

Research Article

Dissipativity Analysis of Linear State/Input Delay Systems

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Received 4 December 2011; Revised 7 February 2012; Accepted 1 March 2012

Academic Editor: Gaston Mandata N'Guerekata

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This paper discusses dissipativity problem for system of linear state/input delay equations. Motivated by dissipativity theory of control systems, we choose a new quadratic supply rate. Using the concept of dissipativity, necessary and sufficient conditions for the linear state/input delay systems to be dissipative and exponentially dissipative are derived. The connection of dissipativity with stability is also considered. Finally, passivity and finite gain are explored, correspondingly. The positive-real and bounded-real lemmas are derived.

1. Introduction

Dissipativity theory is one of major issues in control system theory. It was developed by Willems [1] and further extended by Hill and Moylan [2–4]. And it has become one of the major approaches to study complex systems [1, 5, 6]. In some particular systems with certain physical meaning, for example, electrical network systems, physical system, and so forth, dissipativity, just as the name implies, shows that the energy stored inside the system is no more than energy supplied from outside the system. In the area of dynamical system theory, storage function and supply rate are introduced to express the generalized abstract energy, correspondingly.

For dissipative systems, the storage functions usually provide natural candidates for Lyapunov functions [6]. Therefore, in many cases, stability and stabilization problems can be solved once the dissipativity property is assured [7–14]. Dissipativity theory is applied to impulsive hybrid dynamical systems [8], discontinuous dynamical systems [9], and switched systems [10]. Also, dissipativity theory is an efficient tool for the analysis and

design of composite systems. Especially, it is used to deal with some problems of stability and stabilization for interconnected systems [5, 6, 15]. Passivity and finite gain are two special cases of dissipativity which are important in the stability analysis of dynamical systems. The fact that system is passive is equivalent to that system satisfies positive-real or Kalman-Yakubovich-Popov (KYP) condition [12]. Many versions of positive-real lemmas are given for nonlinear [16, 17] and linear systems [18, 19]. The results about \mathcal{L}_2 -gain are summarized in recent references [15, 20, 21]. In papers [22, 23], bounded real lemmas are presented and are applied to \mathcal{H}_∞ control.

On the other hand, time delay is often encountered in various practical systems [24]. The existence of delays is often one main reason of instability and poor performance of systems. A great number of results on the stability and stabilization problems of control systems with time delays have been reported in the literature (see [25, 26], and the references therein). Lyapunov-Krasovskii functional approach is usually used to cope with various problems [27]. Dissipative theory about time-delay systems is discussed in references [28–31]. Sufficient conditions that guarantee robustly stable and strictly passive for uncertain time-delay systems are derived [28]. Dissipativity of linear and nonlinear time-delay systems is, respectively, discussed in [29, 30]. Dissipative problems of singular time-delay systems are considered in [31]. Unlike the systems discussed in the literatures, we consider a special class of linear state/input delay systems. This paper considers the dissipative problem of linear state/input delay systems with a new supply rate and with a storage function substituted by Lyapunov-Krasovskii functional.

There is no systematic dissipativity theory about time-delay systems because of the existence of delays. We consider linear state/input delay systems in this paper. The cause is not only that it has a simply form but it contains state delay and input delay. Motivated by dissipative theory, the variants of supply rate contain not input and output but delay input. A new quadratic supply rate developed in our paper is a function of input, delay input, and output. Based on it, we give sufficient and necessary conditions of dissipativity and exponential dissipativity and show its relationship to dissipativity and stability of the system. Finally, we discuss passivity and \mathcal{L}_2 -gain and derive position-real lemma and bounded-real lemma for the system.

Notation.

\mathbf{R}^n denotes the n -dimensional Euclidean space, and $\|\cdot\|$ stands for the Euclidean norm of any vector. $\mathbf{R}^{n \times m}$ is the set of $n \times m$ real matrices, and $\lambda(\cdot)$, $\lambda_{\max}(\cdot)$, $\lambda_{\min}(\cdot)$ denote, respectively, the eigenvalues, and the maximum eigenvalues, the minimum eigenvalues of any square matrix. We use $W > 0$ to denote a symmetric positive-definite matrix. In a symmetric block matrix, the expression $*$ will be used to denote the submatrices that lie under the diagonal. Given a positive number $r > 0$, $\mathcal{C}_r = \mathcal{C}([0, r], \mathbf{R}^n)$ denotes the Banach space of continuous functions that maps the interval $[0, r]$ into \mathbf{R}^n with the topology of uniform convergence, and $\|\cdot\|_c$ denotes the norm of continuous function, that is, for $\phi \in \mathcal{C}_r$, $\|\phi\|_c = \sup_{0 \leq \tau \leq r} \|\phi(\tau)\|$. The space of square integrable functions is denoted by \mathcal{L}_2 . And, for any $g(t) \in \mathcal{L}_2[t_0, \infty)$, its norm is defined as

$$\|g(t)\| = \left[\int_{t_0}^{\infty} \|g(t)\|^2 dt \right]^{1/2}. \quad (1.1)$$

The rest of the paper is organized as follows. Problem formulation is presented in Section 2. We give some results about dissipativity analysis of linear state/input delay systems in Section 3. In Section 4, positive-real lemma and bounded-real lemma are derived. Finally, a brief conclusion is provided to summarize the paper.

2. Problem Formulation

Consider the continuous-time systems with a state/input delay

$$\begin{aligned}\dot{x}(t) &= A_1x(t) + A_2x(t-d) + B_1u(t) + B_2(t-d), \\ y(t) &= C_1x(t) + C_2(t-d), \\ x(\theta) &= \varphi(\theta), \quad \theta \in [t_0-d, t_0],\end{aligned}\tag{2.1}$$

where $x(t) \in \mathcal{X} \subset \mathbf{R}^n$ is the state vector, $x(t-d) = x_t \in \mathcal{C}_d$ is the delay state vector, $u(t) \in \mathcal{U} \subset \mathbf{R}^m$ is exogenous input, where \mathcal{U} is the set of admissible input, $y(t) \in \mathcal{Y} \subset \mathbf{R}^p$ is the measured output vector. A_1, A_2, B_1, B_2, C_1 , and C_2 are constant system matrices with appropriate dimensions. $d > 0$ is an unknown constant delay. $\varphi(t) \in \mathcal{C}_d$ which is a real-valued continuous function on $[t_0-d, t_0]$ with the topology of uniform convergence is the initial function. For any continuous initial function $\varphi(t)$ and any admissible input $u(t)$, there exists a unique solution $x(t, t_0, \varphi, u)$ to system (2.1) on $[t_0, \infty)$ which continuously depends on the initial data. If $t \geq t_0$, we denote the segment of trajectory by $x_t(\varphi, u)$

$$x_t(\varphi, u) : \theta \longrightarrow x(t-t_0+\theta, \varphi, u), \quad \theta \in [t_0-d, t_0].\tag{2.2}$$

Assume that a real-valued function w about input and output variants is the supply rate of system (2.1). We assume that for all admissible input and for any $t_0, t \in \mathbf{R}^+$, the output $y(t) = y(t, t_0, \varphi, u)$ of (2.1) is such that $w(t)$ satisfies the following:

$$\int_{t_0}^t w(\tau) d\tau < \infty, \quad \forall t > t_0.\tag{2.3}$$

Definition 2.1. The linear time-delay system (2.1) with supply rate w is said to be dissipative if there exists a nonnegative functional $S : \mathcal{X} \times \mathcal{C}_d \rightarrow \mathbf{R}^+$, called the storage functional, such that for all $t, t_0 \in \mathbf{R}^+$ ($t \geq t_0$), $(x, x_t) \in \mathcal{X} \times \mathcal{C}_d$, and $u \in \mathcal{U}$,

$$s(x, x_t) - s(x_0, x_{t_0}) \leq \int_{t_0}^t w(\tau) d\tau,\tag{2.4}$$

where $x = x(t, t_0, \varphi, u)$, and $w(t)$ is a function about input and output, with $y = y(t_0, \varphi, u)$.

Remark 2.2. The inequality (2.4) is called the dissipation inequality. The dissipation inequality implies that the increase in generalized system energy over a given time interval cannot exceed the generalized energy supply delivered to the system (2.1) during this time interval. Thus, some of the supplied generalized energy to the system is stored, and some is dissipative. The dissipated energy is nonnegative and equals the difference of the right-hand of dissipation inequality and the left-hand.

Remark 2.3. If using equality sign as a substitution for inequality sign in the dissipation inequality, then Definition 2.1 of dissipativity becomes the definition of losslessness. We call system (2.1) lossless with respect to supply rate w .

The next definition shows the notion of available storage, which is the maximum amount of storage energy which may be extracted from a dynamical system.

Definition 2.4. The available storage, s_a , of system (2.1) with supply rate w is the functional $s_a : \mathcal{X} \times \mathcal{C}_d \rightarrow \mathbf{R}$ defined by

$$s_a(x, x_t) = - \inf_{u \in \mathcal{U}, t \geq t_0} \left\{ \int_{t_0}^t w(\tau) d\tau \right\}. \quad (2.5)$$

Note that the available storage, whenever defined, is nonnegative, that is, for all $(x, x_t) \in \mathcal{X} \times \mathcal{C}_d$, $s_a(x, x_t) \geq 0$. The available storage plays an important role in determining whether a given system is dissipative or not. This is shown in the following proposition (see [1]).

Proposition 2.5. *The available storage, s_a , is finite for all $(x, x_t) \in \mathcal{X} \times \mathcal{C}_d$ if and only if system (2.1) with respect to w is dissipative. Moreover, any possible storage function s satisfies $0 \leq s_a \leq s$ and if s_a is continuous, then s_a itself is a possible storage function.*

Since supply rate is a function of input and output and system (2.1) contains delay input, the delay input should be considered as an argument of supply rate. The supply rate should be functional about input, delay input, and output, that is, $w(t) = w(u(t), u(t-d), y(t))$. Motivated by Reference [1, 2], we are interested in studying dissipativity of system (2.1) with the following new quadratic supply rate:

$$\begin{aligned} w(u(t), u(t-d), y(t)) = & y^T(t)T y(t) + 2y^T(t)S_1 u(t) + 2y^T(t)S_2 u(t-d) \\ & + u^T(t)R_1 u(t) + u^T(t-d)R_2 u(t-d) + 2u^T(t)R_3 u(t-d), \end{aligned} \quad (2.6)$$

where $T \in \mathbf{R}^{p \times p}$, $S_i \in \mathbf{R}^{p \times m}$ ($i = 1, 2$), and $R_j \in \mathbf{R}^{m \times m}$ ($j = 1, 2, 3$) are known constant matrices with T and R_i being symmetric. Specially, two important special cases of dissipative system (2.1) are the following.

- (i) When $w(u(t), u(t-d), y(t)) = y^T(t)u(t) + y^T(t)u(t-d)$, $m = p$, dissipative system (2.1) is also called passive system.
- (ii) When $w(u(t), u(t-d), y(t)) = \gamma^2(u^T(t)u(t) + u^T(t-d)u(t-d)) - y(t)y^T(t)$, $\gamma > 0$, dissipative system (2.1) is also called finite gain system.

Remark 2.6. Generally speaking, the storage functional s is nonnegative and needs not to satisfy differentiability. Lyapunov-Krasovskii functional $V(x, x_t)$ can be chosen instead of s because it is an energy functional representing the abstract storage energy of system. If choose $s = V(x, x_t)$, then the dissipation inequality becomes

$$V(x, x_t) - V(x_0, x_{t_0}) \leq \int_{t_0}^t w(\tau) d\tau. \quad (2.7)$$

If functional V is continuous differentiable, then the integral version of dissipation inequality (2.7) can be transformed into a differential version. Differentiating both sides of (2.7), we can get the following:

$$\frac{d}{dt}V(x, x_t) \leq w(u(t), u(t-d), y(t)). \quad (2.8)$$

In this paper, we assume that s has the following form of Lyapunov-Krasovskii functional:

$$V(x_t) = x^T(t)Px(t) + \int_{t-d}^t x^T(\tau)Qx(\tau)d\tau, \quad (2.9)$$

where P, Q are positive-definite matrices.

Before concluding this section, we introduce two definitions which will be used in the development of our results.

Definition 2.7. The linear time-delay system (2.1) is reachable from the origin, if for any given x_1 and t_1 , there exist $t_2 \leq t_1$ and an admissible control $u(t) \in \mathcal{U}$ such that the state can be driven from $x(t_1) = x_1$ to $x(t_2) = 0$.

Remark 2.8. We suppose that the system (2.1) is reachable from the origin.

Definition 2.9. The linear time-delay system (2.1) with $u(t) \equiv 0$ and $u(t-d) \equiv 0$ is zero-state detectable, if for any trajectory $x(t, t_0, \varphi, 0)$, there holds

$$y \equiv 0 \implies \lim_{t \rightarrow \infty} x(t, t_0, \varphi, 0) = 0. \quad (2.10)$$

3. Dissipativity Analysis

Notice that dissipativity defined according to the Definition 2.1 represents an input-output property of the system. The following theorem, which is the central result of our paper, shows that dissipativity can also be characterized in terms of the coefficient matrices of system (2.1).

Theorem 3.1. *System (2.1) is said to be dissipative with respect to supply rate (2.6) if and only if there exist positive definite matrices P, Q and constant matrices $L_i \in \mathbf{R}^{n \times n}, W_i \in \mathbf{R}^{n \times m}$ ($i = 1, 2$) such that*

$$\begin{aligned} L_1^T L_1 &= -PA_1 - A_1^T P - Q + C_1^T T C_1, & L_2^T L_2 &= Q + C_2^T T C_2, \\ L_1^T L_2 &= -PA_2 + C_1^T T C_2, & L_1^T W_i &= -PB_i + C_1^T S_i, \\ L_2^T W_i &= C_2^T S_i, & W_1^T W_2 &= R_3, & W_i^T W_i &= R_i. \end{aligned} \quad (3.1)$$

Proof

Sufficiency. Suppose that $V(x_t)$ of form (2.9), L_i and W_i are given such that (3.1) is satisfied. Then, for any admissible control u and any t_0 and φ , we consider the following performance index:

$$\mathcal{J} = \frac{d}{dt}V(x_t) - w(t). \quad (3.2)$$

Using (3.1) and (2.9), we can get the following:

$$\begin{aligned}
\mathcal{Q} &= \frac{d}{dt}V(x_t) - \left[y^T(t)Ty(t) + 2y^T(t)S_1u(t) + 2y^T(t)S_2u(t-d) \right. \\
&\quad \left. + u^T(t)R_1u(t) + u^T(t-d)R_2u(t-d) \right] \\
&= x^T(t) \left(PA_1 + A_1^T P \right) x(t) + 2x^T(t)PA_2x(t-d) + 2x^T(t)PB_1u(t) \\
&\quad + 2x^T(t)PB_2u(t-d) + x^T(t)Qx(t) - x^T(t-d)Qx(t-d) \\
&\quad - x^T(t)C_1^TTC_1x(t) - 2x^T(t)C_1^TTC_2x(t-d) - x^T(t-d)C_2^TTC_2x(t-d) \\
&\quad - 2x^T(t)C_1^TS_1u(t) - 2x^T(t-d)C_2^TS_2u(t) - 2x^T(t)C_1^TS_2u(t-d) \\
&\quad - 2x^T(t-d)C_2^TS_2u(t-d) - u^T(t)R_1u(t) - u^T(t-d)R_2u(t-d) \\
&= x^T(t) \left(PA_1 + A_1^T P + Q - C_1^TTC_1 \right) x(t) + 2x^T(t) \left(PA_2 - C_1^TTC_2 \right) x(t-d) \\
&\quad + 2x^T(t) \left(PB_1 - C_1^TS_1 \right) u(t) + 2x^T(t) \left(PB_2 - C_2^TS_1 \right) u(t-d) \\
&\quad + x^T(t-d) \left(-Q - C_2^TTC_2 \right) x(t-d) + 2x^T(t-d) \left(-C_2^TS_1 \right) u(t) \\
&\quad + 2x^T(t-d) \left(-C_2^TS_2 \right) u(t-d) + u^T(t) \left(-R_1 \right) u(t) \\
&\quad + 2u^T(t-d) \left(-R_2 \right) u(t-d) \\
&\quad + 2u^T(t) \left(-R_3 \right) u(t-d) \\
&\leq - \left(L_1x(t) + L_2x(t-d) + W_1u(t) + W_2u(t-d) \right)^T \left(L_1x(t) + L_2x(t-d) \right. \\
&\quad \left. + W_1u(t) + W_2u(t-d) \right) \\
&\leq 0.
\end{aligned} \tag{3.3}$$

Obviously,

$$\frac{d}{dt}V(x_t) \leq w(t). \tag{3.4}$$

Integrating both sides of (3.4) from t_0 to t , it yields

$$V(x_t) - V(x_{t_0}) \leq \int_{t_0}^t w(\tau) d\tau. \tag{3.5}$$

Necessity. Suppose that system (2.1) is dissipative with respect to the supply rate (2.6). We will show that the available storage, V_a , is a solution of (3.1) for some appropriate matrices

L_i and W_i . Due to (2.6) and Definition 2.4, it follows from Proposition 2.5 that the available storage V_a of system (2.1) is finite for all $x_t \in \mathcal{C}_d$, $V_a(0) = 0$ and satisfies that

$$V_a(x_t) - V_a(x_{t_0}) \leq \int_{t_0}^t w(\tau) d\tau, \quad (3.6)$$

for all $t \geq t_0$ and $u \in \mathcal{U}$.

Differentiating both sides of the above inequality,

$$\frac{d}{dt} V_a(x_t) \leq w(u(t), u(t-d), y(t)). \quad (3.7)$$

Defining a functional $d : \mathcal{X} \times \mathcal{C}_d \times \mathcal{U} \rightarrow \mathbf{R}$ by

$$d(x(t), x(t-d), u(t), u(t-d)) = -\frac{d}{dt} V_a(x_t) + w(u(t), u(t-d), y(t)). \quad (3.8)$$

From (3.7), $d(x(t), x(t-d), u(t), u(t-d)) \geq 0$. In addition, it is known from (3.8) that $d(x(t), x(t-d), u(t), u(t-d))$ is quadratic in $x(t)$, $x(t-d)$, $u(t)$ and $u(t-d)$.

Thus there exist constant matrices $L_i \in \mathbf{R}^{n \times n}$, and $W_i \in \mathbf{R}^{n \times m}$ ($i = 1, 2$) such that

$$\begin{aligned} d(x(t), x(t-d), u(t), u(t-d)) &= (L_1 x(t) + L_2 x(t-d) + W_1 u(t) + W_2 u(t-d))^T \\ &\quad \cdot (L_1 x(t) + L_2 x(t-d) + W_1 u(t) + W_2 u(t-d)). \end{aligned} \quad (3.9)$$

Substituting (3.9) into (3.8), we have

$$\begin{aligned} &-\frac{d}{dt} V_a(x_t) + y^T(t) T y(t) + 2y^T(t) S_1 u(t) + 2y^T(t) S_2 u(t-d) \\ &\quad + u^T(t) R_1 u(t) + u^T(t-d) R_2 u(t-d) + 2u^T(t) R_3 u(t-d) \\ &= (L_1 x(t) + L_2 x(t-d) + W_1 u(t) + W_2 u(t-d))^T \\ &\quad \cdot (L_1 x(t) + L_2 x(t-d) + W_1 u(t) + W_2 u(t-d)) \end{aligned} \quad (3.10)$$

for all $(x, x_t) \in \mathcal{X} \times \mathcal{C}_d$ and $u(t) \in \mathcal{U}$.

Now, let $V(x_t) = V_a(x_t)$. Equating coefficients of equal powers, (3.1) is obtained.

Theorem 3.1 provides a dissipative criterion for system (2.1). Theorem 3.1 can be represented by linear matrix inequality (LMI). The Corollary 3.2 shows a LMI's version of dissipative criterion. \square

Corollary 3.2. System (2.1) is said to be dissipative with respect to supply rate (2.6) if and only if there exist positive definite matrices P and Q such that

$$\begin{pmatrix} \Sigma_1 & PA_2 - C_1^T TC_2 & PB_1 - C_1^T S_1 & PB_2 - C_1^T S_2 \\ * & -Q - C_2^T TC_2 & -C_2^T S_1 & -C_2^T S_2 \\ * & * & -R_1 & -R_3 \\ * & * & * & -R_2 \end{pmatrix} \leq 0, \quad (3.11)$$

where $\Sigma_1 = PA_1 + A_1^T P + Q - C_1^T TC_1$.

The theory of dissipative systems is used to investigate stability of system via Lyapunov methods. It is shown in the following theorem.

Theorem 3.3. Suppose that system (2.1) is dissipative with smooth storage functional (2.9) with respect to the supply rate (2.6) and zero-state detectable, then the unforced system $\dot{x}(t) = A_1 x(t) + A_2 x(t-d)$ is Lyapunov stable if $T \leq 0$ and asymptotically stable if $T < 0$.

Proof. Suppose that system (2.1) is dissipative, according to the proof part of Theorem 3.1, and (3.10) is satisfied. Let $u(t) = u(t-d) = 0$, we have

$$\frac{d}{dt}V(x_t) = -(L_1 x(t) + L_2 x(t-d))^T (L_1 x(t) + L_2 x(t-d)) + y^T(t)Ty(t). \quad (3.12)$$

If $T \leq 0$, then $(d/dt)V(x_t) \leq 0$. The unforced system is Lyapunov stable (see [22, 23]).

If $T < 0$, then $(d/dt)V(x_t) < 0$. By zero-state detectable and LaSalle's invariance principle (see [22, 23]), the unforced system is asymptotically stable. \square

Next, we give some results of exponential dissipativity. First, the definition of exponential dissipativity is represented.

Definition 3.4. The linear time-delay system (2.1) with supply rate w is said to be exponentially dissipative if there exists a nonnegative storage functional $S : \mathcal{X} \times \mathcal{C}_d \rightarrow \mathbf{R}^+$ and a constant $\varepsilon > 0$, such that for all $t, t_0 \in \mathbf{R}^+$ ($t \geq t_0$), $(x, x_t) \in \mathcal{X} \times \mathcal{C}_d$, and $u \in \mathcal{U}$,

$$e^{\varepsilon t} s(x, x_t) - e^{\varepsilon t_0} s(x_0, x_{t_0}) \leq \int_{t_0}^t e^{\varepsilon \tau} w(\tau) d\tau, \quad (3.13)$$

where $x = x(t, t_0, \varphi, u)$ and $w(t) = w(u(t), u(t-d), y(t))$, with $y = y(t_0, \varphi, u)$.

Remark 3.5. When we choose $\varepsilon = 0$, the definition of exponential dissipativity becomes the definition of dissipativity.

Assume that s has the following form of Lyapunov-Krasovskii functional:

$$V(x_t) = x^T(t)Px(t) + \int_{t-d}^t x^T(\tau)e^{\varepsilon(\tau-t)}Qx(\tau)d\tau, \quad (3.14)$$

where P, Q are positive definite matrices, and ε is a positive constant.

Theorem 3.6. System (2.1) is said to be exponentially dissipative with respect to supply rate (2.6) if and only if there exist positive definite matrices P , Q , constant matrices $L_i \in \mathbf{R}^{n \times n}$, $W_i \in \mathbf{R}^{n \times m}$ ($i = 1, 2$) and a constant $\varepsilon > 0$ such that

$$\begin{aligned} L_1^T L_1 &= -PA_1 - A_1^T P - \varepsilon P - Q + C_1^T T C_1, & L_2^T L_2 &= e^{-\varepsilon d} Q + C_2^T T C_2, \\ L_1^T L_2 &= -PA_2 + C_1^T T C_2, & L_1^T W_i &= -PB_i + C_1^T S_i, \\ L_2^T W_i &= C_2^T S_i, & W_1^T W_2 &= R_3, & W_i^T W_i &= R_i. \end{aligned} \quad (3.15)$$

Proof. Using (3.14), there is

$$\begin{aligned} \frac{d}{dt} \left[e^{\varepsilon t} V(x_t) - e^{\varepsilon t_0} V(x_{t_0}) - \int_{t_0}^t e^{\varepsilon \tau} w(\tau) d\tau \right] &= e^{\varepsilon t} \frac{d}{dt} V(x_t) + \varepsilon e^{\varepsilon t} V(x_t) - e^{\varepsilon t} w(t) \\ &= e^{\varepsilon t} \left[\frac{d}{dt} V(x_t) + \varepsilon V(x_t) - w(t) \right]. \end{aligned} \quad (3.16)$$

Let performance index be

$$\mathcal{J}_1 = \frac{d}{dt} V(x_t) + \varepsilon V(x_t) - w(t). \quad (3.17)$$

Note that $e^{\varepsilon t} \geq 1$. So if $\mathcal{J}_1 \leq 0$, we can directly obtain that

$$e^{\varepsilon t} \left[\frac{d}{dt} V(x_t) + \varepsilon V(x_t) - w(t) \right] \leq 0. \quad (3.18)$$

Substituting $(d/dt)V(x_t)$ and $V(x_t)$ into \mathcal{J}_1 , it yields the following:

$$\begin{aligned} \mathcal{J}_1 &= x^T(t) \left(PA_1 + A_1^T P \right) x(t) + 2x^T(t) PA_2 x(t-d) + 2x^T(t) PB_1 u(t) \\ &\quad + 2x^T(t) PB_2 u(t-d) + x^T(t) Q x(t) - x^T(t-d) e^{-\varepsilon d} Q x(t-d) \\ &\quad - \varepsilon \int_{t-d}^t x^T(\tau) e^{\varepsilon(\tau-t)} Q x(\tau) d\tau + \varepsilon x^T(t) P x(t) \\ &\quad + \varepsilon \int_{t-d}^t x^T(\tau) e^{\varepsilon(\tau-t)} Q x(\tau) d\tau - w(t) \\ &= x^T(t) \left(PA_1 + A_1^T P + Q + \varepsilon P \right) x(t) + 2x^T(t) PA_2 x(t-d) + 2x^T(t) PB_1 u(t) \\ &\quad + 2x^T(t) PB_2 u(t-d) - x^T(t-d) e^{-\varepsilon d} Q x(t-d) - w(t). \end{aligned} \quad (3.19)$$

The sufficiency and necessity proof is similar to the corresponding part of Theorem 3.1. For the sake of brevity, we omit it. \square

We show the result of Theorem 3.6 by LMI.

Corollary 3.7. System (2.1) is said to be exponentially dissipative with respect to supply rate (2.6) if and only if there exist positive definite matrices P , Q and a constant $\varepsilon > 0$ such that

$$\begin{pmatrix} \Sigma_2 & PA_2 - C_1^T TC_2 & PB_1 - C_1^T S_1 & PB_2 - C_1^T S_2 \\ * & -e^{-\varepsilon d} Q - C_2^T TC_2 & -C_2^T S_1 & -C_2^T S_2 \\ * & * & -R_1 & -R_3 \\ * & * & * & -R_2 \end{pmatrix} \leq 0, \quad (3.20)$$

where $\Sigma_2 = PA_1 + A_1^T P + \varepsilon P + Q - C_1^T TC_1$.

Similar to the result about the effect of dissipativity on the stability of unforced system, we will show that the exponentially dissipative system which satisfies that the property of zero-state detectability is exponentially stable.

Theorem 3.8. Suppose that system (2.1) is exponentially dissipative with smooth storage functional (3.14) with respect to the supply rate (2.6) and zero-state detectable, then the unforced system $\dot{x}(t) = A_1 x(t) + A_2 x(t-d)$ is exponentially stable if $T \leq 0$.

Proof. According to the conditions of Theorem 3.8 and the proof of Theorem 3.6, we can get the following:

$$e^{\varepsilon t} \left[\frac{d}{dt} V(x_t) + \varepsilon V(x_t) \right] = -(L_1 x(t) + L_2 x(t-d))^T (L_1 x(t) + L_2 x(t-d)) + e^{\varepsilon t} y^T(t) T y(t). \quad (3.21)$$

If $T \leq 0$, then

$$\frac{d}{dt} V(x_t(\varphi, 0)) + \varepsilon V(x_t(\varphi, 0)) \leq 0. \quad (3.22)$$

Integrating both sides from t_0 to t , it yields that

$$V(x_t(\varphi, 0)) \leq e^{-\varepsilon(t-t_0)} V(\varphi), \quad t \geq t_0. \quad (3.23)$$

From (3.14), we get the following inequality:

$$\alpha_1 \|x(t, \varphi, 0)\|^2 \leq V(x_t(\varphi, 0)) \leq \alpha_2 \|x_t(\varphi, 0)\|_C^2, \quad (3.24)$$

where $\alpha_1 = \lambda_{\min}(P)$ and $\alpha_2 = \lambda_{\max}(P) + d\lambda_{\max}(Q)$.

By (3.23) and (3.24), we have

$$\alpha_1 \|x(t, \varphi, 0)\|^2 \leq V(x_t(\varphi, 0)) \leq e^{-\varepsilon(t-t_0)} V(\varphi) \leq \alpha_2 e^{-\varepsilon(t-t_0)} \|\varphi\|_C^2. \quad (3.25)$$

Any solution $x(t, \varphi, 0)$ of unforced system satisfies that

$$\|x(t, \varphi, 0)\| \leq \sqrt{\frac{\alpha_2}{\alpha_1}} e^{-(1/2)\varepsilon(t-t_0)} \|\varphi\|_{\mathcal{C}}, \quad t \geq t_0. \quad (3.26)$$

Thus, the unforced system is exponentially stable. \square

4. Positive-Real and Bounded-Real Conditions

4.1. Positive-Real Lemma

When $m = p$, we shall focus on studying dissipative systems with supply rate given by the inner product of the following form;

$$w(u(t), u(t-d), y(t)) = y^T(t)u(t) + y^T(t)u(t-d). \quad (4.1)$$

For convenience, we characterize this choice by means of a separate definition.

Definition 4.1. The linear time-delay system (2.1) with supply rate (4.1) is said to be passive if it is dissipative with supply rate (4.1); that is, there exists a nonnegative storage functional $S : \mathcal{X} \times \mathcal{C}_d \rightarrow \mathbf{R}^+$, such that for all $t, t_0 \in \mathbf{R}^+$ ($t \geq t_0$), $(x, x_t) \in \mathcal{X} \times \mathcal{C}_d$, and $u \in \mathcal{U}$,

$$s(x, x_t) - s(x_0, x_{t_0}) \leq \int_{t_0}^t \left(y^T(\tau)u(\tau) + y^T(\tau)u(\tau-d) \right) d\tau. \quad (4.2)$$

The following theorem is positive-real lemma (KYP lemma) for linear time-delay system (2.1).

Theorem 4.2. *System (2.1) is said to be passive with respect to supply rate (4.1) if and only if there exist positive definite matrices P, Q and constant matrices $L_i \in \mathbf{R}^{n \times n}$ ($i = 1, 2$) such that*

$$\begin{aligned} L_1^T L_1 &= -PA_1 - A_1^T P - Q, & L_2^T L_2 &= Q, \\ L_1^T L_2 &= -PA_2, & -PB_i + \frac{1}{2}C_1^T &= 0, & C_2 &= 0. \end{aligned} \quad (4.3)$$

Proof. Let $T = R_1 = R_2 = R_3 = 0$, and $S_1 = S_2 = (1/2)I_n$ in Theorem 3.1. We can get $W_1 = W_2 = 0$. The result can be directly obtained. \square

We rewrite the positive-real condition of (4.3) in terms of LMI.

Corollary 4.3. System (2.1) is said to be passive with respect to supply rate (4.1) if and only if there exists positive definite matrices P and Q such that

$$\begin{pmatrix} \Sigma_3 & PA_2 & PB_1 - \frac{1}{2}C_1^T & PB_2 - \frac{1}{2}C_1^T \\ * & -Q & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix} \leq 0, \quad (4.4)$$

where $\Sigma_3 = PA_1 + A_1^T P + Q$.

4.2. Bounded-Real Lemma

In this subsection, we shall be particularly interested in the quadratic supply rate

$$w(u(t), u(t-d), y(t)) = \gamma^2 \left(u^T(t)u(t) + u^T(t-d)u(t-d) \right) - y(t)y^T(t), \quad (4.5)$$

where positive constant $\gamma > 0$ and $u(t) \in \mathcal{L}_2[t_0, \infty)$.

Definition 4.4. The system (2.1) is said to have finite \mathcal{L}_2 -gain from $u(t)$ to $y(t)$ less than or equal to $\gamma > 0$, if it is dissipative with supply rate (4.5): that is, there exists a nonnegative storage functional $s : \mathcal{X} \times \mathcal{C}_d \rightarrow \mathbf{R}^+$, such that for all $(t_0, t) \in [0, \infty)$ ($t \geq t_0$), $(x, x_t) \in \mathcal{X} \times \mathcal{C}_d$, and any $u(t) \in \mathcal{L}_2[t_0, t]$

$$s(x, x_t) - s(x_0, x_{t_0}) \leq \int_{t_0}^t \left[\gamma^2 \left(u^T(\tau)u(\tau) + u^T(\tau-d)u(\tau-d) \right) - y(\tau)y^T(\tau) \right] d\tau. \quad (4.6)$$

When $u(t) = u(t-d) = 0$, obviously, $s(x, x_t) - s(x_0, x_{t_0}) \leq -\int_{t_0}^t y(\tau)y^T(\tau) d\tau \leq 0$. Inequality (4.6) can be rewritten as

$$\int_{t_0}^t y(\tau)y^T(\tau) d\tau \leq \int_{t_0}^t \gamma^2 \left(u^T(\tau)u(\tau) + u^T(\tau-d)u(\tau-d) \right) d\tau. \quad (4.7)$$

The following theorem is bounded-real lemma for linear time-delay system (2.1).

Theorem 4.5. System (2.1) is said to have finite \mathcal{L}_2 -gain from $u(t)$ to $y(t)$ less than or equal to $\gamma > 0$ with respect to supply rate (4.5) if and only if there exist positive definite matrices P, Q and constant matrices $L_i \in \mathbf{R}^{n \times n}$, $W_i \in \mathbf{R}^{n \times m}$ ($i = 1, 2$) such that

$$\begin{aligned} L_1^T L_1 &= -PA_1 - A_1^T P - Q - C_1^T C_1, & L_2^T L_2 &= Q - C_2^T C_2, \\ L_1^T L_2 &= -PA_2 - C_1^T C_2, & L_1^T W_i &= -PB_i, \\ L_2^T W_i &= W_i^T W_2 = 0, & W_i^T W_i &= \gamma^2 I. \end{aligned} \quad (4.8)$$

Proof. The result can be directly obtained from the consequence of Theorem 3.1 with $T = -I$, $R_1 = R_2 = \gamma^2 I$, and $S_1 = S_2 = R_3 = 0$. \square

By LMI, Theorem 4.5 can be rewritten as the following corollary.

Corollary 4.6. *System (2.1) is said to have finite \mathcal{L}_2 -gain from $u(t)$ to $y(t)$ less than or equal to $\gamma > 0$ with respect to supply rate (4.5) if and only if there exist positive definite matrices P and Q such that*

$$\begin{pmatrix} \Sigma_3 & PA_2 + C_1^T C_2 & PB_1 & PB_2 \\ * & -Q + C_2^T C_2 & 0 & 0 \\ * & * & -\gamma^2 I & 0 \\ * & * & * & -\gamma^2 I \end{pmatrix} \leq 0, \quad (4.9)$$

where $\Sigma_4 = PA_1 + A_1^T P + Q - C_1^T C_1$.

Remark 4.7. The connections between passive property or finite gain property and unforced systems have not been shown in this section. They can be derived according to the results of Theorem 3.3 and Theorem 3.8.

5. Conclusion

This paper considers dissipativity theory for linear state/input delay systems. Motivated by reference [1, 2], we consider a new type quadratic supply rate for linear state/input delay systems. The choice of the new supply rate also depends on the case of system itself. Based on new supply rate, the necessary and sufficient conditions of dissipativity and exponential dissipativity are represented. And, stability results of dissipative linear state/input delay systems are derived. Finally, we focus on passive and finite \mathcal{L}_2 -gain systems and discuss positive-real and bounded-real condition, correspondingly. Positive-real lemma and bounded-real lemma based on new supply rate may help us to analyze the problem of stabilization using output or state feedback for the system with or without external disturbances, that is a future direction of research. Interconnected dissipative delay systems are also considered in future works.

Acknowledgments

This work is supported by the Natural Science Foundation of China (60874006), the Foundation of Henan Educational Committee (12A120008, 2008B120001), the Outstanding Youth Foundation of Henan Province (084100510010), and the 2012 Key Technologies R&D Program of Henan Scientific Committee (122102210061).

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