

Research Article

Approximation Theorems for Generalized Complex Kantorovich-Type Operators

N. I. Mahmudov and M. Kara

Eastern Mediterranean University, Gazimagusa, T.R. North Cyprus, Mersin 10, Turkey

Correspondence should be addressed to N. I. Mahmudov, nazim.mahmudov@emu.edu.tr

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The order of simultaneous approximation and Voronovskaja-type results with quantitative estimate for complex q -Kantorovich polynomials ($q > 0$) attached to analytic functions on compact disks are obtained. In particular, it is proved that for functions analytic in $\{z \in \mathbb{C} : |z| < R\}$, $R > q$, the rate of approximation by the q -Kantorovich operators ($q > 1$) is of order q^{-n} versus $1/n$ for the classical Kantorovich operators.

1. Introduction

For each integer $k \geq 0$, the q -integer $[k]_q$ and the q -factorial $[k]_q!$ are defined by

$$[k]_q := \begin{cases} \frac{1 - q^k}{1 - q}, & \text{if } q \in \mathbb{R}^+ \setminus \{1\}, \\ k, & \text{if } q = 1, \end{cases} \quad \text{for } k \in \mathbb{N}, [0]_q = 0, \quad (1.1)$$

$$[k]_q! := [1]_q [2]_q \cdots [k]_q \quad \text{for } k \in \mathbb{N}, [0]! = 1.$$

For integers $0 \leq k \leq n$, the q -binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q! [n - k]_q!}. \quad (1.2)$$

For fixed $1 \neq q > 0$, we denote the q -derivative $D_q f(z)$ of f by

$$D_q f(z) = \begin{cases} \frac{f(qz) - f(z)}{(q-1)z}, & z \neq 0, \\ f'(0), & z = 0. \end{cases} \quad (1.3)$$

Let \mathbb{D}_R be a disc $\mathbb{D}_R := \{z \in \mathbb{C} : |z| < R\}$ in the complex plane \mathbb{C} . Denote by $H(\mathbb{D}_R)$ the space of all analytic functions on \mathbb{D}_R . For $f \in H(\mathbb{D}_R)$ we assume that $f(z) = \sum_{m=0}^{\infty} a_m z^m$.

In several recent papers, convergence properties of complex q -Bernstein polynomials, proposed by Phillips [1], defined by

$$B_{n,q}(f; z) = \sum_{k=0}^n f\left(\frac{[k]_q}{[n]_q}\right) \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{j=0}^{n-k-1} (1 - q^j x) = \sum_{k=0}^n f\left(\frac{[k]_q}{[n]_q}\right) p_{n,k}(q; z) \quad (1.4)$$

and attached to an analytic function f in closed disks, were intensively studied by many authors; see [2] and references therein. It is known that the cases $0 < q < 1$ and $q > 1$ are not similar to each other. This difference is caused by the fact that, for $0 < q < 1$, $B_{n,q}$ are positive linear operators on $C[0, 1]$ while for $q > 1$, the positivity fails. The lack of positivity makes the investigation of convergence in the case $q > 1$ essentially more difficult than that for $0 < q < 1$. There are few papers [3–6] studying systematically the convergence of the q -Bernstein polynomials in the case $q > 1$. If $q \geq 1$ then qualitative Voronovskaja-type and saturation results for complex q -Bernstein polynomials were obtained in Wang and Wu [5]. Wu [6] studied saturation of convergence on the interval $[0, 1]$ for the q -Bernstein polynomials of a continuous function f for arbitrary fixed $q > 1$. On the other hand, Gal [7, 8], Anastassiou and Gal [9, 10], Mahmudov [11–13], and Mahmudov and Gupta [14] obtained quantitative estimates of the convergence and of the Voronovskaja's theorem in compact disks, for different complex Bernstein-Durrmeyer type operators.

The goal of the present note is to extend these type of results to complex Kantorovich operators based on the q -integers, in the case $q > 0$, defined as follows:

$$K_{n,q}(f; z) = \sum_{k=0}^n p_{n,k}(q; z) \int_0^1 f\left(\frac{q[k]_q + t}{[n+1]_q}\right) dt. \quad (1.5)$$

Notice that in the case $q = 1$, these operators coincide with the classical Kantorovich operators. For $0 < q \leq 1$ the operator $K_{n,q} : C[0, 1] \rightarrow C[0, 1]$ is positive and for $q > 1$, it is not positive. The problems studied in this paper in the case $q = 1$ were investigated in [2, 9].

We start with the following quantitative estimates of the convergence for complex q -Kantorovich-type operators attached to an analytic function in a disk of radius $R > 1$ and center 0.

Theorem 1.1. Let $f \in H(\mathbb{D}_R)$.

(i) Let $0 < q \leq 1$ and $1 \leq r < R$. For all $z \in \mathbb{D}_r$ and $n \in \mathbb{N}$, one has

$$|K_{n,q}(f; z) - f(z)| \leq \frac{3 + q^{-1}}{2[n]_q} \sum_{m=1}^{\infty} |a_m| m(m+1) r^m. \quad (1.6)$$

(ii) Let $1 < q < R < \infty$ and $1 \leq r < R/q$. For all $z \in \mathbb{D}_r$ and $n \in \mathbb{N}$, one has

$$|K_{n,q}(f; z) - f(z)| \leq \frac{2}{[n]_q} \sum_{m=1}^{\infty} |a_m| m(m+1) q^m r^m. \quad (1.7)$$

Remark 1.2. (i) Since $[n]_q \rightarrow (1-q)^{-1}$ as $n \rightarrow \infty$ in the estimate in Theorem 1.1(i) we do not obtain convergence of $K_{n,q}(f; z)$ to $f(z)$. But this situation can be improved by choosing $0 < q = q_n < 1$ with $q_n \nearrow 1$ as $n \rightarrow \infty$. Since in this case $[n]_{q_n} \rightarrow \infty$ as $n \rightarrow \infty$, from Theorem 1.1(i) we get uniform convergence in \mathbb{D}_r .

(ii) Theorem 1.1(ii) says that for functions analytic in \mathbb{D}_R , $R > q$, the rate of approximation by the q -Kantorovich operators ($q > 1$) is of order q^{-n} versus $1/n$ for the classical Kantorovich operators.

Let $f \in H(\mathbb{D}_R)$. Let us define

$$L_q(f; z) := \begin{cases} \frac{1-2z}{2} f'(z) + \frac{(1-z)(D_q f(z) - f'(z))}{1-q^{-1}}, & \text{if } |z| < \frac{R}{q}, R > q > 1, \\ \frac{1-2z}{2} f'(z) + \frac{z(1-z)}{2} f''(z), & \text{if } |z| < R, 0 < q \leq 1. \end{cases} \quad (1.8)$$

It is not difficult to show that

$$\begin{aligned} L_q(f; z) &= q(1-z) \sum_{m=1}^{\infty} a_m \frac{[m]_q - m}{q-1} z^{m-1} + \frac{1-2z}{2} \sum_{m=1}^{\infty} a_m m z^{m-1} \\ &= q \sum_{m=1}^{\infty} a_m ([1]_q + \dots + [m-1]_q) z^{m-1} (1-z) + \frac{1-2z}{2} \sum_{m=1}^{\infty} a_m m z^{m-1}, \quad q > 1. \end{aligned} \quad (1.9)$$

Here we used the identity

$$\frac{[m]_q - m}{q-1} = [1]_q + \dots + [m-1]_q. \quad (1.10)$$

The next theorem gives Voronovskaja-type result in compact disks, for complex q -Kantorovich operators attached to an analytic function in \mathbb{D}_R , $R > 1$ and center 0.

Theorem 1.3. Let $f \in H(\mathbb{D}_R)$.

(i) Let $0 < q \leq 1$ and $1 \leq r < R$. For all $z \in \mathbb{D}_r$ and $n \in \mathbb{N}$ one has

$$\left| K_{n,q}(f; z) - f(z) - \frac{1-2z}{2[n+1]_q} f'(z) - \frac{z(1-z)}{2[n+1]_q} f''(z) \right| \leq \frac{28+q^{-1}}{2[n]_q^2} \sum_{m=2}^{\infty} |a_m| m^2 (m-1)^2 r^m. \quad (1.11)$$

(ii) Let $1 < q < R < \infty$ and $1 \leq r < R/q^2$. For all $z \in \mathbb{D}_r$ and $n \in \mathbb{N}$, one has

$$\left| K_{n,q}(f; z) - f(z) - \frac{1}{[n+1]_q} L_q(f; z) \right| \leq \frac{14}{[n]_q^2} \sum_{m=2}^{\infty} |a_m| m^2 (m-1)^2 q^{2m} r^m. \quad (1.12)$$

Remark 1.4. (i) In the hypothesis on f in Theorem 1.3(i) choosing $0 < q_n < 1$ with $q_n \nearrow 1$ as $n \rightarrow \infty$, it follows that

$$\lim_{n \rightarrow \infty} [n+1]_{q_n} [K_{n,q_n}(f; z) - f(z)] = \frac{1-2z}{2} f'(z) + \frac{z(1-z)}{2} f''(z) \quad (1.13)$$

uniformly in any compact disk included in the open disk \mathbb{D}_R .

(ii) Theorem 1.3(ii) gives explicit formulas of Voronovskaja-type for the q -Kantorovich polynomials for $q > 1$.

(iii) Obviously the best order of approximation that can be obtained from the estimate Theorem 1.3(i) is $O(1/[n]_{q_n}^2)$ and $O(1/n^2)$ for $q = 1$, while the order given by Theorem 1.3(ii) is $O(1/q^{2n})$, $q > 1$, which is essentially better.

Next theorem shows that $L_q(f; z)$, $q \geq 1$, is continuous about the parameter q for $f \in H(\mathbb{D}_R)$, $R > 1$.

Theorem 1.5. Let $R > 1$ and $f \in H(\mathbb{D}_R)$. Then for any r , $0 < r < R$,

$$\lim_{q \rightarrow 1^+} L_q(f; z) = L_1(f; z) \quad (1.14)$$

uniformly on \mathbb{D}_r .

As an application of Theorem 1.3, we present the order of approximation for complex q -Kantorovich operators.

Theorem 1.6. Let $1 < q < R$, $1 \leq r < R/q^2$ (or $0 < q \leq 1$, $1 \leq r < R$) and $f \in H(\mathbb{D}_R)$. If f is not a constant function then the estimate

$$\|K_{n,q}(f) - f\|_r \geq \frac{1}{[n+1]_q} C_{r,q}(f), \quad n \in \mathbb{N} \quad (1.15)$$

holds, where the constant $C_{r,q}(f)$ depends on f , q and r but is independent of n .

2. Auxiliary Results

Lemma 2.1. Let $q > 0$. For all $n \in \mathbb{N}$, $m \in \mathbb{N} \cup \{0\}$, $z \in \mathbb{C}$ one has

$$K_{n,q}(e_m; z) = \sum_{j=0}^m \binom{m}{j} \frac{q^j [n]_q^j}{[n+1]_q^m (m-j+1)} B_{n,q}(e_j; z), \quad (2.1)$$

where $e_m(z) = z^m$.

Proof. The recurrence formula can be derived by direct computation.

$$\begin{aligned} K_{n,q}(e_m; z) &= \sum_{k=0}^n p_{n,k}(z) \sum_{j=0}^m \int_0^1 \binom{m}{j} \frac{q^j [k]_q^j t^{m-j}}{[n+1]_q^m} dt = \sum_{k=0}^n p_{n,k}(z) \sum_{j=0}^m \binom{m}{j} \frac{q^j [k]_q^j}{[n+1]_q^m (m-j+1)} \\ &= \sum_{j=0}^m \binom{m}{j} \frac{q^j [n]_q^j}{[n+1]_q^m (m-j+1)} \sum_{k=0}^n \frac{[k]_q^j}{[n]_q^j} p_{n,k}(z) \\ &= \sum_{j=0}^m \binom{m}{j} \frac{q^j [n]_q^j}{[n+1]_q^m (m-j+1)} B_{n,q}(e_j; z). \end{aligned} \quad (2.2)$$

□

Lemma 2.2. For all $z \in \mathbb{D}_r$, $r \geq 1$ one has

$$|K_{n,q}(e_m; z)| \leq r^m, \quad n, m \in \mathbb{N}. \quad (2.3)$$

Proof. Indeed, using the inequality $|B_{n,q}(e_j; z)| \leq r^j$ (see [3]), we get

$$\begin{aligned} |K_{n,q}(e_m; z)| &\leq \sum_{j=0}^m \binom{m}{j} \frac{q^j [n]_q^j}{[n+1]_q^m (m-j+1)} |B_{n,q}(e_j; z)| \\ &\leq \frac{1}{[n+1]_q^m} \sum_{j=0}^m \binom{m}{j} q^j [n]_q^j r^m = \left(\frac{1+q[n]_q}{[n+1]_q} \right)^m r^m = r^m. \end{aligned} \quad (2.4)$$

□

Lemma 2.3. For all $n, m \in \mathbb{N}$, $z \in \mathbb{C}$, $1 \neq q > 0$ one has

$$\begin{aligned} K_{n,q}(e_{m+1}; z) &= \frac{z(1-z)}{[n]_q} D_q K_{n,q}(e_m; z) + z K_{n,q}(e_m; z) \\ &\quad + \frac{1}{[n+1]_q^{m+1}} \sum_{j=0}^{m+1} \binom{m+1}{j} q^j [n]_q^j \frac{1}{(m-j+2)} \left(1 - \frac{j[n+1]_q}{q(m+1)[n]_q} \right) B_{n,q}(e_j; z). \end{aligned} \quad (2.5)$$

Proof. We know that (see [2])

$$\frac{z(1-z)}{[n]_q} D_q B_{n,q}(e_j; z) = B_{n,q}(e_{j+1}; z) - z B_{n,q}(e_j; z). \quad (2.6)$$

Taking the derivative of the formula (2.1) and using the above formula we have

$$\begin{aligned} \frac{z(1-z)}{[n]_q} D_q K_{n,q}(e_m; z) &= \sum_{j=0}^m \binom{m}{j} \frac{q^j [n]_q^j}{[n+1]_q^m (m-j+1)} \frac{z(1-z)}{[n]_q} D_q B_{n,q}(e_j; z) \\ &= \sum_{j=0}^m \binom{m}{j} \frac{q^j [n]_q^j}{[n+1]_q^m (m-j+1)} (B_{n,q}(e_{j+1}; z) - z B_{n,q}(e_j; z)) \\ &= \sum_{j=1}^{m+1} \binom{m}{j-1} \frac{q^{j-1} [n]_q^{j-1}}{[n+1]_q^m (m-j+2)} B_{n,q}(e_j; z) - z K_{n,q}(e_m; z). \end{aligned} \quad (2.7)$$

It follows that

$$\begin{aligned} K_{n,q}(e_{m+1}; z) &= \frac{z(1-z)}{[n]_q} D_q K_{n,q}(e_m; z) + z K_{n,q}(e_m; z) \\ &\quad + \sum_{j=0}^{m+1} \binom{m+1}{j} \frac{q^j [n]_q^j}{[n+1]_q^{m+1} (m-j+2)} B_{n,q}(e_j; z) \\ &\quad - \sum_{j=1}^{m+1} \binom{m}{j-1} \frac{q^{j-1} [n]_q^{j-1}}{[n+1]_q^m (m-j+2)} B_{n,q}(e_j; z) \\ &= \frac{z(1-z)}{[n]_q} D_q K_{n,q}(e_m; z) + z K_{n,q}(e_m; z) + \frac{1}{[n+1]_q^{m+1} (m+2)} \\ &\quad + \sum_{j=1}^{m+1} \binom{m+1}{j} \frac{q^{j-1} [n]_q^{j-1}}{[n+1]_q^m (m-j+2)} \frac{(m+1)q[n]_q - j[n+1]_q}{(m+1)[n+1]_q} B_{n,q}(e_j; z) \\ &= \frac{z(1-z)}{[n]_q} D_q K_{n,q}(e_m; z) + z K_{n,q}(e_m; z) \\ &\quad + \sum_{j=0}^{m+1} \binom{m+1}{j} \frac{q^{j-1} [n]_q^{j-1}}{[n+1]_q^m (m-j+2)} \frac{(m+1)q[n]_q - j[n+1]_q}{(m+1)[n+1]_q} B_{n,q}(e_j; z). \end{aligned} \quad (2.8)$$

Here we used the identity

$$\binom{m}{j-1} = \binom{m+1}{j} \frac{j}{(m+1)}. \quad (2.9)$$

□

For $m \in \mathbb{N} \cup \{0\}$ define

$$E_{n,m}(z) := \begin{cases} K_{n,q}(e_m; z) - e_m(z) - \frac{1-2z}{2[n+1]_q} m z^{m-1} - \frac{z(1-z)}{2[n+1]_q} m(m-1) z^{m-2}, & \text{if } 0 < q \leq 1, \\ K_{n,q}(e_m; z) - e_m(z) - \frac{1-2z}{2[n+1]_q} m z^{m-1} - \sum_{j=1}^{m-1} [j]_q \frac{q z^{m-1}(1-z)}{[n+1]_q}, & \text{if } q > 1. \end{cases} \quad (2.10)$$

Here it is assumed that $\sum_{j=1}^0 [j]_q = 0$.

Lemma 2.4. Let $n, m \in \mathbb{N}$.

(a) If $0 < q < 1$, one has the following recurrence formula:

$$\begin{aligned} E_{n,m}(z) &= \frac{z(1-z)}{[n]_q} D_q(K_{n,q}(e_{m-1}; z) - e_{m-1}(z)) + z E_{n,m-1}(z) \\ &+ \left(\frac{m-1}{[n+1]_q} - \frac{[m-1]_q}{[n]_q} \right) z^{m-1}(1-z) - \frac{1-2z}{2[n+1]_q} z^{m-1} \\ &+ \frac{1}{[n+1]_q^m} \sum_{j=0}^m \binom{m}{j} \frac{q^j [n]_q^j}{(m-j+1)} \frac{mq[n]_q - j[n+1]_q}{mq[n]_q} B_{n,q}(e_j; z). \end{aligned} \quad (2.11)$$

(b) If $q > 1$, one has

$$\begin{aligned} E_{n,m}(z) &= \frac{z(1-z)}{[n]_q} D_q(K_{n,q}(e_{m-1}; z) - e_{m-1}(z)) + z E_{n,m-1}(z) + \frac{[m-1]_q}{[n]_q [n+1]_q} z^{m-1}(1-z) \\ &- \frac{1-2z}{2[n+1]_q} z^{m-1} + \frac{1}{[n+1]_q^m} \sum_{j=0}^m \binom{m}{j} \frac{q^j [n]_q^j}{(m-j+1)} \frac{mq[n]_q - j[n+1]_q}{mq[n]_q} B_{n,q}(e_j; z). \end{aligned} \quad (2.12)$$

Proof. We give the proof for the case $q > 1$. The case $0 < q < 1$ is similar to that of $q > 1$.

(b) It is immediate that $E_{n,m}(z)$ is a polynomial of degree less than or equal to m and that $E_{n,0}(z) = E_{n,1}(z) = 0$.

Using the formula (2.5), we get

$$\begin{aligned}
 E_{n,m}(z) &= \frac{z(1-z)}{[n]_q} D_q(K_{n,q}(e_{m-1}; z) - e_{m-1}(z)) + \frac{[m-1]_q}{[n]_q} z^{m-1}(1-z) \\
 &\quad + z \left(E_{n,m-1}(z) + \frac{1-2z}{2[n+1]_q} (m-1)z^{m-2} + \sum_{j=1}^{m-2} [j]_q \frac{qz^{m-2}(1-z)}{[n+1]_q} \right) \\
 &\quad - \frac{1-2z}{2[n+1]_q} mz^{m-1} - \sum_{j=1}^{m-1} [j]_q \frac{qz^{m-1}(1-z)}{[n+1]_q} \\
 &\quad + \frac{1}{[n+1]_q^m} \sum_{j=0}^m \binom{m}{j} \frac{q^j [n]_q^j}{(m-j+1)} \frac{mq[n]_q - j[n+1]_q}{mq[n]_q} B_{n,q}(e_j; z).
 \end{aligned} \tag{2.13}$$

A simple calculation leads us to the following relationship:

$$\begin{aligned}
 E_{n,m}(z) &= \frac{z(1-z)}{[n]_q} D_q(K_{n,q}(e_{m-1}; z) - e_{m-1}(z)) + zE_{n,m-1}(z) + \frac{[m-1]_q}{[n]_q[n+1]_q} z^{m-1}(1-z) \\
 &\quad - \frac{1-2z}{2[n+1]_q} z^{m-1} + \frac{1}{[n+1]_q^m} \sum_{j=0}^m \binom{m}{j} \frac{q^j [n]_q^j}{(m-j+1)} \frac{mq[n]_q - j[n+1]_q}{mq[n]_q} B_{n,q}(e_j; z),
 \end{aligned} \tag{2.14}$$

which is the desired recurrence formula. \square

Remark 2.5. Lemmas 2.3 and 2.4 are true in the case $q = 1$. In the formulae, we have to replace q -derivative by the ordinary derivative.

3. Proofs of the Main Results

We give proofs for the case $q > 1$. The case $0 < q < 1$ and $q = 1$ are similar to that of $q > 1$.

Proof of Theorem 1.1. The use of the above recurrence we obtain the following relationship:

$$\begin{aligned}
 K_{n,q}(e_m; z) - e_m(z) &= \frac{z(1-z)}{[n]_q} D_q K_{n,q}(e_{m-1}; z) + z(K_{n,q}(e_{m-1}; z) - e_{m-1}(z)) \\
 &\quad + \frac{1}{[n+1]_q^m} \sum_{j=0}^m \binom{m}{j} \frac{q^j [n]_q^j}{(m-j+1)} \frac{mq[n]_q - j[n+1]_q}{mq[n]_q} B_{n,q}(e_j; z).
 \end{aligned} \tag{3.1}$$

We can easily estimate the sum in the above formula as follows:

$$\begin{aligned}
& \left| \frac{1}{[n+1]_q^m} \sum_{j=0}^m \binom{m}{j} q^j [n]_q^j \frac{1}{(m-j+1)} \left(1 - \frac{j}{m} - \frac{j}{mq[n]_q} \right) B_{n,q}(e_j; z) \right| \\
& \leq \frac{1}{[n+1]_q^m} \left(\sum_{j=0}^{m-1} \binom{m-1}{j} \frac{m}{m-j} \frac{q^j [n]_q^j}{m-j+1} \left| 1 - \frac{j}{m} - \frac{j}{mq[n]_q} \right| \right) |B_{n,q}(e_j; z)| + \frac{q^{m-1} [n]_q^{m-1}}{[n+1]_q^m} r^m \\
& \leq \frac{2m(q[n]_q + 1)^{m-1} + q^{m-1} [n]_q^{m-1}}{[n+1]_q^m} r^m \leq \frac{2m+1}{[n+1]_q} r^m.
\end{aligned} \tag{3.2}$$

It is known that by a linear transformation, the Bernstein inequality in the closed unit disk becomes

$$|P'_m(z)| \leq \frac{m}{qr} \|P_m\|_{qr}, \quad \forall |z| \leq qr, \quad r \geq 1, \tag{3.3}$$

(where $\|P_m\|_{qr} = \max\{|P_m(z)| : |z| \leq qr\}$) which combined with the mean value theorem in complex analysis implies

$$|D_q(P_m; z)| = \left| \frac{P_m(qz) - P_m(z)}{qz - z} \right| \leq \|P'_m\|_{qr} \leq \frac{m}{qr} \|P_m\|_{qr}, \tag{3.4}$$

for all $|z| \leq r$, where $P_m(z)$ is a complex polynomial of degree $\leq m$. From the above recurrence formula (3.1), we get

$$\begin{aligned}
|K_{n,q}(e_m; z) - e_m(z)| & \leq \frac{|z||1-z|}{[n]_q} |D_q K_{n,q}(e_{m-1}; z)| + |z| |K_{n,q}(e_{m-1}; z) - e_{m-1}(z)| + \frac{2m+1}{[n+1]_q} r^m \\
& \leq \frac{r(1+r)}{[n]_q} \frac{m-1}{qr} \|K_{n,q}(e_{m-1})\|_{qr} + r |K_{n,q}(e_{m-1}; z) - e_{m-1}(z)| + \frac{2m+1}{[n+1]_q} r^m \\
& \leq r |K_{n,q}(e_{m-1}; z) - e_{m-1}(z)| + \frac{2(m-1)}{[n]_q} q^{m-1} r^m + \frac{2m+1}{[n+1]_q} r^m \\
& \leq r |K_{n,q}(e_{m-1}; z) - e_{m-1}(z)| + \frac{4m}{[n]_q} q^m r^m.
\end{aligned} \tag{3.5}$$

By writing the last inequality for $m = 1, 2, \dots$, we easily obtain, step by step, the following:

$$\begin{aligned} |K_n(e_m; z) - e_m(z)| &\leq \frac{4m}{[n]_q} q^m r^m + r \frac{4(m-1)}{[n]_q} q^{m-1} r^{m-1} + r^2 \frac{4(m-2)}{[n]_q} q^{m-2} r^{m-2} + \dots + r^{m-1} \frac{4}{[n]_q} q r \\ &= \frac{4}{[n]_q} q^m r^m (m + m - 1 + \dots + 1) \leq \frac{2m(m+1)}{[n]_q} q^m r^m. \end{aligned} \quad (3.6)$$

Since $K_{n,q}(f; z)$ is analytic in \mathbb{D}_R , we can write

$$K_{n,q}(f; z) = \sum_{m=0}^{\infty} a_m K_{n,q}(e_m; z), \quad z \in \mathbb{D}_R, \quad (3.7)$$

which together with (3.6) immediately implies for all $|z| \leq r$

$$|K_{n,q}(f; z) - f(z)| \leq \sum_{m=0}^{\infty} |a_m| |K_{n,q}(e_m; z) - e_m(z)| \leq \frac{2}{[n]_q} \sum_{m=1}^{\infty} |c_m| m(m+1) (qr)^m. \quad (3.8) \quad \square$$

Proof of Theorem 1.3. A simple calculation and the use of the recurrence formula (2.5) lead us to the following relationship:

$$\begin{aligned} E_{n,m}(z) &= \frac{z(1-z)}{[n]_q} D_q(K_{n,q}(e_{m-1}; z) - e_{m-1}(z)) + zE_{n,m-1}(z) + \frac{[m-1]_q}{[n]_q [n+1]_q} z^{m-1} (1-z) \\ &\quad + \frac{1}{[n+1]_q} (z^m - B_{n,q}(e_m; z)) + \frac{1}{[n+1]_q} \left(1 - \frac{q^{m-1} [n]_q^{m-1}}{[n+1]_q^{m-1}} \right) B_{n,q}(e_m; z) \\ &\quad + \frac{1}{2[n+1]_q} \left(\frac{q^{m-1} [n]_q^{m-1}}{[n+1]_q^{m-1}} - 1 \right) B_{n,q}(e_{m-1}; z) + \frac{1}{2[n+1]_q} (B_{n,q}(e_{m-1}; z) - z^{m-1}) \\ &\quad - \frac{(m-1)q^{m-2} [n]_q^{m-2}}{2[n+1]_q^m} B_{n,q}(e_{m-1}; z) \\ &\quad + \frac{1}{[n+1]_q^m} \sum_{j=0}^{m-2} \binom{m}{j} \frac{q^j [n]_q^j}{(m-j+1)} \frac{mq[n]_q - j[n+1]_q}{mq[n]_q} B_{n,q}(e_j; z) \\ &:= \sum_{k=1}^9 I_k. \end{aligned} \quad (3.9)$$

Firstly, we estimate I_3, I_8 . It is clear that

$$|I_3| \leq \frac{[m-1]_q}{[n]_q [n+1]_q} r^{m-1} (1+r),$$

$$|I_8| \leq \frac{(m-1)}{2[n+1]_q^2} |B_{n,q}(e_{m-1}; z)| \leq \frac{(m-1)}{2[n+1]_q^2} r^{m-1}.$$
(3.10)

Secondly, using the known inequality

$$1 - \prod_{k=1}^m x_k \leq \sum_{k=1}^m (1 - x_k), \quad 0 \leq x_k \leq 1,$$
(3.11)

to estimate I_5, I_6, I_9 .

$$|I_5| \leq \frac{1}{[n+1]_q} \left(1 - \frac{q^{m-1} [n]_q^{m-1}}{[n+1]_q^{m-1}} \right) |B_{n,q}(e_m; z)| \leq \frac{m-1}{[n+1]_q^2} r^m,$$

$$|I_6| \leq \frac{1}{2[n+1]_q} \left(1 - \frac{q^{m-1} [n]_q^{m-1}}{[n+1]_q^{m-1}} \right) |B_{n,q}(e_{m-1}; z)| \leq \frac{m-1}{2[n+1]_q^2} r^{m-1},$$

$$|I_9| \leq \frac{1}{[n+1]_q^m} \sum_{j=0}^{m-2} \binom{m-2}{j} \frac{m(m-1)}{(m-j)(m-j-1)} \frac{q^j [n]_q^j}{(m-j+1)} \left(1 - \frac{j}{m} - \frac{j}{mq[n]_q} \right) r^j$$

$$\leq \frac{2m(m-1)[n+1]_q^{m-2}}{[n+1]_q^m} r^m = \frac{2m(m-1)}{[n+1]_q^2} r^m.$$
(3.12)

Finally, we estimate I_4, I_7 . We use [2, Theorem 1.1.2]

$$|I_4| + |I_7| \leq \frac{1}{[n+1]_q} |z^m - B_{n,q}(e_m; z)| + \frac{1}{2[n+1]_q} |B_n(e_{m-1}; z) - z^{m-1}|$$

$$\leq \frac{2[m-1]_q(m-1)}{[n]_q [n+1]_q} r^m + \frac{[m-2]_q(m-2)}{[n]_q [n+1]_q} r^{m-1}.$$
(3.13)

Using (3.6), (3.10), (3.12), and (3.13) in (3.9) finally we have ($m \geq 3$)

$$\begin{aligned}
|E_{n,m}(z)| &\leq \frac{r(1+r)}{[n]_q} |D_q(K_{n,q}(e_{m-1}; z) - e_{m-1}(z))| + r|E_{n,m-1}(z)| + \frac{[m-1]_q}{[n]_q[n+1]_q} r^{m-1}(1+r) \\
&\quad + \frac{2[m-1]_q(m-1)}{[n]_q[n+1]_q} r^m + \frac{m-1}{[n+1]_q^2} r^m + \frac{m-1}{2[n+1]_q^2} r^{m-1} + \frac{[m-2]_q(m-2)}{[n]_q[n+1]_q} r^{m-1} \\
&\quad + \frac{(m-1)}{2[n+1]_q^2} r^{m-1} + \frac{2m(m-1)}{[n+1]_q^2} r^m \\
&\leq \frac{r(1+r)}{[n]_q} \frac{m-1}{qr} \|K_{n,q}(e_{m-1}) - e_{m-1}\|_{qr} + r|E_{n,m-1}(z)| + \frac{10m[m-1]_q}{[n]_q^2} r^m \\
&\leq \frac{(m-1)(1+r)}{[n]_q} \frac{2(m-1)m}{[n]_q} q^{2(m-1)} r^{m-1} + r|E_{n,m-1}(z)| + \frac{10m[m-1]_q}{[n]_q^2} r^m \\
&\leq r|E_{n,m-1}(z)| + \frac{4m(m-1)^2}{[n]_q^2} q^{2m} r^m + \frac{10m(m-1)}{[n]_q^2} q^m r^m \\
&\leq r|E_{n,m-1}(z)| + \frac{14m^2(m-1)^2}{[n]_q^2} q^{2m} r^m.
\end{aligned} \tag{3.14}$$

As a consequence, we get

$$|E_{n,m}(z)| \leq \frac{14m^2(m-1)^2}{[n]_q^2} q^{2m} r^m. \tag{3.15}$$

This inequality combined with

$$\left| K_{n,q}(f; z) - f(z) - \frac{1}{[n+1]_q} L_q(f, z) \right| \leq \sum_{m=1}^{\infty} |a_m| |E_{n,m}(z)| \tag{3.16}$$

immediately implies the required estimate in statement.

Note that since $f^{(4)} = \sum_{m=4}^{\infty} a_m m(m-1)(m-2)(m-3)z^{m-4}$ and the series is absolutely convergent for all $|z| < R$, it easily follows the finiteness of the involved constants in the statement. \square

Proof of Theorem 1.6. For all $z \in \mathbb{D}_R$ and $n \in \mathbb{N}$, we get

$$K_{n,q}(f; z) - f(z) = \frac{1}{[n+1]_q} \left\{ L_q(f; z) + [n+1]_q \left(K_{n,q}(f; z) - f(z) - \frac{1}{[n+1]_q} L_q(f; z) \right) \right\}. \tag{3.17}$$

We apply

$$\|F + G\|_r \geq \| \|F\|_r - \|G\|_r \| \geq \|F\|_r - \|G\|_r \quad (3.18)$$

to get

$$\|K_{n,q}(f) - f\|_r \geq \frac{1}{[n+1]_q} \left\{ \|L_q(f; z)\|_r - [n+1]_q \left\| K_{n,q}(f; z) - f(z) - \frac{1}{[n+1]_q} L_q(f; z) \right\|_r \right\}. \quad (3.19)$$

Because by hypothesis f is not a constant in \mathbb{D}_R , it follows $\|L_q(f; z)\|_r > 0$. Indeed, assuming the contrary, it follows that $L_q(f; z) = 0$ for all $z \in \overline{\mathbb{D}}_R$ that is

$$\begin{aligned} \sum_{m=1}^{\infty} a_m \left(\frac{1}{2} - z \right) m z^{m-1} + \sum_{m=1}^{\infty} a_m \sum_{j=1}^{m-1} [j]_q z^{m-1} (1-z) &= 0, \\ \frac{1}{2} a_1 + a_1 + \sum_{m=1}^{\infty} \left(\frac{1}{2} (m+1) a_{m+1} - a_m + a_{m+1} \sum_{j=1}^m [j]_q - a_m \sum_{j=1}^{m-1} [j]_q \right) z^m &= 0 \end{aligned} \quad (3.20)$$

for all $z \in \overline{\mathbb{D}}_R \setminus \{0\}$. Thus $a_m = 0$, $m = 1, 2, 3, \dots$. Thus, f is constant, which is contradiction with the hypothesis.

Now, by Theorem 1.3, we have

$$\begin{aligned} [n+1]_q \left| K_{n,q}(f; z) - f(z) - \frac{1}{[n+1]_q} L_q(f; z) \right| \\ \leq \frac{[n+1]_q}{[n]_q} \frac{14}{[n]_q} \sum_{m=2}^{\infty} |a_m| m^2 (m-1)^2 q^{2m} r^m \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned} \quad (3.21)$$

Consequently, there exists n_1 (depending only on f and r) such that for all $n \geq n_1$ we have

$$\|L_q(f; z)\|_r - [n+1]_q \left\| K_{n,q}(f; z) - f(z) - \frac{1}{[n+1]_q} L_q(f; z) \right\|_r \geq \frac{1}{2} \|L_q(f; z)\|_r, \quad (3.22)$$

which implies

$$\|K_{n,q}(f) - f\|_r \geq \frac{1}{[n+1]_q} \frac{1}{2} \|L_q(f; z)\|_r, \quad \forall n \geq n_1. \quad (3.23)$$

For $1 \leq n \leq n_1 - 1$, we have

$$\|K_{n,q}(f) - f\|_r \geq \frac{1}{[n+1]_q} \left([n+1]_q \|K_{n,q}(f) - f\|_r \right) = \frac{1}{[n+1]_q} M_{r,n}(f) > 0, \quad (3.24)$$

which finally implies that

$$\|K_{n,q}(f) - f\|_r \geq \frac{1}{[n+1]_q} C_{r,q}(f), \quad (3.25)$$

for all n , with $C_{r,q}(f) = \min\{M_{r,1}(f), \dots, M_{r,n_1-1}(f), (1/2)\|L_q(f; z)\|_r\}$. □

Proof of Theorem 1.5. Proof is similar to that of Theorem 1.3 [5]. □

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