-5-\epsilon}(1,1+x) < G^{\alpha}(1,1+x)H^{1-\alpha}(1,1+x)$ for $x \in (0,\delta)$.

On the other hand, for any $\epsilon \in (0,(2/\alpha)-1)$ we have

$$\begin{aligned}&L_{-(2/\alpha)+\epsilon}(1,t)-G^{\alpha}(1,t)H^{1-\alpha}(1,t)\\&\quad=t^{\alpha/(2-\epsilon\alpha)}\left\{\left[\frac{1-t^{-2/\alpha+\epsilon+1}}{(2/\alpha-\epsilon-1)(1-1/t)}\right]^{-\alpha/(2-\epsilon\alpha)}-t^{-\epsilon\alpha^2/2(2-\epsilon\alpha)}\left(\frac{2t}{1+t}\right)^{1-\alpha}\right\},\\&\lim_{t\to+\infty}\left\{\left[\frac{1-t^{-2/\alpha+\epsilon+1}}{(2/\alpha-\epsilon-1)(1-1/t)}\right]^{-\alpha/(2-\epsilon\alpha)}-t^{-\epsilon\alpha^2/2(2-\epsilon\alpha)}\left(\frac{2t}{1+t}\right)^{1-\alpha}\right\}\\&\quad=\left(\frac{2}{\alpha}-\epsilon-1\right)^{\alpha/(2-\epsilon\alpha)}>0.\end{aligned}\tag{2.20}$$

From (2.20) we know that for any $\alpha \in (0,2/3)$ and $\epsilon \in (0,2/\alpha-1)$ there exists $T=T(\epsilon,\alpha)>1$, such that $L_{-2/\alpha+\epsilon}(1,t) > G^{\alpha}(1,t)H^{1-\alpha}(1,t)$ for $t\in(T,\infty)$.

Case 2 ($\alpha \in (2/3,1)$). For any $\epsilon \in (0,4-3\alpha)$ and $x \in (0,1)$, from (1.1) one has

$$\begin{aligned}&\left[L_{3\alpha-5+\epsilon}(1,1+x)\right]^{5-3\alpha-\epsilon}-\left[G^{\alpha}(1,1+x)H^{1-\alpha}(1,1+x)\right]^{5-3\alpha-\epsilon}\\&\quad=\frac{f_2(x)}{(1+x/2)^{(1-\alpha)(5-3\alpha-\epsilon)}\left[(1+x)^{4-3\alpha-\epsilon}-1\right]},\end{aligned}\tag{2.21}$$

where $f_2(x)=(4-3\alpha-\epsilon)x(1+x)^{4-3\alpha-\epsilon}(1+x/2)^{(1-\alpha)(5-3\alpha-\epsilon)}-(1+x)^{(1-\alpha/2)(5-3\alpha-\epsilon)}[(1+x)^{4-3\alpha-\epsilon}-1]$.

Let $x \to 0$; making use of the Taylor expansion, we have

$$f_2(x)=\frac{\epsilon}{24}(4-3\alpha-\epsilon)(5-3\alpha-\epsilon)x^3+o\left(x^3\right).\tag{2.22}$$

Equations (2.21) and (2.22) imply that for any $\alpha \in (2/3, 1)$ and $\epsilon \in (0, 4 - 3\alpha)$ there exists $\delta = \delta(\epsilon, \alpha) \in (0, 1)$, such that $L_{3\alpha-5+\epsilon}(1, 1+x) > G^{\alpha}(1, 1+x)H^{1-\alpha}(1, 1+x)$ for $x \in (0, \delta)$.

On the other hand, for any $\epsilon > 0$, we have

$$\begin{aligned} &G^{\alpha}(1,t)H^{1-\alpha}(1,t) - L_{-(2/\alpha)-\epsilon}(1,t) \\ &\quad = t^{\alpha/2}\left\{\left(\frac{2t}{1+t}\right)^{1-\alpha} - t^{-\epsilon\alpha^2/2(2+\epsilon\alpha)}\left[\frac{1-t^{-(2/\alpha+\epsilon-1)}}{(2/\alpha+\epsilon-1)(1-1/t)}\right]^{-\alpha/(2+\epsilon\alpha)}\right\}, \\ &\lim_{t\to+\infty}\left\{\left(\frac{2t}{1+t}\right)^{1-\alpha} - t^{-\epsilon\alpha^2/2(2+\epsilon\alpha)}\left[\frac{1-t^{-(2/\alpha+\epsilon-1)}}{(2/\alpha+\epsilon-1)(1-1/t)}\right]^{-\alpha/(2+\epsilon\alpha)}\right\} = 2^{1-\alpha} > 0. \end{aligned} \tag{2.23}$$

From (2.23) we know that for any $\alpha \in (2/3, 1)$ and $\epsilon > 0$ there exists $T = T(\epsilon, \alpha) > 1$, such that $L_{-(2/\alpha)-\epsilon}(1, t) < G^{\alpha}(1, t)H^{1-\alpha}(1, t)$ for $t \in (T, \infty)$. □

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References

[1] K. B. Stolarsky, "The power and generalized logarithmic means," *The American Mathematical Monthly*, vol. 87, no. 7, pp. 545–548, 1980.

[2] F. Qi and B.-N. Guo, "An inequality between ratio of the extended logarithmic means and ratio of the exponential means," *Taiwanese Journal of Mathematics*, vol. 7, no. 2, pp. 229–237, 2003.

[3] C.-P. Chen and F. Qi, "Monotonicity properties for generalized logarithmic means," *The Australian Journal of Mathematical Analysis and Applications*, vol. 1, no. 2, article 2, p. 4, 2004.

[4] X. Li, C.-P. Chen, and F. Qi, "Monotonicity result for generalized logarithmic means," *Tamkang Journal of Mathematics*, vol. 38, no. 2, pp. 177–181, 2007.

[5] F. Qi, S.-X. Chen, and C.-P. Chen, "Monotonicity of ratio between the generalized logarithmic means," *Mathematical Inequalities & Applications*, vol. 10, no. 3, pp. 559–564, 2007.

[6] C.-P. Chen, "The monotonicity of the ratio between generalized logarithmic means," *Journal of Mathematical Analysis and Applications*, vol. 345, no. 1, pp. 86–89, 2008.

[7] B.-N. Guo and F. Qi, "A simple proof of logarithmic convexity of extended mean values," *Numerical Algorithms*, vol. 52, no. 1, pp. 89–92, 2009.

[8] Y.-M. Chu and W.-F. Xia, "Inequalities for generalized logarithmic means," *Journal of Inequalities and Applications*, vol. 2009, Article ID 763252, 7 pages, 2009.

[9] M.-Y. Shi, Y.-M. Chu, and Y.-P. Jiang, "Optimal inequalities among various means of two arguments," *Abstract and Applied Analysis*, vol. 2009, Article ID 694394, 10 pages, 2009.

[10] W.-F. Xia and Y.-M. Chu, "Optimal inequalities related to the logarithmic, identric, arithmetic and harmonic means," *Revue d'Analyse Numérique et de Théorie de l'Approximation*, vol. 39, no. 2, pp. 176–183, 2010.

[11] Y.-M. Chu and W.-F. Xia, "Two optimal double inequalities between power mean and logarithmic mean," *Computers & Mathematics with Applications*, vol. 60, no. 1, pp. 83–89, 2010.

[12] W.-F. Xia, Y.-M. Chu, and G.-D. Wang, "The optimal upper and lower power mean bounds for a convex combination of the arithmetic and logarithmic means," *Abstract and Applied Analysis*, vol. 2010, Article ID 604804, 9 pages, 2010.

[13] B.-Y. Long and Y.-M. Chu, "Optimal inequalities for generalized logarithmic, arithmetic, and geometric means," *Journal of Inequalities and Applications*, vol. 2010, Article ID 806825, 10 pages, 2010.

[14] Y.-M. Chu, S.-S. Wang, and C. Zong, "Optimal lower power mean bound for the convex combination of harmonic and logarithmic means," *Abstract and Applied Analysis*, vol. 2011, Article ID 520648, 9 pages, 2011.
[15] M.-Y. Shi, Y.-M. Chu, and Y.-P. Jiang, "Optimal inequalities related to the power, harmonic and identric means," *Acta Mathematica Scientia A*, vol. 31, no. 5, pp. 1377–1384, 2011 (Chinese).
[16] Y.-F. Qiu, M.-K. Wang, Y.-M. Chu, and G.-D. Wang, "Two sharp inequalities for Lehmer mean, identric mean and logarithmic mean," *Journal of Mathematical Inequalities*, vol. 5, no. 3, pp. 301–306, 2011.
[17] Y.-M. Chu, M.-K. Wang, and Z.-K. Wang, "A sharp double inequality between harmonic and identric means," *Abstract and Applied Analysis*, vol. 2011, Article ID 657935, 7 pages, 2011.
[18] Y.-M. Chu and M.-K. Wang, "Optimal inequalities between harmonic, geometric, logarithmic, and arithmetic-geometric means," *Journal of Applied Mathematics*, vol. 2011, Article ID 618929, 9 pages, 2011.
[19] Y.-M. Chu, S.-W. Hou, and W.-M. Gong, "Inequalities between logarithmic, harmonic, arithmetic and centroidal means," *SIAM Journal on Mathematical Analysis*, vol. 2, no. 2, pp. 1–5, 2011.
[20] H.-N. Hu, S.-S. Wang, and Y.-M. Chu, "Optimal upper power mean bound for the convex combiantion of harmonic and logarithmic means," *Pacific Journal of Applied Mathematics*, vol. 4, no. 1, pp. 35–44, 2011.
[21] Y.-F. Qiu, M.-K. Wang, and Y.-M. Chu, "The sharp combination bounds of arithmetic and logarithmic means for Seiffert's mean," *International Journal of Pure and Applied Mathematics*, vol. 72, no. 1, pp. 11–18, 2011.
[22] Y.-F. Qiu, M.-K. Wang, and Y.-M. Chu, "The optimal generalized Heronian mean bounds for the identric mean," *International Journal of Pure and Applied Mathematics*, vol. 72, no. 1, pp. 19–26, 2011.
[23] M.-K. Wang, Z.-K. Wang, and Y.-M. Chu, "An optimal double inequality between geometric and identric means," *Applied Mathematics Letters*, vol. 25, pp. 471–475, 2012.
[24] B.-N. Guo and F. Qi, "Inequalities for generalized weighted mean values of convex function," *Mathematical Inequalities & Applications*, vol. 4, no. 2, pp. 195–202, 2001.
[25] A. O. Pittenger, "The logarithmic mean in n variables," *The American Mathematical Monthly*, vol. 92, no. 2, pp. 99–104, 1985.
[26] P. Kahlig and J. Matkowski, "Functional equations involving the logarithmic mean," *Zeitschrift für Angewandte Mathematik und Mechanik*, vol. 76, no. 7, pp. 385–390, 1996.
[27] G. Pólya and G. Szegö, *Isoperimetric Inequalities in Mathematical Physics*, Princeton University Press, Princeton, NJ, USA, 1951.
[28] E. B. Leach and M. C. Sholander, "Extended mean values. II," *Journal of Mathematical Analysis and Applications*, vol. 92, no. 1, pp. 207–223, 1983.
[29] J. Sándor, "A note on some inequalities for means," *Archiv der Mathematik*, vol. 56, no. 5, pp. 471–473, 1991.
[30] B. C. Carlson, "The logarithmic mean," *The American Mathematical Monthly*, vol. 79, pp. 615–618, 1972.
[31] H. Alzer, "Ungleichungen für Mittelwerte," *Archiv der Mathematik*, vol. 47, no. 5, pp. 422–426, 1986.
[32] P. S. Bullen, D. S. Mitrinović, and P. M. Vasić, *Means and Their Inequalities*, vol. 31, D. Reidel, Dordrecht, The Netherlands, 1988.
[33] H. Alzer, "Ungleichungen fur $(e/a)^a(b/e)^b$," *Elemente der Mathematik*, vol. 40, no. 5, pp. 120–123, 1985.
[34] F. Burk, "Notes: the geometric, logarithmic, and arithmetic mean inequality," *The American Mathematical Monthly*, vol. 94, no. 6, pp. 527–528, 1987.
[35] T. P. Lin, "The power mean and the logarithmic mean," *The American Mathematical Monthly*, vol. 81, pp. 879–883, 1974.
[36] A. O. Pittenger, "Inequalities between arithmetic and logarithmic means," *Univerzitet u Beogradu*, vol. 678–715, pp. 15–18, 1980.
[37] A. O. Pittenger, "The symmetric, logarithmic and power means," *Univerzitet u Beogradu*, vol. 678–715, pp. 19–23, 1980.
[38] H. Alzer and S.-L. Qiu, "Inequalities for means in two variables," *Archiv der Mathematik*, vol. 80, no. 2, pp. 201–215, 2003.
[39] Y.-M. Chu and B.-Y. Long, "Best possible inequalities between generalized logarithmic mean and classical means," *Abstract and Applied Analysis*, vol. 2010, Article ID 303286, 13 pages, 2010.