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Research Article

Weighted Composition Operators on the Zygmund Space

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We characterize the boundedness and compactness of the weighted composition operator on the Zygmund space $\mathcal{Z} = \{f \in H(D) : \sup_{z\in D}(1-|z|^2)|f''(z)| < \infty\}$ and the little Zygmund space $\mathcal{Z}_0$.

1. Introduction

Let $D = \{z : |z| < 1\}$ be the open unit disk in the complex plane $\mathbf{C}$, let $T = \{z : |z| = 1\}$ be its boundary, and let $H(D)$ denote the set of all analytic functions by D. For $f \in H(D)$, let

$$\|f\|_{\mathcal{Z}} = \sup\left\{\left(1-|z|^2\right)\left|f''(z)\right| : z \in D\right\}. \tag{1.1}$$

An analytic function $f \in H(D)$ is said to belong to the Zygmund space $\mathcal{Z}$ if $\|f\|_{\mathcal{Z}} < +\infty$, and the little Zygmund space $\mathcal{Z}_0$ consists of all $f \in \mathcal{Z}$ satisfying $\lim_{|z|\to 1^-}(1-|z|^2)|f''(z)| = 0$. From a theorem of Zygmund (see [1, vol. I, page 263] or [2, Theorem 5.3]), we see that $f \in \mathcal{Z}$ if and only if f is continuous in the close unit disk $\overline{D} = \{z : |z| \le 1\}$ and the boundary function $f(e^{i\theta})$ such that

$$\frac{\left|f\left(e^{i(\theta+h)}\right) + f\left(e^{i(\theta-h)}\right) - 2f\left(e^{i\theta}\right)\right|}{h} < \infty, \tag{1.2}$$

for all $e^{i\theta} \in T$ and all $h > 0$. It can easily proved that $\mathcal{Z}$ is a Banach space under the norm:

$$\|f\|_{*} = \left|f(0)\right| + \left|f'(0)\right| + \|f\|_{\mathcal{Z}} \tag{1.3}$$

and that $\mathcal{Z}_0$ is a closed subspace of $\mathcal{Z}$. It is easily obtained that

$$\left|f'(z) - f'(0)\right| \leq \frac{1}{2}\|f\|_{\mathcal{Z}} \log\frac{1+|z|}{1-|z|} \quad \text{for } f \in \mathcal{Z}, \tag{1.4}$$

$$\lim_{|z|\to 1^-} \frac{\left|f'(z)\right|}{\log(1/(1-|z|))} = 0 \quad \text{for } f \in \mathcal{Z}_0. \tag{1.5}$$

For some other information on this space and some operators on it, see, for example, [3–5].

An analytic self-map $\varphi : D \to D$ induces the composition operator C_φ on $H(D)$, defined by $C_\varphi(f) = f(\varphi(z))$ for f analytic on D. It is a well-known consequence of Littlewood's subordination principle that the composition operator C_φ is bounded on the classical Hardy, Bergman, and Bloch spaces (see, e.g., [6–9]).

Recall that a linear operator is said to be bounded if the image of a bounded set is a bounded set, while a linear operator is compact if it takes bounded sets to sets with compact closure. It is interesting to provide a function theoretic characterization of when φ induces a bounded or compact composition operator on various spaces. The book [10] contains plenty of information on this topic.

Let u be a fixed analytic function on the open unit disk. Define a linear operator uC_φ on the space of analytic functions on D, called a weighted composition operator, by $uC_\varphi f = u\cdot(f\circ\varphi)$, where f is an analytic function on D. We can regard this operator as a generalization of a multiplication operator and a composition operator. In recent years, the weighted composition operator has been received much attention and appears in various settings in the literature. For example, it is known that isometries of many analytic function spaces are weighted composition operators (e.g., see [11]). The boundedness and compactness of it has been studied on various Banach spaces of analytic functions, such as Hardy, Bergman, BMOA, Bloch-type spaces, see, for example, [12–16]. Also, it has been studied from one Banach space of analytic functions to another, one may see in [17–26].

The purpose of this paper is to consider the weighted composition operators on the Zygmund space $\mathcal{Z}$ and the little Zygmund space $\mathcal{Z}_0$. Our main goal is to characterize boundedness and compactness of the operators uC_φ on $\mathcal{Z}$ in terms of function theoretic properties of the symbols u and φ. We also characterize boundedness and compactness of uC_φ on $\mathcal{Z}_0$.

Throughout this paper, constants are denoted by C, they are positive and may differ from one occurrence to the other.

2. Auxiliary Results

In order to prove the main results of this paper, we need some auxiliary results.

Lemma 2.1. *If $f \in \mathcal{Z}$, then*

(i) $|f(z)| \leq \|f\|_*$ *for every $z \in D$;*

(ii) $|f'(z)| \leq \log\ (e/(1-|z|^2))\|f\|_*$ *for every $z \in D$.*

Proof. Suppose $f \in \mathcal{Z}$, $z \in D$ and $0 < t < 1$, then

$$\left|f'(zt)\right| \leq \left|f'(0)\right| + \frac{1}{2}\|f\|_{\mathcal{Z}} \log \frac{1+|zt|}{1-|zt|}, \tag{2.1}$$

by (1.4). It follows that

$$\begin{aligned} \left|f(z) - f(0)\right| = \left|z\int_0^1 f'(zt)dt\right| &\leq |z| \int_0^1 \left(\left|f'(0)\right| + \frac{1}{2}\|f\|_{\mathcal{Z}} \log \frac{1+|zt|}{1-|zt|}\right)dt \\ &\leq |z|\left|f'(0)\right| + \frac{1}{2}\ \|f\|_{\mathcal{Z}} \int_0^{|z|} \log \frac{1+s}{1-s} ds \\ &\leq |z|\left|f'(0)\right| + \log(1+|z|)\|f\|_{\mathcal{Z}}, \end{aligned} \tag{2.2}$$

hence

$$\left|f(z)\right| \leq \left|f(0)\right| + \left|f'(0)\right| + \|f\|_{\mathcal{Z}} \log 2 \leq \|f\|_{*}. \tag{2.3}$$

One may easily prove (ii) by (1.4). The details are omitted here. □

Lemma 2.2. *Suppose $f \in \mathcal{Z}$, then $\|f_t\|_* \leq \|f\|_*$, $0 < t < 1$, where $f_t(z) = f(tz)$.*

One may easily obtain it by a calculation.

Lemma 2.3. *Suppose $uC_\varphi : \mathcal{Z}_0 \to \mathcal{Z}_0$ is a bounded operator. Then $uC_\varphi : \mathcal{Z} \to \mathcal{Z}$ is a bounded operator.*

Proof. Suppose uC_φ is bounded in $\mathcal{Z}_0$. It is clear that for any $f \in \mathcal{Z}$, we have $f_t \in \mathcal{Z}_0$ for every $0 < t < 1$. According to Lemma 2.2, we obtain that

$$\left\|uC_\varphi(f_t)\right\|_* \leq \left\|uC_\varphi\right\| \left\|f_t\right\|_* \leq \left\|uC_\varphi\right\| \left\|f\right\|_* < +\infty. \tag{2.4}$$

Then

$$\left\|uC_\varphi(f)\right\|_* = \lim_{t\to 1^-} \left\|uC_\varphi(f_t)\right\|_* \leq \sup_{0<t<1} \left\|uC_\varphi(f_t)\right\|_* \leq \left\|uC_\varphi\right\| \left\|f\right\|_* < +\infty. \tag{2.5}$$

Hence, $uC_\varphi : \mathcal{Z} \to \mathcal{Z}$ is a bounded operator. □

3. Boundedness of uC_φ

In this section, we characterize bounded weighted composition operators on the Zygmund space $\mathcal{Z}$ and the little Zygmund space $\mathcal{Z}_0$.

Theorem 3.1. *Let u be an analytic function on the unit disc D and φ an analytic self-map of D. Then uC_φ is bounded on the Zygmund space $\mathcal{Z}$ if and only if $u \in \mathcal{Z}$ and the following are satisfied:*

$$\sup_{z\in D} \frac{\left(1-|z|^2\right)\left|u(z)\left(\varphi'(z)\right)^2\right|}{1-|\varphi(z)|^2} < \infty; \tag{3.1}$$

$$\sup_{z\in D}\left(1-|z|^2\right)\left|2\varphi'(z)u'(z)+\varphi''(z)u(z)\right| \log\frac{1}{1-|\varphi(z)|^2} < \infty. \tag{3.2}$$

Proof. Suppose uC_φ is bounded on the Zygmund space $\mathcal{Z}$. Then we can easily obtain the following results by taking $f(z)=1$ and $f(z)=z$ in $\mathcal{Z}$, respectively:

$$u \in \mathcal{Z}; \tag{3.3}$$

$$\sup_{z\in D}\left(1-|z|^2\right)\left|2\varphi'(z)u'(z)+\varphi''(z)u(z)+\varphi(z)u''(z)\right| < +\infty. \tag{3.4}$$

By (3.3), (3.4), and the boundedness of the function $\varphi(z)$, we get

$$K_1 = \sup_{z\in D}\left(1-|z|^2\right)\left|2\varphi'(z)u'(z)+\varphi''(z)u(z)\right| < +\infty. \tag{3.5}$$

Let $f(z)=z^2$ in $\mathcal{Z}$ again, in the same way we have

$$\sup_{z\in D}\left(1-|z|^2\right)\left|4\varphi(z)\varphi'(z)u'(z)+\varphi^2(z)u''(z)+2u(z)\left(\varphi(z)\varphi''(z)+\left(\varphi'(z)\right)^2\right)\right| < \infty. \tag{3.6}$$

Using these facts and the boundedness of the function $\varphi(z)$ again, we get

$$K_2 = \sup_{z\in D}\left(1-|z|^2\right)\left|\left(\varphi'(z)\right)^2 u(z)\right| < +\infty. \tag{3.7}$$

Fix $a \in D$ with $|a| > 1/2$, we take the test functions:

$$f_a(z) = \frac{h(\overline{a}z)}{\overline{a}}\left(\log\frac{1}{1-|a|^2}\right)^{-1} - \int_0^z \log\frac{1}{1-\overline{a}\omega}\,d\omega, \tag{3.8}$$

for $z \in D$, where

$$h(z) = (z-1)\left(\left(1+\log\frac{1}{1-z}\right)^2+1\right). \tag{3.9}$$

Then we have

$$
\begin{aligned}
f_a'(z) &= \left(\log \frac{1}{1-\overline{a}z}\right)^2 \left(\log \frac{1}{1-|a|^2}\right)^{-1} - \log \frac{1}{1-\overline{a}z}, \\
f_a''(z) &= \frac{2\overline{a}}{1-\overline{a}z} \log \frac{1}{1-\overline{a}z} \left(\log \frac{1}{1-|a|^2}\right)^{-1} - \frac{\overline{a}}{1-\overline{a}z},
\end{aligned}
\tag{3.10}
$$

and $\sup_{1/2<|a|<1} \|f_a\|_* \le C$ by [3], where C is not dependent on a. Therefore, for all $\lambda \in D$ with $|\varphi(\lambda)| > 1/2$, we have

$$
\begin{aligned}
C\|f_a\|_* \ge \|uC_\varphi f_a\|_* &\ge \sup_{z\in D}\left(1-|z|^2\right)\left|(uC_\varphi f_a)''(z)\right| \\
&= \sup_{z\in D}\left(1-|z|^2\right)\Big|\left(2\varphi'(z)u'(z) + \varphi''(z)u(z)\right) f_a'(\varphi(z)) \\
&\qquad + f_a''(\varphi(z))\left(\varphi'(z)\right)^2 u(z) + u''(z) f_a(\varphi(z))\Big|.
\end{aligned}
\tag{3.11}
$$

Let $a = \varphi(\lambda)$, it follows that

$$
\begin{aligned}
C\|f_a\|_* &\ge \left(1-|\lambda|^2\right)\Big|\left(2\varphi'(\lambda)u'(\lambda) + \varphi''(\lambda)u(\lambda)\right) f_{\varphi(\lambda)}'(\varphi(\lambda)) \\
&\qquad + f_{\varphi(\lambda)}''(\varphi(\lambda))\left(\varphi'(\lambda)\right)^2 u(\lambda) + u''(\lambda) f_{\varphi(\lambda)}(\varphi(\lambda))\Big| \\
&= \left(1-|\lambda|^2\right)\left|\left(\varphi'(\lambda)\right)^2 u(\lambda)\frac{\overline{\varphi(\lambda)}}{1-|\varphi(\lambda)|^2} + u''(\lambda) f_{\varphi(\lambda)}(\varphi(\lambda))\right| \\
&\ge \left(1-|\lambda|^2\right)\left|\left(\varphi'(\lambda)\right)^2 u(\lambda)\frac{\overline{\varphi(\lambda)}}{1-|\varphi(\lambda)|^2}\right| - \left(1-|\lambda|^2\right)\left|u''(\lambda) f_{\varphi(\lambda)}(\varphi(\lambda))\right|.
\end{aligned}
\tag{3.12}
$$

Then, by Lemma 2.1 and (3.3), we have

$$
\begin{aligned}
\left(1-|\lambda|^2\right)\left|\left(\varphi'(\lambda)\right)^2 u(\lambda)\frac{\overline{\varphi(\lambda)}}{1-|\varphi(\lambda)|^2}\right| &\le \left(1-|\lambda|^2\right)\left|u''(\lambda) f_{\varphi(\lambda)}(\varphi(\lambda))\right| + C\|f_a\|_* \\
&\le \|u\|_{\mathcal{Z}}\|f_a\|_* + C\|f_a\|_*.
\end{aligned}
\tag{3.13}
$$

Hence

$$
\begin{aligned}
\sup_{|\varphi(\lambda)|>1/2} \frac{\left(1-|\lambda|^2\right)\left|\left(\varphi'(\lambda)\right)^2 u(\lambda)\right|}{1-|\varphi(\lambda)|^2} &\le 2\sup_{|\varphi(\lambda)|>1/2}\left(1-|\lambda|^2\right)\left|\left(\varphi'(\lambda)\right)^2 u(\lambda)\frac{\overline{\varphi(\lambda)}}{1-|\varphi(\lambda)|^2}\right| \\
&\le C\|f_a\|_* < \infty.
\end{aligned}
\tag{3.14}
$$

For all $\lambda \in D$ with $|\varphi(\lambda)| \le 1/2$, by (3.7), we have

$$\sup_{\lambda\in D}\frac{\left(1-|\lambda|^2\right)\left|u(\lambda)\left(\varphi'(\lambda)\right)^2\right|}{1-\left|\varphi(\lambda)\right|^2} \le \frac{4}{3}\sup_{\lambda\in D}\left(1-|\lambda|^2\right)\left|u(\lambda)\left(\varphi'(\lambda)\right)^2\right| < +\infty. \tag{3.15}$$

Hence (3.1) holds.

Next, we will show that (3.2) holds. Fix $a \in D$ with $|a| > 1/2$, we take another test functions:

$$g_a(z) = \frac{h(\overline{a}z)}{\overline{a}}\left(\log\frac{1}{1-|a|^2}\right)^{-1} \tag{3.16}$$

for $z \in D$. It is proved that $\sup_{1/2<|a|<1}\|g_a\|_* \le C$ above, where C is not dependent on a. Therefore, for all $\lambda \in D$ with $|\varphi(\lambda)| > 1/2$, we have

$$\begin{aligned}
C\left\|g_a\right\|_* &\ge \left\|uC_\varphi g_a\right\|_* \ge \sup_{z\in D}\left(1-|z|^2\right)\left|\left(uC_\varphi g_a\right)''(z)\right| \\
&= \sup_{z\in D}\left(1-|z|^2\right)\Big|\left(2\varphi'(z)u'(z)+\varphi''(z)u(z)\right)g_a'(\varphi(z)) \\
&\qquad\qquad + g_a''(\varphi(z))\left(\varphi'(z)\right)^2u(z)+u''(z)g_a(\varphi(z))\Big| \\
&= \sup_{z\in D}\left(1-|z|^2\right)\left|\left(2\varphi'(z)u'(z)+\varphi''(z)u(z)\right)\left(\log\frac{1}{1-\overline{a}\varphi(z)}\right)^2\left(\log\frac{1}{1-|a|^2}\right)^{-1}\right. \\
&\qquad\qquad + \frac{2\overline{a}}{1-\overline{a}\varphi(z)}\log\frac{1}{1-\overline{a}\varphi(z)}\left(\log\frac{1}{1-|a|^2}\right)^{-1}\left(\varphi'(z)\right)^2u(z) \\
&\qquad\qquad \left. + g_a(\varphi(z))u''(z)\right|.
\end{aligned} \tag{3.17}$$

Let $a = \varphi(\lambda)$, it follows that

$$\begin{aligned}
C\left\|g_a\right\|_* \ge \left(1-|\lambda|^2\right)&\left|\left(2\varphi'(\lambda)u'(\lambda)+\varphi''(\lambda)u(\lambda)\right)\left(\log\frac{1}{1-\left|\varphi(\lambda)\right|^2}\right)^2\left(\log\frac{1}{1-\left|\varphi(\lambda)\right|^2}\right)^{-1}\right. \\
&\left. + \frac{2\overline{\varphi(\lambda)}}{1-\left|\varphi(\lambda)\right|^2}\left(\varphi'(\lambda)\right)^2u(\lambda)+u''(\lambda)g_{\varphi(\lambda)}(\varphi(\lambda))\right|
\end{aligned}$$

$$\geq \left(1-|\lambda|^2\right)\left|(2\varphi'(\lambda)u'(\lambda)+\varphi''(\lambda)u(\lambda))\left(\log\frac{1}{1-|\varphi(\lambda)|^2}\right)\right|$$
$$-\left(1-|\lambda|^2\right)\frac{2|\varphi(\lambda)|}{1-|\varphi(\lambda)|^2}\left|(\varphi'(\lambda))^2u(\lambda)\right|-\left(1-|\lambda|^2\right)\left|u''(\lambda)g_{\varphi(\lambda)}(\varphi(\lambda))\right|. \tag{3.18}$$

Hence

$$\begin{aligned}\left(1-|\lambda|^2\right)\left|2\varphi'(\lambda)u'(\lambda)+\varphi''(\lambda)u(\lambda)\right|\log\frac{1}{1-|\varphi(\lambda)|^2} \leq{}& \left(1-|\lambda|^2\right)\frac{2|\varphi(\lambda)|}{1-|\varphi(\lambda)|^2}\left|(\varphi'(\lambda))^2u(\lambda)\right| \\ &+\left(1-|\lambda|^2\right)\left|u''(\lambda)g_{\varphi(\lambda)}(\varphi(\lambda))\right| \\ &+C\|g_a\|_*.\end{aligned} \tag{3.19}$$

By (3.1), Lemma 2.1, and the boundedness of the function $\varphi(z)$, we get

$$\begin{aligned}&\sup_{|\varphi(\lambda)|>1/2}\left(1-|\lambda|^2\right)\left|2\varphi'(\lambda)u'(\lambda)+\varphi''(\lambda)u(\lambda)\right|\log\frac{1}{1-|\varphi(\lambda)|^2} \\ &\leq \sup_{|\varphi(\lambda)|>1/2}\left(1-|\lambda|^2\right)\frac{2}{1-|\varphi(\lambda)|^2}\left|(\varphi'(\lambda))^2u(\lambda)\right|+\sup_{|\varphi(\lambda)|>1/2}\|u\|_{\mathcal{Z}}\|g_{\varphi(\lambda)}\|_*+C\|g_a\|_*<\infty.\end{aligned} \tag{3.20}$$

For all $\lambda\in D$ with $|\varphi(\lambda)|\leq 1/2$, by (3.5), we have

$$\begin{aligned}&\sup_{|\varphi(\lambda)|\leq 1/2}\left(1-|\lambda|^2\right)\left|2\varphi'(\lambda)u'(\lambda)+\varphi''(\lambda)u(\lambda)\right|\log\frac{1}{1-|\varphi(\lambda)|^2} \\ &\leq \log\frac{4}{3}\sup_{|\varphi(\lambda)|\leq 1/2}\left(1-|\lambda|^2\right)\left|2\varphi'(\lambda)u'(\lambda)+\varphi''(\lambda)u(\lambda)\right|<\infty.\end{aligned} \tag{3.21}$$

Hence (3.2) holds.

Conversely, suppose that $u\in\mathcal{Z}$, (3.1) and (3.2) hold. For $f\in\mathcal{Z}$, by Lemma 2.1, we have the following inequality:

$$\begin{aligned}\left(1-|z|^2\right)\left|(uC_\varphi f)''(z)\right| = \left(1-|z|^2\right)\Big|&(2\varphi'(z)u'(z)+\varphi''(z)u(z))f'(\varphi(z)) \\ &+f''(\varphi(z))(\varphi'(z))^2u(z)+u''(z)f(\varphi(z))\Big|\end{aligned}$$

$$
\begin{aligned}
&\leq \left(1-|z|^2\right)\left|\left(2\varphi'(z)u'(z)+\varphi''(z)u(z)\right)f'(\varphi(z))\right| \\
&\quad + \left(1-|z|^2\right)\left|f''(\varphi(z))\left(\varphi'(z)\right)^2 u(z)\right| + \left(1-|z|^2\right)\left|u''(z)f(\varphi(z))\right| \\
&\leq \left(1-|z|^2\right)\left|2\varphi'(z)u'(z)+\varphi''(z)u(z)\right| \log\frac{e}{1-|\varphi(z)|^2}\|f\|_* \\
&\quad + \frac{\left(1-|z|^2\right)\left|\left(\varphi'(z)\right)^2 u(z)\right|}{1-|\varphi(z)|^2}\left(1-|\varphi(z)|^2\right)\left|f''(\varphi(z))\right| \\
&\quad + \left(1-|z|^2\right)\left|u''(z)\right|\|f\|_* \\
&\leq \left(1-|z|^2\right)\left|2\varphi'(z)u'(z)+\varphi''(z)u(z)\right| \log\frac{e}{1-|\varphi(z)|^2}\|f\|_* \\
&\quad + \frac{\left(1-|z|^2\right)\left|\left(\varphi'(z)\right)^2 u(z)\right|}{1-|\varphi(z)|^2}\|f\|_{\mathcal{Z}} + \|u\|_{\mathcal{Z}}\|f\|_* \\
&\leq C\|f\|_*.
\end{aligned}
\tag{3.22}
$$

This shows that uC_φ is bounded. This completes the proof of Theorem 3.1. □

Theorem 3.2. *Let u be an analytic function on the unit disc D and φ an analytic self-map of D. Then uC_φ is bounded on the little Zygmund space $\mathcal{Z}_0$ if and only if $u \in \mathcal{Z}_0$,* (3.1) *and* (3.2) *hold, and the following are satisfied:*

$$\lim_{|z|\to 1^-}\left(1-|z|^2\right)\left|u(z)\left(\varphi'(z)\right)^2\right| = 0; \tag{3.23}$$

$$\lim_{|z|\to 1^-}\left(1-|z|^2\right)\left|2\varphi'(z)u'(z)+\varphi''(z)u(z)\right| = 0. \tag{3.24}$$

Proof. Suppose that uC_φ is bounded on the little Zygmund space $\mathcal{Z}_0$. Then $u = uC_\varphi 1 \in \mathcal{Z}_0$. Also $u\varphi = uC_\varphi z \in \mathcal{Z}_0$, thus

$$\left(1-|z|^2\right)\left|2\varphi'(z)u'(z)+\varphi''(z)u(z)+\varphi(z)u''(z)\right| \longrightarrow 0 \quad \left(|z| \longrightarrow 1^-\right). \tag{3.25}$$

Since $|\varphi| \leq 1$ and $u \in \mathcal{Z}_0$, we have $\lim_{|z|\to 1^-}(1-|z|^2)|2\varphi'(z)u'(z)+\varphi''(z)u(z)| = 0$. Hence (3.24) holds.

Similarly, $uC_\varphi z^2 \in \mathcal{Z}_0$, then

$$\left(1-|z|^2\right)\left|4\varphi(z)\varphi'(z)u'(z)+\varphi^2(z)u''(z)+2u(z)\left(\varphi(z)\varphi''(z)+\left(\varphi'(z)\right)^2\right)\right| \longrightarrow 0 \quad \left(|z| \longrightarrow 1^-\right). \tag{3.26}$$

By (3.24), $|\varphi| \le 1$ and $u \in \mathcal{Z}_0$, we get that $\lim_{|z|\to 1^-}(1-|z|^2)|u(z)(\varphi'(z))^2| = 0$, that is, (3.23) holds.

On the other hand, by Lemma 2.3 and Theorem 3.1, we obtain that (3.1) and (3.2) hold.

Conversely, let

$$M_1 = \sup_{z\in D} \frac{\left(1-|z|^2\right)\left|u(z)(\varphi'(z))^2\right|}{1-|\varphi(z)|^2} < \infty;$$
$$M_2 = \sup_{z\in D}\left(1-|z|^2\right)\left|2\varphi'(z)u'(z)+\varphi''(z)u(z)\right| \log\frac{1}{1-|\varphi(z)|^2} < \infty. \tag{3.27}$$

For all $f \in \mathcal{Z}_0$, we have both $(1-|z|^2)|f''(z)| \to 0$ and $|f'(z)|/\log(1/(1-|z|^2)) \to 0$ as $|z| \to 1^-$ by (1.5). Since $u \in \mathcal{Z}_0$, given that $\epsilon > 0$, there is a $0<\delta<1$ such that $(1-|z|^2)|u''(z)| < \epsilon/3\|f\|_*$, $(1-|z|^2)|f''(z)| < \epsilon/3M_1$ and $|f(z)|/\log(1/(1-|z|^2)) < \epsilon/3M_2$ for all z with $\delta < |z| < 1$.

If $|\varphi(z)| > \delta$, it follows that

$$\begin{aligned}
\left(1-|z|^2\right)\left|(uC_\varphi f)''(z)\right| &= \left(1-|z|^2\right)\Big|\left(2\varphi'(z)u'(z)+\varphi''(z)u(z)\right)f'(\varphi(z)) \\
&\qquad + f''(\varphi(z))(\varphi'(z))^2u(z) + u''(z)f(\varphi(z))\Big| \\
&\le \left(1-|z|^2\right)\left|\left(2\varphi'(z)u'(z)+\varphi''(z)u(z)\right)f'(\varphi(z))\right| \\
&\quad + \left(1-|z|^2\right)\left|f''(\varphi(z))(\varphi'(z))^2u(z)\right| + \left(1-|z|^2\right)\left|u''(z)f(\varphi(z))\right| \\
&\le M_2\frac{|f(\varphi(z))|}{\log\left(1/\left(1-|\varphi(z)|^2\right)\right)} + M_1\left(1-|\varphi(z)|^2\right)|f''(\varphi(z))| \\
&\quad + \left(1-|z|^2\right)|u''(z)|\|f\|_* \\
&\le \frac{\epsilon}{3}+\frac{\epsilon}{3}+\frac{\epsilon}{3} = \epsilon.
\end{aligned} \tag{3.28}$$

We know that there exists a constant M_3 such that $|f(z)| \le M_3$, $|f'(z)| \le M_3$ and $|f''(z)| \le M_3$ for all $|z| \le \delta$.

If $|\varphi(z)| \le \delta$, it follows that

$$\begin{aligned}
\left(1-|z|^2\right)\left|(uC_\varphi f)''(z)\right| &= \left(1-|z|^2\right)\Big|\left(2\varphi'(z)u'(z)+\varphi''(z)u(z)\right)f'(\varphi(z)) \\
&\qquad + f''(\varphi(z))(\varphi'(z))^2u(z) + u''(z)f(\varphi(z))\Big| \\
&\le M_3\left(1-|z|^2\right)\left|2\varphi'(z)u'(z)+\varphi''(z)u(z)\right| \\
&\quad + M_3\left(1-|z|^2\right)\left|(\varphi'(z))^2u(z)\right| + M_3\left(1-|z|^2\right)|u''(z)|.
\end{aligned} \tag{3.29}$$

Thus, we conclude that $(1-|z|^2)|(uC_\varphi(f))''(z)| \to 0$ as $|z| \to 1^-$. Hence $uC_\varphi f \in \mathcal{Z}_0$ for all $f \in \mathcal{Z}_0$. On the other hand, uC_φ is bounded on $\mathcal{Z}$ by Theorem 3.1. Hence uC_φ is a bounded operator on the little Zygmund space $\mathcal{Z}_0$. □

The following corollary is just as Theorem 2.2 in [27].

Corollary 3.3. *Let φ be an analytic self-map of D. Then C_φ is a bounded operator on $\mathcal{Z}$ if and only if*

$$\sup_{z\in D} \frac{\left(1-|z|^2\right)\left|(\varphi'(z))^2\right|}{1-|\varphi(z)|^2} < \infty, \tag{3.30}$$

$$\sup_{z\in D}\left(1-|z|^2\right)|\varphi''(z)| \log \frac{1}{1-|\varphi(z)|^2} < \infty. \tag{3.31}$$

Corollary 3.4. *Let φ be an analytic self-map of D. Then C_φ is a bounded operator on $\mathcal{Z}_0$ if and only if $\varphi \in \mathcal{Z}_0$,* (3.30) *and* (3.31) *hold.*

Proof. By Theorem 3.2, C_φ is a bounded operator on $\mathcal{Z}_0$ if and only if $\varphi \in \mathcal{Z}_0$, $\lim_{|z|\to 1^-}(1-|z|^2)|(\varphi'(z))^2| = 0$, (3.30) and (3.31) hold. However, by (1.5), $\varphi \in \mathcal{Z}_0$ implies that $\lim_{|z|\to 1^-}(1-|z|^2)|(\varphi'(z))^2| = 0$. Then, C_φ is a bounded operator on $\mathcal{Z}_0$ if and only if $\varphi \in \mathcal{Z}_0$, (3.30) and (3.31) hold. □

4. Compactness of uC_φ

In order to prove the compactness of uC_φ on the Zygmund space $\mathcal{Z}$, we require the following lemmas.

Lemma 4.1. *Suppose that uC_φ be a bounded operator on $\mathcal{Z}$. Then uC_φ is compact if and only if for any bounded sequence $\{f_n\}$ in $\mathcal{Z}$ which converges to 0 uniformly on compact subsets of D, we have $\|uC_\varphi(f_n)\|_* \to 0$ as $n \to \infty$.*

The proof is similar to that of Proposition 3.11 in [10]. The details are omitted.

Lemma 4.2. *Let $\{f_n\}$ be a bounded sequence in $\mathcal{Z}$ which converges to 0 uniformly on compact subsets of D. Then $\lim_{n\to\infty}\sup_{z\in D}|f_n(z)| = 0$.*

Proof. Let $K = \sup_n \|f_n\|_* < \infty$. Given any $\varepsilon > 0$, there exist $0 < t < 1$ such that $(1-t)^{1/2} < \varepsilon$. If $t < |z| < 1$, by Lemma 2.1, it follows that

$$\begin{aligned} \left|f_n(z) - f_n\left(\frac{t}{|z|}z\right)\right| &= \left|\int_{t/|z|}^1 z f_n'(zt)dt\right| \le K\int_{t/|z|}^1 |z|\log\frac{e}{1-|zt|^2}dt \\ &\le 2e^{-1/2}K\int_{t/|z|}^1 \frac{|z|}{\left(1-|zt|^2\right)^{1/2}}dt \le Ke^{-1/2}(1-t)^{1/2} < Ke^{-1/2}\varepsilon, \end{aligned} \tag{4.1}$$

where we use the fact that $x^{1/2}\log(e/x) \le 2e^{-1/2}$ for all $x \in (0,1]$. Then

$$\sup_{t<|z|<1}\left|f_n(z)\right| \le Ke^{-1/2}\varepsilon + \sup_{|z|=t}\left|f_n(z)\right|. \tag{4.2}$$

Noting that $\{f_n\}$ converges to 0 uniformly on compact subsets of D, we get

$$\lim_{n\to\infty}\sup_{z\in D}\left|f_n(z)\right| \le \lim_{n\to\infty}\sup_{z\in D}\left(Ke^{-1/2}\varepsilon + \sup_{|z|\le t}\left|f_n(z)\right|\right) = Ke^{-1/2}\varepsilon. \tag{4.3}$$

Hence, $\lim_{n\to\infty}\sup_{z\in D}|f_n(z)| = 0$. □

Theorem 4.3. *Let u be an analytic function on the unit disc D and φ an analytic self-map of D. Suppose that uC_φ be a bounded operator on $\mathcal{Z}$. Then uC_φ is compact if and only if the following are satisfied:*

$$\begin{aligned}
&\text{(i)} \lim_{|\varphi(z)|\to 1^-} \frac{\left(1-|z|^2\right)\left|u(z)\left(\varphi'(z)\right)^2\right|}{1-\left|\varphi(z)\right|^2} = 0;\\
&\text{(ii)} \lim_{|\varphi(z)|\to 1^-}\left(1-|z|^2\right)\left|2\varphi'(z)u'(z)+\varphi''(z)u(z)\right|\log\frac{1}{1-\left|\varphi(z)\right|^2} = 0.
\end{aligned} \tag{4.4}$$

Proof. Suppose that uC_φ is compact on the Zygmund space $\mathcal{Z}$. Let $\{z_n\}$ be a sequence in D such that $|\varphi(z_n)| \to 1$ as $n \to \infty$. Without loss of generality, we may suppose that $|\varphi(z_n)| > 1/2$ for all n. We take the test functions:

$$f_n(z) = \frac{\overline{\varphi(z_n)}z-1}{\overline{\varphi(z_n)}}\left(\left(1+\log\frac{1}{1-\overline{\varphi(z_n)}z}\right)^2+1\right)\left(\log\frac{1}{1-\left|\varphi(z_n)\right|^2}\right)^{-1} - a_n, \tag{4.5}$$

where

$$a_n = \frac{\left|\varphi(z_n)\right|^2-1}{\overline{\varphi(z_n)}}\left(\left(1+\log\frac{1}{1-\left|\varphi(z_n)\right|^2}\right)^2+1\right)\left(\log\frac{1}{1-\left|\varphi(z_n)\right|^2}\right)^{-1} \tag{4.6}$$

such that $\lim_{n\to\infty}a_n = 0$. By a direct calculation, we may easily prove that $\{f_n\}$ converges to 0 uniformly on compact subsets of D. From the proof of Theorem 3.1, we see that $\sup_n\|f_n\|_* < \infty$. Then $\{f_n\}$ is a bounded sequence in $\mathcal{Z}$ which converges to 0 uniformly on compact subsets of D. Then $\lim_{n\to\infty}\|uC_\varphi(f_n)\|_* = 0$ by Lemma 4.1. Note that

$$f_n(\varphi(z_n)) = 0, \qquad f_n'(\varphi(z_n)) = \log\frac{1}{1-\left|\varphi(z_n)\right|^2}, \qquad f_n''(\varphi(z_n)) = \frac{2\overline{\varphi(z_n)}}{1-\left|\varphi(z_n)\right|^2}, \tag{4.7}$$

it follows that

$$
\begin{aligned}
\left\|uC_\varphi f_n\right\|_* &\geq \left\|uC_\varphi f_n\right\|_{\mathcal{Z}} \\
&\geq \left(1-|z_n|^2\right)\Big|\left(2u'(z_n)\varphi'(z_n)+\varphi''(z_n)u(z_n)\right)f_n'(\varphi(z_n)) \\
&\qquad +u(z_n)f_n''(\varphi(z_n))\left(\varphi'(z_n)\right)^2+u''(z_n)f_n(\varphi(z_n))\Big| \\
&= \left(1-|z_n|^2\right)\Big|\left(2u'(z_n)\varphi'(z_n)+\varphi''(z_n)u(z_n)\right)\log\frac{1}{1-\left|\varphi(z_n)\right|^2} \\
&\qquad +\left(\varphi''(z_n)\right)^2u(z_n)\frac{2\overline{\varphi(z_n)}}{1-\left|\varphi(z_n)\right|^2}\Big| \\
&\geq \left(1-|z_n|^2\right)\left|\left(2u'(z_n)\varphi'(z_n)+\varphi''(z_n)u(z_n)\right)\log\frac{1}{1-\left|\varphi(z_n)\right|^2}\right| \\
&\quad -\frac{2\left(1-|z_n|^2\right)\left|\overline{\varphi(z_n)}u(z_n)\left(\varphi''(z_n)\right)^2\right|}{1-\left|\varphi(z_n)\right|^2}.
\end{aligned}
\tag{4.8}
$$

Then

$$
\begin{aligned}
&\lim_{n\to\infty}\left(1-|z_n|^2\right)\left|\left(2u'(z_n)\varphi'(z_n)+\varphi''(z_n)u(z_n)\right)\log\frac{1}{1-\left|\varphi(z_n)\right|^2}\right| \\
&\quad= \lim_{n\to\infty}\left(1-|z_n|^2\right)\frac{2\left(1-|z_n|^2\right)\left|\overline{\varphi(z_n)}u(z_n)\left(\varphi''(z_n)\right)^2\right|}{1-\left|\varphi(z_n)\right|^2},
\end{aligned}
\tag{4.9}
$$

if one of these two limits exits.

On the other hand, let

$$
h_n(z)=\frac{h\left(\overline{\varphi(z_n)}z\right)}{\overline{\varphi(z_n)}}\left(\log\frac{1}{1-\left|\varphi(z_n)\right|^2}\right)^{-1}-\int_0^z\log^3\frac{1}{1-\overline{\varphi(z_n)}\omega}d\omega\left(\log\frac{1}{1-\left|\varphi(z_n)\right|^2}\right)^{-2},
\tag{4.10}
$$

so

$$
h_n'(z)=\left(\log\frac{1}{1-\overline{\varphi(z_n)}z}\right)^2\left(\log\frac{1}{1-\left|\varphi(z_n)\right|^2}\right)^{-1}-\log^3\frac{1}{1-\overline{\varphi(z_n)}z}\left(\log\frac{1}{1-\left|\varphi(z_n)\right|^2}\right)^{-2},
$$

$$h_n''(z) = \frac{2\overline{\varphi(z_n)}}{1-\overline{\varphi(z_n)}z}\log\frac{1}{1-\overline{\varphi(z_n)}z}\left(\log\frac{1}{1-|\varphi(z_n)|^2}\right)^{-1} - \frac{3\overline{\varphi(z_n)}}{1-\overline{\varphi(z_n)}z}\log^2\frac{1}{1-\overline{\varphi(z_n)}z}\left(\log\frac{1}{1-|\varphi(z_n)|^2}\right)^{-2}. \tag{4.11}$$

One may obtain that $h_n \rightrightarrows 0\,(n \to \infty)$ on compact subsets of D by a direct calculation and $\sup_n\|h_n\|_* \le C < \infty$ by the proof of Theorem 3.1. Consequently, $\{h_n\}$ is a bounded sequence in $\mathcal{Z}$ which converges to 0 uniformly on compact subsets of D. Then $\lim_{n\to\infty}\|uC_\varphi(h_n)\|_* = 0$ by Lemma 4.1. Note that $u \in \mathcal{Z}, h_n'(\varphi(z_n)) \equiv 0$ and $\lim_{n\to\infty}\sup_{z\in D}|h_n(z)| = 0$ by Lemma 4.2, it follows that

$$\begin{aligned} 0 \longleftarrow \|uC_\varphi h_n\|_* &\ge \|uC_\varphi h_n\|_{\mathcal{Z}} \\ &\ge \left(1-|z_n|^2\right)\left|u(z_n)h_n''(\varphi(z_n))(\varphi'(z_n))^2 + u''(z_n)h_n(\varphi(z_n))\right| \\ &\ge \left(1-|z_n|^2\right)\left|u(z_n)(\varphi'(z_n))^2\frac{|\varphi(z_n)|}{1-|\varphi(z_n)|^2}\right| - \left(1-|z_n|^2\right)\left|u''(z_n)h_n(z_n)\right| \\ &\longrightarrow \left(1-|z_n|^2\right)\frac{\left|u(z_n)(\varphi'(z_n))^2\right|}{1-|\varphi(z_n)|^2}, \end{aligned} \tag{4.12}$$

as $n \to \infty$. Then $\lim_{n\to\infty}(1-|z_n|^2)(|u(z_n)(\varphi'(z_n))^2|/(1-|\varphi(z_n)|^2)) = 0$. The proof of the necessary is completed.

Conversely, Suppose that (i) and (ii) hold. Let $\{f_n\}$ be a bounded sequence in $\mathcal{Z}$ which converges to 0 uniformly on compact subsets of D. Let $M = \sup_n\|f_n\|_* < +\infty$. We only prove $\lim_{n\to\infty}\|uC_\varphi(f_n)\|_* = 0$ by Lemma 4.1. This amounts to showing that

$$\begin{gathered} \sup_{w\in D}\left(1-|w|^2\right)\left|(2\varphi'(w)u'(w) + \varphi''(w)u(w))f_n'(\varphi(w))\right| \longrightarrow 0, \\ \sup_{w\in D}\left(1-|w|^2\right)\left|u(w)(\varphi'(w))^2 f_n''(\varphi(w))\right| \longrightarrow 0, \qquad \sup_{w\in D}\left(1-|w|^2\right)\left|u''(w)f_n(\varphi(w))\right| \longrightarrow 0. \end{gathered} \tag{4.13}$$

By Lemma 4.2 and uC_φ bounded on $\mathcal{Z}$, which implies that $u \in \mathcal{Z}$, then

$$\sup_{w\in D}\left(1-|w|^2\right)\left|u''(w)f_n(\varphi(w))\right| \le \|u\|_{\mathcal{Z}}\sup_{z\in D}\left|f_n(z)\right| \longrightarrow 0. \tag{4.14}$$

If $|\varphi(w)| \le r < 1$, by (3.5), then

$$\left(1-|w|^2\right)\left|(2\varphi'(w)u'(w) + \varphi''(w)u(w))f_n'(\varphi(w))\right| \le K_1\max_{|z|\le r}\left|f_n'(z)\right|. \tag{4.15}$$

If $|\varphi(w)| > r$, by Lemma 2.1, then

$$\begin{aligned}&\left(1-|w|^2\right)\left|\left(2\varphi'(w)u'(w)+\varphi''(w)u(w)\right)f_n'(\varphi(w))\right|\\&\quad\le M\left(1-|w|^2\right)\left|\left(2\varphi'(w)u'(w)+\varphi''(w)u(w)\right)\right|\log\frac{e}{1-|\varphi(w)|^2}.\end{aligned}\tag{4.16}$$

Thus,

$$\begin{aligned}&\sup_{w\in D}\left(1-|w|^2\right)\left|\left(2\varphi'(w)u'(w)+\varphi''(w)u(w)\right)f_n'(\varphi(w))\right|\\&\quad\le K_1\max_{|w|\le r}\left|f_n'(w)\right|+M\sup_{|\varphi(w)|>r}\left(1-|w|^2\right)\left|\left(2\varphi'(w)u'(w)+\varphi''(w)u(w)\right)\right|\log\frac{e}{1-|\varphi(w)|^2}.\end{aligned}\tag{4.17}$$

First, letting n tend to infinity and subsequently r increase to 1, one obtains that

$$\sup_{w\in D}\left(1-|w|^2\right)\left|\left(2\varphi'(w)u'(w)+\varphi''(w)u(w)\right)f_n'(\varphi(w))\right|\longrightarrow 0,\tag{4.18}$$

as $n\to\infty$. The third statement is proved similarly.

If $|\varphi(w)|\le r<1$, by (3.7), then

$$\left(1-|w|^2\right)\left|u(w)\left(\varphi'(w)\right)^2 f_n''(\varphi(w))\right|\le K_2\max_{|z|\le r}\left|f_n''(z)\right|.\tag{4.19}$$

If $|\varphi(w)| > r$, then

$$\left(1-|w|^2\right)\left|u(w)\left(\varphi'(w)\right)^2 f_n''(\varphi(w))\right|\le M\frac{\left(1-|w|^2\right)\left|u(w)\left(\varphi'(w)\right)^2\right|}{1-|\varphi(w)|^2}.\tag{4.20}$$

Thus,

$$\begin{aligned}\sup_{w\in D}\left(1-|w|^2\right)\left|u(w)\left(\varphi'(w)\right)^2 f_n''(\varphi(w))\right|&\le K_2\max_{|z|\le r}\left|f_n''(z)\right|\\&\quad+M\sup_{|\varphi(w)|>r}\frac{\left(1-|w|^2\right)\left|u(w)\left(\varphi'(w)\right)^2\right|}{1-|\varphi(w)|^2},\end{aligned}\tag{4.21}$$

which also implies that

$$\sup_{w\in D}\left(1-|w|^2\right)\left|u(w)\left(\varphi'(w)\right)^2 f_n''(\varphi(w))\right|\longrightarrow 0,\tag{4.22}$$

as $n\to\infty$. This completes the proof of Theorem 4.3. □

In order to prove the compactness of uC_φ on the little Zygmund space $\mathcal{Z}_0$, we require the following lemma.

Lemma 4.4. *Let $U \subset \mathcal{Z}_0$. Then U is compact if and only if it is closed, bounded, and satisfies*

$$\lim_{|z| \longrightarrow 1} \sup_{f \in U} \left(1 - |z|^2\right) \left| f''(z) \right| = 0. \tag{4.23}$$

The proof is similar to that of Lemma 1 in [6], we omit it.

Theorem 4.5. *Let u be an analytic function on the unit disc D and φ an analytic self-map of D. Then uC_φ is compact on the little Zygmund space $\mathcal{Z}_0$ if and only if $u \in \mathcal{Z}_0$ and the following are satisfied:*

$$\begin{aligned} &\text{(i)} \lim_{|z| \longrightarrow 1^-} \frac{\left(1 - |z|^2\right) \left| u(z) (\varphi'(z))^2 \right|}{1 - |\varphi(z)|^2} = 0; \\ &\text{(ii)} \lim_{|z| \to 1^-} \left(1 - |z|^2\right) \left|2\varphi'(z) u'(z) + \varphi''(z) u(z)\right| \log \frac{1}{1 - |\varphi(z)|^2} = 0. \end{aligned} \tag{4.24}$$

Proof. Assume that (i) and (ii) hold, and $u \in \mathcal{Z}_0$. By Theorem 3.2, we know that uC_φ is bounded on the little Zygmund space $\mathcal{Z}_0$. From (ii), we can show that

$$\lim_{|z| \to 1^-} \left(1 - |z|^2\right) \left|2\varphi'(z) u'(z) + \varphi''(z) u(z)\right| = 0. \tag{4.25}$$

Suppose that $f \in \mathcal{Z}_0$ with $\|f\|_* \le 1$. We obtain that

$$\begin{aligned} \left(1 - |z|^2\right) \left| (uC_\varphi f)''(z) \right| &\le \left(1 - |z|^2\right) \left| (2\varphi'(z) u'(z) + \varphi''(z) u(z)) f'(\varphi(z)) \right| \\ &\quad + \left(1 - |z|^2\right) \left| f''(\varphi(z)) (\varphi'(z))^2 u(z) \right| + \left(1 - |z|^2\right) \left| u''(z) f(\varphi(z)) \right| \\ &\le \left(1 - |z|^2\right) \left|2\varphi'(z) u'(z) + \varphi''(z) u(z)\right| \log \frac{e}{1 - |\varphi(z)|^2} \|f\|_* \\ &\quad + \frac{\left(1 - |z|^2\right) \left| (\varphi'(z))^2 u(z) \right|}{1 - |\varphi(z)|^2} \left(1 - |\varphi(z)|^2\right) \left| f''(\varphi(z)) \right| + \left(1 - |z|^2\right) |u''(z)| \|f\|_* \\ &\le \left(1 - |z|^2\right) \left|2\varphi'(z) u'(z) + \varphi''(z) u(z)\right| \left(1 + \log \frac{1}{1 - |\varphi(z)|^2}\right) \\ &\quad + \frac{\left(1 - |z|^2\right) \left| (\varphi'(z))^2 u(z) \right|}{1 - |\varphi(z)|^2} + \left(1 - |z|^2\right) |u''(z)|, \end{aligned} \tag{4.26}$$

thus,

$$\begin{aligned}
&\sup\left\{\left|\left(1-|z|^2\right)\left(uC_\varphi f\right)''(z)\right| : f\in\mathcal{Z}_0, \|f\|_*\le 1\right\} \\
&\le \left(1-|z|^2\right)\left|2\varphi'(z)u'(z)+\varphi''(z)u(z)\right|\left(1+\log\frac{1}{1-|\varphi(z)|^2}\right) \\
&\quad+\frac{\left(1-|z|^2\right)\left|\left(\varphi'(z)\right)^2u(z)\right|}{1-|\varphi(z)|^2}+\left(1-|z|^2\right)\left|u''(z)\right|,
\end{aligned} \tag{4.27}$$

and it follows that

$$\lim_{|z|\to 1^-}\sup\left\{\left|\left(1-|z|^2\right)\left(uC_\varphi f\right)''(z)\right| : f\in\mathcal{Z}_0, \|f\|_*\le 1\right\}=0, \tag{4.28}$$

hence, uC_φ is compact on $\mathcal{Z}_0$ by Lemma 4.1.

Conversely, suppose that uC_φ is compact on $\mathcal{Z}_0$.

First, it is obvious uC_φ is bounded on $\mathcal{Z}_0$, then by Theorem 3.2, we have $u\in\mathcal{Z}_0$ and that (3.24) holds. On the other hand, by Lemma 4.1 we have

$$\lim_{|z|\to 1^-}\sup\left\{\left|\left(1-|z|^2\right)\left(uC_\varphi f\right)''(z)\right| : f\in\mathcal{Z}_0, \|f\|_*\le M\right\}=0, \tag{4.29}$$

for some $M>0$.

Next, note that the proof of Theorem 3.1 and the fact that the functions given in (3.8) are in $\mathcal{Z}_0$ and have norms bounded independently of a, we obtain that

$$\lim_{|z|\to 1^-}\frac{\left(1-|z|^2\right)\left|u(z)\left(\varphi'(z)\right)^2\right|}{1-|\varphi(z)|^2}=0. \tag{4.30}$$

Similarly, note that the functions given in (3.16) are in $\mathcal{Z}_0$ and have norms bounded independently of a, we obtain that

$$\begin{aligned}
&\lim_{|z|\to 1^-}\left(1-|z|^2\right)\left|2\varphi'(z)u'(z)+\varphi''(z)u(z)\right|\log\frac{1}{1-|\varphi(z)|^2} \\
&\le C\lim_{|z|\to 1^-}\left(1-|z|^2\right)\left|\left(uC_\varphi g_a\right)''(z)\right|+\lim_{|z|\to 1^-}\left(1-|z|^2\right)\left|u''(z)\right|\|g_a\|_* \\
&\quad+\lim_{|z|\to 1^-}\left(1-|z|^2\right)\frac{2|\varphi(z)|}{1-|\varphi(z)|^2}\left|u(z)\left(\varphi'(z)\right)^2\right|,
\end{aligned} \tag{4.31}$$

for $|\varphi(z)|>1/2$. So by (4.30) and $u\in\mathcal{Z}_0$, it follows that

$$\lim_{|z|\to 1^-}\left(1-|z|^2\right)\left|2\varphi'(z)u'(z)+\varphi''(z)u(z)\right|\log\frac{1}{1-|\varphi(z)|^2}=0, \tag{4.32}$$

for $|\varphi(z)| > 1/2$. However, if $|\varphi(z)| \le 1/2$, by (3.24), we easily have

$$\begin{aligned}
&\lim_{|z|\to 1^-}\left(1-|z|^2\right)\left|2\varphi'(z)u'(z)+\varphi''(z)u(z)\right|\log\frac{1}{1-\left|\varphi(z)\right|^2} \\
&\le \log\frac{4}{3}\lim_{|z|\longrightarrow 1^-}\left(1-|z|^2\right)\left|2\varphi'(z)u'(z)+\varphi''(z)u(z)\right| = 0.
\end{aligned} \tag{4.33}$$

This completes the proof of Theorem 4.5. □

Corollary 4.6. *Let φ be an analytic self-map of D. Then C_φ is a compact operator on $\mathcal{Z}_0$ if and only if*

$$\begin{aligned}
&\lim_{|z|\longrightarrow 1^-}\frac{\left(1-|z|^2\right)\left|\left(\varphi'(z)\right)^2\right|}{1-\left|\varphi(z)\right|^2} = 0, \\
&\lim_{|z|\longrightarrow 1^-}\left(1-|z|^2\right)\left|\varphi''(z)\right|\log\frac{1}{1-\left|\varphi(z)\right|^2} = 0.
\end{aligned} \tag{4.34}$$

In the formulation of corollary, we use the notation M_u on $H(D)$ defined by $M_u f = uf$ for $f \in H(D)$.

Corollary 4.7. *Let u be an analytic function on the unit disc D. Then the pointwise multiplier M_u : $\mathcal{Z}(resp. \mathcal{Z}_0) \to \mathcal{Z}(resp. \mathcal{Z}_0)$ is a compact operator if and only if $u \equiv 0$.*

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