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Research Article

Strong Convergence Theorems for the Split Common Fixed Point Problem for Countable Family of Nonexpansive Operators

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We introduce a new iterative algorithm for solving the split common fixed point problem for countable family of nonexpansive operators. Under suitable assumptions, we prove that the iterative algorithm strongly converges to a solution of the problem.

1. Introduction

Let H_1 and H_2 be two real Hilbert spaces and let $A : H_1 \to H_2$ be a bounded linear operator. The split feasibility problem (SFP), see [1], is to find a point x^* with the property:

$$x^* \in C, \qquad Ax^* \in Q, \tag{1.1}$$

where C and Q are nonempty closed convex subsets of H_1 and H_2, respectively. A more general form of the SFP is the so-called multiple-set split feasibility problem (MSSFP) which was recently introduced by Censor et al. [2]. Given integers $p, r \geq 1$, the MSSFP is to find a point x^* with the property:

$$x^* \in \bigcap_{i=1}^{p} C_i, \qquad Ax^* \in \bigcap_{j=1}^{r} Q_j, \tag{1.2}$$

where $\{C_i\}_{i=1}^{p}$ and $\{Q_j\}_{j=1}^{r}$ are nonempty closed convex subsets of H_1 and H_2, respectively. The SFP (1.1) and the MSSFP (1.2) serve as a model for many inverse problems where constraints are imposed on the solutions in the domain of a linear operator as well as in this operator's ranges. Recently, the SFP (1.1) and the MSSFP (1.2) are widely applied in the image reconstructions [1, 3], the intensity-modulated radiation therapy [4, 5], and many other areas. The problems have been investigated by many researchers, for instance, [6–13]. The SFP (1.1) can be viewed as a special case of the convex feasibility problem (CFP) since the SFP (1.1) can be rewritten as

$$x^* \in C, \qquad x^* \in A^{-1}Q. \tag{1.3}$$

However, the methods for study the SFP (1.1) are actually different from those for the CFP in order to avoid the usage of the inverse A^{-1}. Byrne [6] introduced a so-called CQ algorithm:

$$x_{n+1} = P_C\left(x_n - \gamma A^*(I - P_Q)Ax_n\right), \quad n \geq 0, \tag{1.4}$$

where the operator A^{-1} is not relevant.

Censor and Segal in [14] firstly introduced the concept of the split common fixed point problem (SCFPP) in finite-dimensional Hilbert spaces. The SCFPP is a generalization of the convex feasibility problem (CFP) and the split feasibility problem (SFP). The SCFPP considers to find a common fixed point of a family of operators in H_1 such that its image under a linear transformation A is a common fixed point of another family of operators in H_2. That is, the SCFPP is to find a point x^* with the property:

$$x^* \in \bigcap_{i=1}^{p} \mathrm{Fix}(U_i), \qquad Ax^* \in \bigcap_{j=1}^{r} \mathrm{Fix}(T_j), \tag{1.5}$$

where $U_i : H_1 \to H_1$ $(i = 1, 2, \ldots, p)$ and $T_j : H_2 \to H_2$ $(j = 1, 2, \ldots, r)$ are nonlinear operators. If $p = r = 1$, the problem (1.5) deduces to the so-called two-set SCFPP, which is to find a point x^* such that

$$x^* \in \mathrm{Fix}(U), \qquad Ax^* \in \mathrm{Fix}(T), \tag{1.6}$$

where $U : H_1 \to H_1$ and $T : H_2 \to H_2$ are nonlinear operators.

Censor and Segal in [14] considered the following iterative algorithm for the SCFPP (1.6) for Class-$\mathfrak{T}$ operators in finite-dimensional Hilbert spaces:

$$x_{n+1} = U\left(x_n - \gamma A^*(I - T)Ax_n\right), \quad n \geq 0, \tag{1.7}$$

where $x_0 \in H_1$, $0 < \gamma < 2/\|A\|^2$ and I is the identity operator.

Recently, in the infinite-dimensional Hilbert space, Wang and Xu [15] studied the SCFPP (1.5) and introduced the following iterative algorithm for Class-$\mathfrak{T}$ operators:

$$x_{n+1} = U_{[n]}\left(x_n - \gamma A^*(I - T_{[n]})Ax_n\right), \quad n \geq 0 \tag{1.8}$$

where $[n] = n \bmod p$ and $p = r$. Under some mild conditions, they proved that $\{x_n\}$ converges weakly to a solution of the SCFPP (1.5), extended and improved Censor and Segal's results.

Moreover, they proved that the SCFPP (1.5) for the Class-$\mathfrak{T}$ operators is equivalent to a common fixed point problem. This is also a classical method. Many problems eventually converted to a common fixed point problem, see [16–18]. Very recently, the split common fixed point problems for various types of operators were studied in [19–21].

The above-mentioned results are about a finite number of operators; that is, the constraints are finite imposed on the solutions. In this paper, we consider the constraints are infinite, but countable. That is, we consider the generalized case of SCFPP for two countable families of operators (denoted GSCFPP), which is to find a point x^* such that

$$x^* \in \bigcap_{i=1}^{\infty} \mathrm{Fix}(U_i), \qquad Ax^* \in \bigcap_{j=1}^{\infty} \mathrm{Fix}(T_j). \tag{1.9}$$

Of course, the GSCFPP is more general and widely used than the SCFPP. This is a novelty of this paper. At the same time, we consider the nonexpansive operator. The nonexpansive operator is important because it includes many types of nonlinear operator arising in applied mathematics. For instance, the projection and the identity operator are nonexpansive. We prove that the GSCFPP (1.9) for the nonexpansive operators is equivalent to a common fixed point problem. Very recently, Gu et al. [22] introduced a new iterative method for dealing with the countable family of operators. They studied the following iterative algorithm:

$$\begin{aligned} y_n &= P_C\left[\beta_n S x_n + (1-\beta_n)x_n\right], \\ x_{n+1} &= P_C\left[\alpha_n f(x_n) + \sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)T_i y_n\right], \end{aligned} \tag{1.10}$$

where S and $\{T_i\}_{i=1}^{\infty}$ are nonexpansive, $\alpha_0 = 1$, $\{\alpha_n\}$ is strictly decreasing sequence in $(0,1)$, and $\{\beta_n\}$ is a sequence in $(0,1)$. Under some certain conditions on parameters, they proved that the sequence $\{x_n\}$ converges strongly to $x^* \in \bigcap_{i=1}^{\infty} F(T_i)$. On the other hand, from weakly convergence to strongly convergence, the viscosity approximation method is also one of the classical methods, see [22–24].

Motivated and inspired by the above results, we introduce the following algorithm:

$$x_{n+1} = \alpha_n f(x_n) + \sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)U_i\left(x_n + \gamma A^*(T_i - I)Ax_n\right). \tag{1.11}$$

Under some certain conditions, we prove that the sequence $\{x_n\}$ generated by (1.11) converges strongly to the solution of the GSCFPP (1.9).

2. Preliminaries

Throughout this paper, we write $x_n \rightharpoonup x$ and $x_n \to x$ to indicate that $\{x_n\}$ converges weakly to x and converges strongly to x, respectively.

Let H be a real Hilbert space. An operator $T : H \to H$ is said to be nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in H$. The set of fixed points of T is denoted by $\mathrm{Fix}(T)$. It is known that $\mathrm{Fix}(T)$ is closed and convex, see [25]. An operator $f : H \to H$ is called contraction if there exists a constant $\rho \in [0,1)$ such that $\|f(x) - f(y)\| \le \rho\|x - y\|$ for all $x, y \in H$. Let C be a nonempty closed convex subset of H. For each $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that $\|x - P_C x\| \le \|x - y\|$ for every $y \in C$. P_C is called the metric projection of H onto C. It is known that, for each $x \in H$,

$$\langle x - P_C x, y - P_C x \rangle \le 0 \tag{2.1}$$

for all $y \in C$.

In order to prove our main results, we collect the following lemmas in this section.

Lemma 2.1 (see [26]). *Let H be a Hilbert space, C a closed convex subset of H, and $T : C \to C$ a nonexpansive operator with $\mathrm{Fix}(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C weakly converging to $x \in C$ and $\{(I - T)x_n\}$ converges strongly to $y \in C$, then $(I - T)x = y$. In particular, if $y = 0$, then $x \in \mathrm{Fix}(T)$.*

Lemma 2.2 (see [23]). *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1 - \gamma_n) a_n + \delta_n, \quad n \ge 0, \tag{2.2}$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$,

(ii) $\limsup_{n\to\infty} \delta_n/\gamma_n \le 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3. Main Results

Now we state and prove our main results of this paper.

Theorem 3.1. *Let $\{U_n\}$ and $\{T_n\}$ be sequences of nonexpansive operators on real Hilbert spaces H_1 and H_2, respectively. Let $f : H_1 \to H_1$ be a contraction with coefficient $\rho \in [0,1)$. Suppose that the solution set Ω of GSCFPP* (1.9) *is nonempty. Let $x_1 \in H_1$ and $0 < \gamma < 2/\|A\|^2$. Set $\alpha_0 = 1$, and let $\{\alpha_n\} \subset (0,1]$ be a strictly decreasing sequence satisfying the following conditions:*

(i) $\lim_{n\to\infty} \alpha_n = 0$;

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

Then the sequence $\{x_n\}$ generated by (1.11) *converges strongly to $w \in \Omega$, where $w = P_\Omega f(w)$.*

Proof. We proceed with the following steps.
Step 1. First show that there exists $w \in \Omega$ such that $w = P_\Omega f(w)$.

In fact, since f is a contraction with coefficient ρ, we have

$$\|P_\Omega f(x) - P_\Omega f(y)\| \le \|f(x) - f(y)\| \le \rho\|x - y\| \tag{3.1}$$

for every $x, y \in H_1$. Hence $P_\Omega f$ is also a contraction of H_1 into itself. Therefore, there exists a unique $w \in H_1$ such that $w = P_\Omega f(w)$. At the same time, we note that $w \in \Omega$.

Step 2. Now we show that $\{x_n\}$ is bounded.

For simplicity, we set $V_i = I + \gamma A^*(T_i - I)A$. Then we can rewrite (1.11) to

$$x_{n+1} = \alpha_n f(x_n) + \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i) U_i V_i x_n. \tag{3.2}$$

Observe that

$$\begin{aligned} \|(T_i - I)Ax - (T_i - I)Ay\|^2 &= \|Ax - Ay\|^2 + \|T_iAx - T_iAy\|^2 - 2\langle Ax - Ay, T_iAx - T_iAy\rangle \\ &\le 2\|Ax - Ay\|^2 - 2\langle Ax - Ay, T_iAx - T_iAy\rangle \\ &= -2\langle Ax - Ay, (T_i - I)Ax - (T_i - I)Ay\rangle \end{aligned} \tag{3.3}$$

for all $x, y \in H_1$. Thus it follows that

$$\begin{aligned} \|V_ix - V_iy\| &= \|(I + \gamma A^*(T_i - I)A)x - (I + \gamma A^*(T_i - I)A)y\|^2 \\ &\le \|x - y\|^2 + \gamma^2\|A\|^2\|(T_i - I)Ax - (T_i - I)Ay\|^2 \\ &\quad + 2\gamma\langle Ax - Ay, (T_i - I)Ax - (T_i - I)Ay\rangle \\ &\le \|x - y\|^2 + \gamma\left(\gamma\|A\|^2 - 1\right)\|(T_i - I)Ax - (T_i - I)Ay\|^2. \end{aligned} \tag{3.4}$$

For $0 < \gamma < 1/\|A\|^2$, we can immediately obtain that V_i is a nonexpansive operator for every $i \in \mathbb{N}$.

Let $p \in \Omega$, then $U_ip = p$ and $T_iAp = Ap$ for every $i \ge 1$. Thus $(T_i - I)Ap = 0$, which implies that $V_ip = p$. Since $\sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i) = 1 - \alpha_n$, we have

$$\begin{aligned} \|x_{n+1} - p\| &\le \alpha_n\|f(x_n) - p\| + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\|U_iV_ix_n - p\| \\ &\le \alpha_n\|f(x_n) - f(p)\| + \alpha_n\|f(p) - p\| + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\|V_ix_n - p\| \\ &\le \alpha_n\rho\|x_n - p\| + \alpha_n\|f(p) - p\| + (1 - \alpha_n)\|x_n - p\| \\ &= (1 - \alpha_n(1 - \rho))\|x_n - p\| + \alpha_n(1 - \rho)\frac{1}{1 - \rho}\|f(p) - p\| \\ &\le \max\left\{\|x_n - p\|, \frac{1}{1 - \rho}\|f(p) - p\|\right\}. \end{aligned} \tag{3.5}$$

Then it follows that

$$\|U_nV_nx_{n-1} - p\| \le \|V_nx_{n-1} - p\| \le \|x_{n-1} - p\| \le \max\left\{\|x_1 - p\|, \frac{1}{1 - \rho}\|f(p) - p\|\right\} \tag{3.6}$$

for every $n \in \mathbb{N}$. This shows that $\{x_n\}$ and $\{U_nV_nx_{n-1}\}$ is bounded. Hence, $\{f(x_n)\}$ is also bounded.

Step 3. We show $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$.

From (3.2), we have

$$\begin{aligned}\|x_{n+1}-x_n\| &= \left\|\alpha_n f(x_n)+\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)U_iV_ix_n-\alpha_{n-1}f(x_{n-1})-\sum_{i=1}^{n-1}(\alpha_{i-1}-\alpha_i)U_iV_ix_{n-1}\right\| \\ &\le \alpha_n\left\|f(x_n)-f(x_{n-1})\right\|+(\alpha_{n-1}-\alpha_n)\left\|f(x_{n-1})\right\| \\ &\quad+\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\|U_iV_ix_n-U_iV_ix_{n-1}\|+(\alpha_{n-1}-\alpha_n)\|U_nV_nx_{n-1}\| \\ &\le \alpha_n\rho\|x_n-x_{n-1}\|+(\alpha_{n-1}-\alpha_n)\left(\left\|f(x_{n-1})\right\|+\|U_nV_nx_{n-1}\|\right)+(1-\alpha_n)\|x_n-x_{n-1}\| \\ &\le \left(1-\alpha_n(1-\rho)\right)\|x_n-x_{n-1}\|+(\alpha_{n-1}-\alpha_n)M,\end{aligned} \tag{3.7}$$

where M is a constant such that

$$M=\sup_{n\ge1}\left\{\left\|f(x_{n-1})\right\|+\|U_nV_nx_{n-1}\|\right\}. \tag{3.8}$$

From (i), (ii), (iii), and Lemma 2.2, it follows that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$

Step 4. We show $\lim_{n\to\infty}\|U_ix_n-x_n\|=0$ and $\lim_{n\to\infty}\|U_iV_ix_n-x_n\|=0$ for $i\in\mathbb{N}$.

We first show $\lim_{n\to\infty}\|U_iV_ix_n-x_n\|=0$ for $i\in\mathbb{N}$. Since $p\in\Omega$, we note that

$$\begin{aligned}\left\|x_n-p\right\|^2 &\ge \left\|V_ix_n-V_ip\right\|^2\ge\left\|U_iV_ix_n-U_iV_ip\right\|^2 \\ &= \left\|U_iV_ix_n-p\right\|^2=\left\|U_iV_ix_n-x_n+x_n-p\right\|^2 \\ &= \|U_iV_ix_n-x_n\|^2+\left\|x_n-p\right\|^2+2\langle U_iV_ix_n-x_n,x_n-p\rangle,\end{aligned} \tag{3.9}$$

which implies that

$$\frac{1}{2}\|U_iV_ix_n-x_n\|^2\le\langle x_n-U_iV_ix_n,x_n-p\rangle. \tag{3.10}$$

Using (3.2) and (3.10), we deduce

$$\begin{aligned}\frac{1}{2}\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\|U_iV_ix_n-x_n\|^2 &\le \sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\langle x_n-U_iV_ix_n,x_n-p\rangle \\ &= \left\langle (1-\alpha_n)x_n-\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)U_iV_ix_n,x_n-p\right\rangle \\ &= \langle (1-\alpha_n)x_n-x_{n+1}+\alpha_nf(x_n),x_n-p\rangle \\ &= \langle x_n-x_{n+1},x_n-p\rangle+\alpha_n\langle f(x_n)-x_n,x_n-p\rangle \\ &\le \|x_n-x_{n+1}\|\left\|x_n-p\right\|+\alpha_n\left\|f(x_n)-x_n\right\|\left\|x_n-p\right\|.\end{aligned} \tag{3.11}$$

Noting that $\lim_{n\to\infty}\|x_n - x_{n+1}\| = 0$ and $\lim_{n\to\infty}\alpha_n = 0$, then we immediately obtain

$$\sum_{i=1}^{\infty}(\alpha_{i-1}-\alpha_i)\|U_iV_ix_n - x_n\|^2 = \lim_{n\to\infty}\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\|U_iV_ix_n - x_n\|^2 = 0. \tag{3.12}$$

Since $\{\alpha_n\}$ is strictly decreasing, it follows that

$$\lim_{n\to\infty}\|U_iV_ix_n - x_n\| = 0, \quad \text{for every } i \in \mathbb{N}. \tag{3.13}$$

Next we show $\lim_{n\to\infty}\|T_iAx_n - Ax_n\| = 0$, for every $i \in \mathbb{N}$. Note for every $i \in \mathbb{N}$,

$$\begin{aligned}\|Ax_n - Ap\|^2 &\geq \|T_iAx_n - T_iAp\|^2 = \|T_iAx_n - Ap\|^2\\ &= \|T_iAx_n - Ax_n + Ax_n - Ap\|^2\\ &= \|T_iAx_n - Ax_n\|^2 + \|Ax_n - Ap\|^2 + 2\langle T_iAx_n - Ax_n, Ax_n - Ap\rangle,\end{aligned} \tag{3.14}$$

which follows that

$$\langle T_iAx_n - Ax_n, Ax_n - Ap\rangle \leq -\frac{1}{2}\|T_iAx_n - Ax_n\|^2, \tag{3.15}$$

for every $i \in \mathbb{N}$. From (3.2), we have

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \left\|\alpha_n f(x_n) + \sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)U_iV_ix_n - p\right\|^2\\ &\leq \alpha_n\|f(x_n) - p\|^2 + \sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\|U_iV_ix_n - p\|^2\\ &\leq \alpha_n\|f(x_n) - p\|^2 + \sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\|x_n + \gamma A^*(T_i - I)Ax_n - p\|^2\\ &\leq \alpha_n\|f(x_n) - p\|^2 + \sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\Big[\|x_n - p\|^2 + \gamma^2\|A\|^2\|T_iAx_n - Ax_n\|^2\\ &\qquad +2\gamma\langle Ax_n - Ap, T_iAx_n - Ax_n\rangle\Big].\end{aligned} \tag{3.16}$$

By (3.15), it follows that

$$\begin{aligned}\|x_{n+1} - p\|^2 &\leq \alpha_n\|f(x_n) - p\|^2 + (1-\alpha_n)\|x_n - p\|^2\\ &\quad + \gamma\left(\gamma\|A\|^2 - 1\right)\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\|T_iAx_n - Ax_n\|^2.\end{aligned} \tag{3.17}$$

Thus,

$$\begin{aligned}
&\gamma\left(1-\gamma\|A\|^2\right)\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\|T_iAx_n-Ax_n\|^2\\
&\quad\le \alpha_n\left(\|f(x_n)-p\|^2-\|x_n-p\|^2\right)+\|x_n-p\|^2-\|x_{n+1}-p\|^2\\
&\quad= \alpha_n\left(\|f(x_n)-p\|^2-\|x_n-p\|^2\right)+\left(\|x_n-p\|+\|x_{n+1}-p\|\right)\left(\|x_n-p\|-\|x_{n+1}-p\|\right)\\
&\quad\le \alpha_n\left(\|f(x_n)-p\|^2-\|x_n-p\|^2\right)+\left(\|x_n-p\|+\|x_{n+1}-p\|\right)\|x_n-x_{n+1}\|.
\end{aligned} \tag{3.18}$$

Using $\lim_{n\to\infty}\alpha_n=0$ and $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$, we have

$$\lim_{n\to\infty}\gamma\left(1-\gamma\|A\|^2\right)\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\|T_iAx_n-Ax_n\|^2=0. \tag{3.19}$$

By $0<\gamma<1/\|A\|^2$, there holds

$$\sum_{i=1}^{\infty}(\alpha_{i-1}-\alpha_i)\|T_iAx_n-Ax_n\|^2=\lim_{n\to\infty}\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\|T_iAx_n-Ax_n\|^2=0. \tag{3.20}$$

Since $\{\alpha_n\}$ is strictly decreasing, we obtain

$$\lim_{n\to\infty}\|T_iAx_n-Ax_n\|=0,\quad \forall i\in\mathbb{N}. \tag{3.21}$$

Last we show $\lim_{n\to\infty}\|U_ix_n-x_n\|=0$ for every $i\in\mathbb{N}$. In fact, we note that for every $i\in\mathbb{N}$,

$$\begin{aligned}
\|U_ix_n-x_n\|&\le\|U_ix_n-U_iV_ix_n\|+\|U_iV_ix_n-x_n\|\\
&\le\|x_n-V_ix_n\|+\|U_iV_ix_n-x_n\|\\
&=\left\|x_n-x_n-\gamma A^*(T_i-I)Ax_n\right\|+\|U_iV_ix_n-x_n\|\\
&\le\gamma\|A\|\|(T_i-I)Ax_n\|+\|U_iV_ix_n-x_n\|.
\end{aligned} \tag{3.22}$$

Then by (3.13) and (3.21), we obtain

$$\lim_{n\to\infty}\|U_ix_n-x_n\|=0,\quad \forall i\in\mathbb{N}. \tag{3.23}$$

Step 5. Show $\limsup_{n\to\infty}\langle f(w)-w,x_n-w\rangle\le 0$, where $w=P_\Omega f(w)$.

Since $\{x_n\}$ is bounded, there exist a point $v \in H_1$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle f(w)-w, x_n-w\rangle = \lim_{k\to\infty}\langle f(w)-w, x_{n_k}-w\rangle \tag{3.24}$$

and $x_{n_k} \rightharpoonup v$. Since A is a bounded linear operator, we have $Ax_{n_k} \rightharpoonup Av$. Now applying (3.21), (3.23), and Lemma 2.1, we conclude that $v \in \operatorname{Fix}(U_i)$ and $Av \in \operatorname{Fix}(T_i)$ for every i. Hence, $v \in \Omega$. Since Ω is closed and convex, by (2.1), we get

$$\limsup_{n\to\infty}\langle f(w)-w, x_n-w\rangle = \lim_{k\to\infty}\langle f(w)-w, x_{n_k}-w\rangle = \langle f(w)-w, v-w\rangle \le 0. \tag{3.25}$$

Step 6. Show $x_n \to w = P_\Omega f(w)$.

Since $w \in \Omega$, we have $U_i w = w$ and $T_i A w = Aw$ for every $i \in \mathbb{N}$. It follows that $V_i w = w$. Using (3.2), we have

$$\begin{aligned}
\|x_{n+1}-w\|^2 &= \left\langle \alpha_n\big(f(x_n)-w\big) + \sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)(U_iV_ix_n-w), x_{n+1}-w\right\rangle \\
&= \alpha_n\langle f(x_n)-f(w), x_{n+1}-w\rangle + \alpha_n\langle f(w)-w, x_{n+1}-w\rangle \\
&\quad + \sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\langle U_iV_ix_n-w, x_{n+1}-w\rangle \\
&\le \alpha_n\rho\|x_n-w\|\|x_{n+1}-w\| + \alpha_n\langle f(w)-w, x_{n+1}-w\rangle \\
&\quad + \sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\|U_iV_ix_n-w\|\|x_{n+1}-w\| \\
&\le \frac{1}{2}\alpha_n\rho\Big(\|x_n-w\|^2+\|x_{n+1}-w\|^2\Big) + \alpha_n\langle f(w)-w, x_{n+1}-w\rangle \\
&\quad + \frac{1}{2}(1-\alpha_n)\Big(\|x_n-w\|^2+\|x_{n+1}-w\|^2\Big) \\
&\le \frac{1}{2}\big[1-\alpha_n(1-\rho)\big]\|x_n-w\|^2 + \frac{1}{2}\|x_{n+1}-w\|^2 + \alpha_n\langle f(w)-w, x_{n+1}-w\rangle,
\end{aligned} \tag{3.26}$$

which implies that

$$\|x_{n+1}-w\|^2 \le \big[1-\alpha_n(1-\rho)\big]\|x_n-w\|^2 + 2\alpha_n(1-\rho)\frac{1}{1-\rho}\langle f(w)-w, x_{n+1}-w\rangle, \tag{3.27}$$

for every $n \in \mathbb{N}$. Consequently, according to (3.25), $\rho \in [0,1)$, and Lemma 2.2, we deduce that $\{x_n\}$ converges strongly to $w = P_\Omega(w)$. This completes the proof. □

Remark 3.2. If we set $\alpha_n = 1/n$ and $f(x) = u$ for all $x \in H_1$, where u is an arbitrary point in H_1, it is easily seen that our conditions are satisfied.

Corollary 3.3. *Let $U : H_1 \to H_1$ and $T : H_2 \to H_2$ be nonexpansive operators. Let $f : H_1 \to H_1$ be a contraction with coefficient $\rho \in [0,1)$. Suppose that the solution set Ω of SCFPP* (1.6) *is nonempty. Let $x_1 \in H_1$ and define a sequence $\{x_n\}$ by the following algorithm:*

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) U\left(x_n + \gamma A^*(T-I)Ax_n\right), \tag{3.28}$$

where $0 < \gamma < 1/\|A\|^2$, $\alpha_0 = 1$ and $\{\alpha_n\} \subset (0,1]$ is a strictly decreasing sequence satisfying the following conditions:

(i) $\lim_{n\to\infty}\alpha_n = 0$;

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

Then $\{x_n\}$ converges strongly to $w \in \Omega$, where $w = P_\Omega f(w)$.

Proof. Set $\{U_n\}$ and $\{T_n\}$ to be sequences of operators defined by $U_n = U$ and $T_n = T$ for all $n \in \mathbb{N}$ in Theorem 3.1. Then by Theorem 3.1 we obtain the desired result. □

Remark 3.4. By adding more operators to the families $\{U_n\}$ and $\{T_n\}$ by setting $U_i = I$ for $i \geq p+1$ and $T_j = I$ for $j \geq r+1$, the SCFPP (1.5) can be viewed as a special case of the GSCFPP (1.9).

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