

## Research Article

# Strong Convergence of the Viscosity Approximation Process for the Split Common Fixed-Point Problem of Quasi-Nonexpansive Mappings

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Very recently, Moudafi (2011) introduced an algorithm with weak convergence for the split common fixed-point problem. In this paper, we will continue to consider the split common fixed-point problem. We discuss the strong convergence of the viscosity approximation method for solving the split common fixed-point problem for the class of quasi-nonexpansive mappings in Hilbert spaces. Our results improve and extend the corresponding results announced by many others.

## 1. Introduction and Preliminary

Throughout this paper, we always assume that  $H$  is a real Hilbert space with inner product  $\langle \cdot, \cdot \rangle$  and norm  $\| \cdot \|$ . Let  $I$  denote the identity operator on  $H$ . Let  $C$  and  $Q$  be nonempty closed convex subset of real Hilbert spaces  $H_1$  and  $H_2$ , respectively. The split feasibility problem (SFP) is to find a point

$$x \in C \quad \text{such that} \quad Ax \in Q, \quad (1.1)$$

where  $A : H_1 \rightarrow H_2$  is a bounded linear operator. The SFP in finite-dimensional Hilbert spaces was first introduced by Censor and Elfving [1] for modeling inverse problems which arise from phase retrievals and in medical image reconstruction [2]. The SFP attracts many authors' attention due to its application in signal processing. Various algorithms have been invented to solve it (see [3–9] and references therein).

Note that the split feasibility problem (1.1) can be formulated as a fixed-point equation by using the fact

$$P_C(I - \gamma A^*(I - P_Q)A)x^* = x^*; \quad (1.2)$$

that is,  $x^*$  solves the SFP (1.1) if and only if  $x^*$  solves the fixed point equation (1.2) (see [10] for the details). This implies that we can use fixed-point algorithms (see [11–13]) to solve SFP. A popular algorithm that solves the SFP (1.1) is due to Byrne's CQ algorithm [2] which is found to be a gradient-projection method (GPM) in convex minimization. Subsequently, Byrne [3] applied KM iteration to the CQ algorithm, and Zhao and Yang [14] applied KM iteration to the perturbed CQ algorithm to solve the SFP. It is well known that the CQ algorithm and the KM algorithm for a split feasibility problem do not necessarily converge strongly in the infinite-dimensional Hilbert spaces.

The split common fixed-point problem (SCFP) is a generalization of the split feasibility problem (SFP) and the convex feasibility problem (CFP); see [15]. In this paper, we introduce and study the convergence properties of a viscosity approximation algorithm for solving the SCFP for the class of quasi-nonexpansive operators  $S$  such that  $I - S$  is demiclosed at the origin.

Now let us first recall the definition of quasi-nonexpansive operators which appear naturally when using subgradient projection operator techniques in solving some feasibility problems, and also some definitions of classes of operators often used in fixed-point theory and which are commonly encountered in the literature.

Let  $T : H \rightarrow H$  be a mapping. A point  $x \in H$  is said to be a fixed point of  $T$  provided that  $Tx = x$ . In this paper, we use  $F(T)$  to denote the fixed-point set and use  $\rightarrow$  and  $\rightharpoonup$  to denote the strong convergence and weak convergence, respectively. We use  $\omega_w(x_k) = \{x : \exists x_{k_j} \rightharpoonup x\}$  stand for the weak  $\omega$ -limit set of  $\{x_k\}$ .

- (i) A mapping  $T : H \rightarrow H$  belongs to the general class  $\Phi_Q$  of (possibly discontinuous) quasi-nonexpansive mappings if

$$\|Tx - q\| \leq \|x - q\|, \quad \forall (x, q) \in H \times F(T). \quad (1.3)$$

- (ii) A mapping  $T : H \rightarrow H$  belongs to the set  $\Phi_N$  of nonexpansive mappings if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall (x, y) \in H \times H. \quad (1.4)$$

- (iii) A mapping  $T : H \rightarrow H$  belongs to the set  $\Phi_{FN}$  of firmly nonexpansive mappings if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(x - y) - (Tx - Ty)\|^2, \quad \forall (x, y) \in H \times H. \quad (1.5)$$

- (iv) A mapping  $T : H \rightarrow H$  belongs to the set  $\Phi_{FQ}$  of firmly quasi-nonexpansive mappings if

$$\|Tx - q\|^2 \leq \|x - q\|^2 - \|x - Tx\|^2, \quad \forall (x, q) \in H \times F(T). \quad (1.6)$$

It is easily observed that  $\Phi_{\text{FN}} \subset \Phi_N \subset \Phi_Q$  and that  $\Phi_{\text{FN}} \subset \Phi_{\text{FQ}} \subset \Phi_Q$ . Furthermore,  $\Phi_{\text{FN}}$  is well known to include resolvents and projection operators, while  $\Phi_{\text{FQ}}$  contains subgradient projection operators (see, e.g., [16] and the reference therein).

A mapping  $T : H \rightarrow H$  is called demiclosed at the origin if any sequence  $\{x_n\}$  weakly converges to  $x$ , and if the sequence  $\{Tx_n\}$  strongly converges to 0, then  $Tx = 0$ . A mapping  $f : H \rightarrow H$  is called a contraction of modulus  $\rho \in [0, 1)$  if

$$\|fx - fy\| \leq \rho \|x - y\|, \quad \forall (x, y) \in H \times H. \quad (1.7)$$

In what follows, we will focus our attention on the following general two-operator split common fixed-point problem:

$$\text{find } x^* \in C \quad \text{such that } Ax^* \in Q, \quad (1.8)$$

where  $A : H_1 \rightarrow H_2$  is a bounded linear operator,  $U : H_1 \rightarrow H_1$  and  $S : H_2 \rightarrow H_2$  are two quasi-nonexpansive operators with nonempty fixed-point sets  $F(U) = C$  and  $F(S) = Q$ , and denote the solution set of the two-operator SCFP by

$$\Gamma = \{y \in C; Ay \in Q\}. \quad (1.9)$$

Recall that  $F(U)$  and  $F(S)$  are nonempty closed convex subsets of  $H_1$  and  $H_2$ , respectively. If  $\Gamma \neq \emptyset$ , we have  $\Gamma$  which is close convex subset of  $H_1$ . To solve (1.8), Censor and Segal [15] proposed and proved, in infinite-dimensional spaces, the convergence of the following algorithm:

$$x_{k+1} = U(x_k + \gamma A^t(S - I)Ax_k), \quad k \in N, \quad (1.10)$$

where  $\gamma \in (0, 2/\lambda)$ , with  $\lambda$  being the largest eigenvalue of the matrix  $A^t A$  ( $A^t$  stands for matrix transposition). Very recently, Moudafi [17] introduced the following relaxed algorithm:

$$x_{k+1} = (1 - \alpha_k)u_k + \alpha_k U(u_k), \quad k \in N, \quad (1.11)$$

where  $u_k = x_k + \gamma \beta A^*(S - I)Ax_k$ ,  $\beta \in (0, 1)$ ,  $\alpha_k \in (0, 1)$ , and  $\gamma \in (0, 1/\lambda\beta)$ , with  $\lambda$  being the spectral radius of the operator  $A^*A$ . Moudafi proved weak convergence result of the algorithm in Hilbert spaces.

Inspired by their work, we introduce the following viscosity approximation algorithm.

*Algorithm 1.* Initialization: Let  $x_0 \in H$  be arbitrary.

Iterative step: Set  $T = U(I + \gamma A^*(S - I)A)$ . For  $k \in N$ , let

$$x_{k+1} = \alpha_k f(x_k) + (1 - \alpha_k)((1 - \omega_k)x_k + \omega_k T x_k), \quad (1.12)$$

where  $f : H \rightarrow H$  is a contraction of modulus  $\rho$ ,  $\omega_k \in (0, 1/2)$ ,  $\gamma \in (0, 1/\lambda)$  with  $\lambda$  being the spectral radius of the operator  $A^*A$ , and  $\alpha_k \in (0, 1)$ .

This paper establishes the strong convergence of the sequence given by (1.12) to the unique solution of the variational inequality problem  $\text{VIP}(I - f, \Gamma)$  :

$$\text{find } x^* \in \Gamma \text{ such that } \langle (I - f)x^*, v - x^* \rangle \geq 0, \quad \forall v \in \Gamma. \quad (1.13)$$

Now we give a series of preliminary results needed for the convergence analysis of algorithm (1.12).

**Lemma 1.1.** *Let  $H$  be a real Hilbert space and  $T : H \rightarrow H$  a quasi-nonexpansive mapping. Then, the following properties are reached:*

$$\begin{aligned} \text{(i)} \quad \langle x, y \rangle &= -\frac{1}{2}\|x - y\|^2 + \frac{1}{2}\|x\|^2 + \frac{1}{2}\|y\|^2, \quad \forall (x, y) \in H \times H; \\ \text{(ii)} \quad \langle x - Tx, x - q \rangle &\geq \frac{1}{2}\|x - Tx\|^2 \text{ and } \langle x - Tx, q - Tx \rangle \leq \frac{1}{2}\|x - Tx\|^2, \quad \forall (x, q) \in H \times F(T). \end{aligned}$$

*Remark 1.2.* Let  $F := I - f$ , where  $f$  is the contraction defined in (1.7). It is a simple matter to see that the operator  $F$  is  $(1 - \rho)$  strongly monotone over  $H$ ; that is,

$$\langle Fx - Fy, x - y \rangle \geq (1 - \rho)\|x - y\|^2, \quad \forall (x, y) \in H \times H. \quad (1.14)$$

The next result is of fundamental importance for the techniques of analysis used in this paper. It was established in [18], and its proof is given for the sake of completeness.

**Lemma 1.3** (see [18, Lemma 1.3]). *Let  $\{\delta_n\}$  be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence  $\{\delta_{n_j}\}_{j \geq 0}$  of  $\{\delta_n\}$  which satisfies  $\delta_{n_j} < \delta_{n_{j+1}}$  for all  $j \geq 0$ . Also consider the sequence of integers  $\{\tau(n)\}_{n \geq n_0}$  defined by*

$$\tau(n) = \max\{k \leq n \mid \delta_k < \delta_{k+1}\}. \quad (1.15)$$

*Then  $\{\tau(n)\}_{n \geq n_0}$  is a nondecreasing sequence verifying  $\lim_{n \rightarrow \infty} \tau(n) = \infty$ , and, for all  $n \geq n_0$ , it holds that  $\delta_{\tau(n)} \leq \delta_{\tau(n)+1}$  and one has*

$$\delta_n \leq \delta_{\tau(n)+1}. \quad (1.16)$$

*Proof.* Clearly, we can see that  $\{\tau(n)\}$  is a well-defined sequence, and the fact that it is nondecreasing is obvious as well as  $\lim_{n \rightarrow \infty} \tau(n) = \infty$  and  $\delta_{\tau(n)} \leq \delta_{\tau(n)+1}$ . Let us prove (1.16). It is easily observed that  $\tau(n) \leq n$ . Consequently, we prove (1.16) by distinguishing the three cases: (c1)  $\tau(n) = n$ ; (c2)  $\tau(n) = n - 1$ ; (c3)  $\tau(n) < n - 1$ . In the first case (i.e.,  $\tau(n) = n$ ), (1.16) is immediately given by  $\delta_{\tau(n)} \leq \delta_{\tau(n)+1}$ . In the second case (i.e.,  $\tau(n) = n - 1$ ), (1.16) becomes obvious. In the third case (i.e.,  $\tau(n) \leq n - 2$ ), by (1.15) and for any integer  $n \geq n_0$ , we easily observe that  $\delta_j \geq \delta_{j+1}$  for  $\tau(n) + 1 \leq j \leq n - 1$ ; namely,

$$\delta_{\tau(n)+1} \geq \delta_{\tau(n)+2} \geq \dots \geq \delta_{n-1} \geq \delta_n, \quad (1.17)$$

which entails the desired result. □

## 2. Main Results

**Theorem 2.1.** *Given a bounded linear operator  $A : H_1 \rightarrow H_2$ , let  $U : H_1 \rightarrow H_1$  and  $S : H_2 \rightarrow H_2$  be quasi-nonexpansive mappings with nonempty fixed-point set  $F(U) = C$  and  $F(S) = Q$ . Assume that  $U - I$  and  $S - I$  are demiclosed at origin. Let  $\{x_k\}$  be the sequence given by (1.12) with  $\gamma \in (0, 1/\lambda)$ ,  $\omega_k \in (0, 1/2)$  such that  $0 < \liminf_{k \rightarrow \infty} \omega_k \leq \limsup_{k \rightarrow \infty} \omega_k < 1/2$  and  $\{\alpha_k\} \subset (0, 1)$  such that  $\lim_{k \rightarrow \infty} \alpha_k = 0$  and  $\sum_k \alpha_k = \infty$ . If  $\Gamma \neq \emptyset$ , then the sequence  $\{x_k\}$  strongly converges to a split common fixed-point  $x^* \in \Gamma$ , verifying  $x^* = P_\Gamma f(x^*)$  which equivalently solves the following variational inequality problem:*

$$x^* \in \Gamma, \quad \langle (I - f)x^*, v - x^* \rangle \geq 0, \quad \forall v \in \Gamma. \quad (2.1)$$

*Proof.* Set  $T_{\omega_k} = (1 - \omega_k)I + \omega_k T$ . Then  $x_{k+1} = \alpha_k f(x_k) + (1 - \alpha_k)T_{\omega_k} x_k$ .

Firstly, we prove that  $\{x_k\}$  is bounded. Taking  $y \in \Gamma$ , that is,  $y \in F(U)$ ,  $Ay \in F(S)$ . We have

$$\begin{aligned} \|x_{k+1} - y\| &= \|\alpha_k(f(x_k) - f(y)) + \alpha_k(f(y) - y) + (1 - \alpha_k)(T_{\omega_k} x_k - y)\| \\ &\leq \alpha_k \|f(x_k) - f(y)\| + \alpha_k \|f(y) - y\| + (1 - \alpha_k) \|T_{\omega_k} x_k - y\| \\ &\leq \alpha_k \rho \|x_k - y\| + \alpha_k \|f(y) - y\| + (1 - \alpha_k) \|T_{\omega_k} x_k - y\|. \end{aligned} \quad (2.2)$$

From the definition of  $T_{\omega_k}$ , we get

$$\begin{aligned} \|T_{\omega_k} x_k - y\|^2 &= \|(1 - \omega_k)x_k + \omega_k T x_k - y\|^2 \\ &= \|x_k - y + \omega_k(T x_k - x_k)\|^2 \\ &= \|x_k - y\|^2 - 2\omega_k \langle x_k - y, x_k - T x_k \rangle + \omega_k^2 \|T x_k - x_k\|^2. \end{aligned} \quad (2.3)$$

On the other hand, we have

$$\begin{aligned} \|T x_k - y\|^2 &= \|U(I + \gamma A^*(S - I)A)x_k - y\|^2 \\ &\leq \|(I + \gamma A^*(S - I)A)x_k - y\|^2 \\ &= \|x_k - y\|^2 + \gamma^2 \|A^*(S - I)A x_k\|^2 + 2\gamma \langle x_k - y, A^*(S - I)A x_k \rangle \\ &= \|x_k - y\|^2 + \gamma^2 \langle (S - I)A x_k, A A^*(S - I)A x_k \rangle + 2\gamma \langle x_k - y, A^*(S - I)A x_k \rangle. \end{aligned} \quad (2.4)$$

From the definition of  $\lambda$ , it follows that

$$\begin{aligned} \gamma^2 \langle (S - I)A x_k, A A^*(S - I)A x_k \rangle &\leq \lambda \gamma^2 \langle (S - I)A x_k, (S - I)A x_k \rangle \\ &= \lambda \gamma^2 \|(S - I)A x_k\|^2. \end{aligned} \quad (2.5)$$

Now, by using property (ii) of Lemma 1.1, we obtain

$$\begin{aligned}
2\gamma\langle x_k - y, A^*(S - I)Ax_k \rangle &= 2\gamma\langle A(x_k - y), (S - I)Ax_k \rangle \\
&= 2\gamma\langle A(x_k - y) + (S - I)Ax_k - (S - I)Ax_k, (S - I)Ax_k \rangle \\
&= 2\gamma\left(\langle S(Ax_k) - Ay, (S - I)Ax_k \rangle - \|(S - I)Ax_k\|^2\right) \\
&\leq 2\gamma\left(\frac{1}{2}\|(S - I)Ax_k\|^2 - \|(S - I)Ax_k\|^2\right) \\
&= -\gamma\|(S - I)Ax_k\|^2.
\end{aligned} \tag{2.6}$$

Combining (2.4)–(2.6), we have

$$\begin{aligned}
\|Tx_k - y\|^2 &\leq \|x_k - y\|^2 + \lambda\gamma^2\|(S - I)Ax_k\|^2 - \gamma\|(S - I)Ax_k\|^2 \\
&= \|x_k - y\|^2 - \gamma(1 - \lambda\gamma)\|(S - I)Ax_k\|^2 \\
&\leq \|x_k - y\|^2.
\end{aligned} \tag{2.7}$$

From property (i) of Lemma 1.1, we have

$$\begin{aligned}
\langle x_k - y, x_k - Tx_k \rangle &= -\frac{1}{2}\|Tx_k - y\|^2 + \frac{1}{2}\|x_k - y\|^2 + \frac{1}{2}\|x_k - Tx_k\|^2 \\
&\geq \frac{1}{2}\|x_k - Tx_k\|^2.
\end{aligned} \tag{2.8}$$

From (2.3) and (2.8), we have

$$\begin{aligned}
\|T_{\omega_k}x_k - y\|^2 &\leq \|x_k - y\|^2 - \omega_k\|x_k - Tx_k\|^2 + \omega_k^2\|x_k - Tx_k\|^2 \\
&= \|x_k - y\|^2 - \omega_k(1 - \omega_k)\|x_k - Tx_k\|^2 \\
&\leq \|x_k - y\|^2,
\end{aligned} \tag{2.9}$$

Combining (2.2), (2.3), and (2.9), it follows that

$$\begin{aligned}
\|x_{k+1} - y\| &\leq \alpha_k\rho\|x_k - y\| + \alpha_k\|f(y) - y\| + (1 - \alpha_k)\|x_k - y\| \\
&= [1 - \alpha_k(1 - \rho)]\|x_k - y\| + \alpha_k\|f(y) - y\| \\
&\leq \max\left\{\|x_k - y\|, \frac{1}{1 - \rho}\|f(y) - y\|\right\}.
\end{aligned} \tag{2.10}$$

It is obviously that

$$\|x_k - y\| \leq \max \left\{ \|x_0 - y\|, \frac{1}{1-\rho} \|f(y) - y\| \right\}, \quad (2.11)$$

and hence  $\{x_k\}$  is bounded. Let  $x^* = P_{\Gamma} f(x^*)$ . We have

$$x_{k+1} - x_k + \alpha_k (x_k - f(x_k)) = (1 - \alpha_k)(T_{\omega_k} x_k - x_k), \quad (2.12)$$

and hence

$$\langle x_{k+1} - x_k + \alpha_k (I - f)x_k, x_k - x^* \rangle = -(1 - \alpha_k) \langle x_k - T_{\omega_k} x_k, x_k - x^* \rangle. \quad (2.13)$$

By (2.9) we obtain that

$$\begin{aligned} \langle x_k - T_{\omega_k} x_k, x_k - x^* \rangle &= \frac{1}{2} \|x_k - T_{\omega_k} x_k\|^2 + \frac{1}{2} \|x_k - x^*\|^2 - \frac{1}{2} \|T_{\omega_k} x_k - x^*\|^2 \\ &\geq \frac{\omega_k^2}{2} \|x_k - Tx_k\|^2 + \frac{1}{2} \|x_k - x^*\|^2 - \frac{1}{2} \|x_k - x^*\|^2 + \frac{\omega_k}{2} (1 - \omega_k) \|x_k - Tx_k\|^2 \\ &= \frac{\omega_k}{2} \|x_k - Tx_k\|^2. \end{aligned} \quad (2.14)$$

It follows from (2.13) that

$$\langle x_{k+1} - x_k + \alpha_k (I - f)x_k, x_k - x^* \rangle \leq -\frac{\omega_k}{2} (1 - \alpha_k) \|x_k - Tx_k\|^2, \quad (2.15)$$

and hence

$$-\langle x_k - x_{k+1}, x_k - x^* \rangle \leq -\alpha_k \langle (I - f)x_k, x_k - x^* \rangle - \frac{\omega_k}{2} (1 - \alpha_k) \|x_k - Tx_k\|^2. \quad (2.16)$$

Setting  $\delta_k = \frac{1}{2} \|x_k - x^*\|^2$ , we have

$$\begin{aligned} \langle x_k - x_{k+1}, x_k - x^* \rangle &= -\frac{1}{2} \|x_{k+1} - x^*\|^2 + \frac{1}{2} \|x_k - x^*\|^2 + \frac{1}{2} \|x_k - x_{k+1}\|^2 \\ &= -\delta_{k+1} + \delta_k + \frac{1}{2} \|x_k - x_{k+1}\|^2, \end{aligned} \quad (2.17)$$

so that (2.16) can be rewritten as

$$\delta_{k+1} - \delta_k - \frac{1}{2} \|x_k - x_{k+1}\|^2 \leq -\alpha_k \langle (I - f)x_k, x_k - x^* \rangle - \frac{\omega_k}{2} (1 - \alpha_k) \|x_k - Tx_k\|^2. \quad (2.18)$$

Now using (2.12) again, we have

$$\begin{aligned}
\|x_{k+1} - x_k\|^2 &= \|\alpha_k(f(x_k) - x_k) + (1 - \alpha_k)(Tx_k - x_k)\|^2 \\
&\leq (\alpha_k\|f(x_k) - x_k\| + (1 - \alpha_k)\|Tx_k - x_k\|)^2 \\
&\leq 2\alpha_k^2\|f(x_k) - x_k\|^2 + 2(1 - \alpha_k)^2\|Tx_k - x_k\|^2 \\
&\leq 2\alpha_k^2\|f(x_k) - x_k\|^2 + 2(1 - \alpha_k)\omega_k^2\|Tx_k - x_k\|^2,
\end{aligned} \tag{2.19}$$

which yields

$$\frac{1}{2}\|x_{k+1} - x_k\|^2 \leq \alpha_k^2\|f(x_k) - x_k\|^2 + (1 - \alpha_k)\omega_k^2\|Tx_k - x_k\|^2. \tag{2.20}$$

From (2.18) and (2.20), we obtain

$$\delta_{k+1} - \delta_k + \omega_k(1 - \alpha_k)\left(\frac{1}{2} - \omega_k\right)\|Tx_k - x_k\|^2 \leq \alpha_k\left[\alpha_k\|f(x_k) - x_k\|^2 - \langle(I - f)x_k, x_k - x^*\rangle\right]. \tag{2.21}$$

It follows from Remark 1.2 that

$$\langle(I - f)x_k - (I - f)x^*, x_k - x^*\rangle \geq (1 - \rho)\|x_k - x^*\|^2 = 2(1 - \rho)\delta_k, \tag{2.22}$$

and hence

$$2(1 - \rho)\delta_k + \langle(I - f)x^*, x_k - x^*\rangle \leq \langle(I - f)x_k, x_k - x^*\rangle. \tag{2.23}$$

The rest of the proof will be divided into two parts.

*Case 1.* Suppose that there exists  $k_0$  such that  $\{\delta_k\}_{k \geq k_0}$  is nonincreasing. In this situation,  $\{\delta_k\}$  is convergent because it is nonnegative, so that  $\lim_{k \rightarrow \infty}(\delta_{k+1} - \delta_k) = 0$ ; hence, in light of (2.21) together with  $\alpha_k \rightarrow 0$ , the boundedness of  $\{x_k\}$ , and  $0 < \liminf_{k \rightarrow \infty}\omega_k \leq \limsup_{k \rightarrow \infty}\omega_k < 1/2$ , we obtain

$$\lim_{k \rightarrow \infty}\|x_k - Tx_k\| = 0. \tag{2.24}$$

From (2.21) again, we have

$$\alpha_k\left[-\alpha_k\|f(x_k) - x_k\|^2 + \langle(I - f)x_k, x_k - x^*\rangle\right] \leq \delta_k - \delta_{k+1}. \tag{2.25}$$

By  $\sum_k \alpha_k = \infty$ , we deduce that

$$\liminf_{k \rightarrow \infty}\left(-\alpha_k\|f(x_k) - x_k\|^2 + \langle(I - f)x_k, x_k - x^*\rangle\right) \leq 0 \tag{2.26}$$



and hence (as  $\alpha_k \|f(x_k) - x_k\|^2 \rightarrow 0$ )

$$\liminf_{k \rightarrow \infty} \langle (I - f)x_k, x_k - x^* \rangle \leq 0. \quad (2.27)$$

By (2.23) and (2.27), we have

$$\liminf_{k \rightarrow \infty} (2(1 - \rho)\delta_k + \langle (I - f)x^*, x_k - x^* \rangle) \leq 0; \quad (2.28)$$

recalling that  $\lim_{k \rightarrow \infty} \delta_k$  exists, we obtain

$$2(1 - \rho) \lim_{k \rightarrow \infty} \delta_k + \liminf_{k \rightarrow \infty} \langle (I - f)x^*, x_k - x^* \rangle \leq 0. \quad (2.29)$$

Now we prove that

$$\liminf_{k \rightarrow \infty} \langle (I - f)x^*, x_k - x^* \rangle \geq 0. \quad (2.30)$$

It follows from (2.7) and (2.24) that

$$\begin{aligned} \gamma(1 - \lambda\gamma) \|(S - I)Ax_k\|^2 &\leq \|x_k - y\|^2 - \|Tx_k - y\|^2 \\ &= (\|x_k - y\| - \|Tx_k - y\|)(\|x_k - y\| + \|Tx_k - y\|) \\ &\leq \|x_k - Tx_k\|(\|x_k - y\| + \|Tx_k - y\|) \\ &\rightarrow 0 \quad (k \rightarrow \infty), \end{aligned} \quad (2.31)$$

and hence

$$\lim_{k \rightarrow \infty} \|(S - I)Ax_k\| = 0. \quad (2.32)$$

Taking  $y \in \omega_w(x_k)$ , from the demiclosedness of  $S - I$  at 0, we obtain

$$S(Ay) = Ay. \quad (2.33)$$

Now, by setting  $u_k = x_k + \gamma A^*(S - I)Ax_k$ , it follows that  $y \in \omega_w(u_k)$ . On the other hand,

$$\|U(u_k) - u_k\| = \|Tx_k - x_k - \gamma A^*(S - I)Ax_k\| \leq \|Tx_k - x_k\| + \gamma \|A^*\| \cdot \|(S - I)Ax_k\| \rightarrow 0, \quad (2.34)$$

which, combined with the demiclosedness of  $U - I$  at 0, yields

$$Uy = y. \quad (2.35)$$

Hence,  $y \in C$  and  $y \in \Gamma$ . We can take subsequence  $\{x_{k_j}\}$  of  $\{x_k\}$  such that  $x_{k_j} \rightarrow y$  as  $j \rightarrow \infty$  and

$$\liminf_{k \rightarrow \infty} \langle (I-f)x^*, x_k - x^* \rangle = \lim_{j \rightarrow \infty} \langle (I-f)x^*, x_{k_j} - x^* \rangle, \quad (2.36)$$

which leads to

$$\liminf_{k \rightarrow \infty} \langle (I-f)x^*, x_k - x^* \rangle = \langle (I-f)x^*, y - x^* \rangle \geq 0. \quad (2.37)$$

By (2.29), we have  $\lim_{k \rightarrow \infty} \delta_k = 0$ , and hence  $\{x_k\}$  converges strongly to  $x^*$ .

*Case 2.* Suppose there exists a subsequence  $\{\delta_{k_j}\}_{j \geq 0}$  of  $\{\delta_k\}$  such that  $\delta_{k_j} < \delta_{k_{j+1}}$  for all  $j \geq 0$ . In this situation, we consider the sequence of indices  $\{\tau(k)\}$  as defined in Lemma 1.3. It follows that  $\delta_{\tau(k)+1} - \delta_{\tau(k)} > 0$ , which by (2.21) amounts to

$$\begin{aligned} \omega_k(1 - \alpha_{\tau(k)}) \left( \frac{1}{2} - \omega_k \right) \|Tx_{\tau(k)} - x_{\tau(k)}\|^2 &\leq \alpha_{\tau(k)} \left[ \alpha_{\tau(k)} \|f(x_{\tau(k)}) - x_{\tau(k)}\|^2 \right. \\ &\quad \left. - \langle (I-f)x_{\tau(k)}, x_{\tau(k)} - x^* \rangle \right]. \end{aligned} \quad (2.38)$$

By the boundedness of  $\{x_k\}$  and  $\alpha_k \rightarrow 0$ , we immediately obtain

$$\lim_{k \rightarrow \infty} \|Tx_{\tau(k)} - x_{\tau(k)}\| = 0. \quad (2.39)$$

Similar to Case 1, we have

$$\liminf_{k \rightarrow \infty} \langle (I-f)x^*, x_{\tau(k)} - x^* \rangle \geq 0. \quad (2.40)$$

It follows from (2.38) that

$$\langle (I-f)x_{\tau(k)}, x_{\tau(k)} - x^* \rangle \leq \alpha_{\tau(k)} \|f(x_{\tau(k)}) - x_{\tau(k)}\|^2, \quad (2.41)$$

which in the light of (2.23) yields

$$2(1 - \rho)\delta_{\tau(k)} + \langle (I-f)x^*, x_{\tau(k)} - x^* \rangle \leq \alpha_{\tau(k)} \|f(x_{\tau(k)}) - x_{\tau(k)}\|^2; \quad (2.42)$$

hence (as  $\alpha_{\tau(k)} \|f(x_{\tau(k)}) - x_{\tau(k)}\|^2 \rightarrow 0$ ) it follows that

$$2(1 - \rho) \limsup_{k \rightarrow \infty} \delta_{\tau(k)} \leq -\liminf_{k \rightarrow \infty} \langle (I-f)x^*, x_{\tau(k)} - x^* \rangle. \quad (2.43)$$

From (2.40) we have  $\limsup_{k \rightarrow \infty} \delta_{\tau(k)} = 0$ , so that  $\lim_{k \rightarrow \infty} \delta_{\tau(k)} = 0$ , and hence  $\lim_{k \rightarrow \infty} \|x_{\tau(k)} - x^*\| = 0$ . On the other hand, it follows that

$$\begin{aligned} \|x_{\tau(k)+1} - x_{\tau(k)}\| &= \|\alpha_{\tau(k)}(f(x_{\tau(k)}) - x_{\tau(k)}) + (1 - \alpha_{\tau(k)})(T_{\omega_k}x_{\tau(k)} - x_{\tau(k)})\| \\ &\leq \alpha_{\tau(k)}\|f(x_{\tau(k)}) - x_{\tau(k)}\| + (1 - \alpha_{\tau(k)})\omega_k\|Tx_{\tau(k)} - x_{\tau(k)}\|, \end{aligned} \quad (2.44)$$

which, by (2.39), implies that

$$\lim_{k \rightarrow \infty} \|x_{\tau(k)+1} - x_{\tau(k)}\| = 0. \quad (2.45)$$

So we have

$$\lim_{k \rightarrow \infty} \delta_{\tau(k)+1} = \frac{1}{2} \|x_{\tau(k)+1} - x^*\| = 0. \quad (2.46)$$

Then, recalling that  $\delta_k \leq \delta_{\tau(k)+1}$  (by Lemma 1.3), we get  $\lim_{k \rightarrow \infty} \delta_k = 0$ , so that the sequence  $\{x_k\}$  converges strongly to  $x^*$ .  $\square$

**Theorem 2.2.** *Given a bounded linear operator  $A : H_1 \rightarrow H_2$ , let  $U : H_1 \rightarrow H_1$  and  $S : H_2 \rightarrow H_2$  be quasi-nonexpansive mappings with nonempty fixed-point set  $F(U) = C$  and  $F(S) = Q$ . Assume that  $U - I$  and  $S - I$  are demiclosed at origin. Let  $x_0 \in H$  be arbitrary and  $\{x_k\}$  the sequence given by*

$$x_{k+1} = \alpha_k f(x_k) + (1 - \alpha_k)((1 - \omega)x_k + \omega Tx_k), \quad (2.47)$$

where  $T = U(I + \gamma A^*(S - I)A)$ ,  $f : H \rightarrow H$  a contraction of modulus  $\rho$ ,  $\gamma \in (0, 1/\lambda)$ ,  $\omega \in (0, 1/2)$ , and  $\{\alpha_k\} \subset (0, 1)$  such that  $\lim_{k \rightarrow \infty} \alpha_k = 0$  and  $\sum_k \alpha_k = \infty$ . If  $\Gamma \neq \emptyset$ , then the sequence  $\{x_k\}$  strongly converges to a split common fixed-point  $x^* \in \Gamma$ , verifying  $x^* = P_{\Gamma}f(x^*)$  which equivalently solves the following variational inequality problem:

$$x^* \in \Gamma, \quad \langle (I - f)x^*, v - x^* \rangle \geq 0, \quad \forall v \in \Gamma. \quad (2.48)$$

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