

## Research Article

# On New Inequalities via Riemann-Liouville Fractional Integration

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We extend the Montgomery identities for the Riemann-Liouville fractional integrals. We also use these Montgomery identities to establish some new integral inequalities. Finally, we develop some integral inequalities for the fractional integral using differentiable convex functions.

## 1. Introduction

The inequality of Ostrowski [1] gives us an estimate for the deviation of the values of a smooth function from its mean value. More precisely, if  $f : [a, b] \rightarrow \mathbb{R}$  is a differentiable function with bounded derivative, then

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[ \frac{1}{4} + \frac{(x - (a+b)/2)^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty, \quad (1.1)$$

for every  $x \in [a, b]$ . Moreover, the constant  $1/4$  is the best possible.

For some generalizations of this classic fact see ([2], (pages 468–484)) by Mitrinović et al. A simple proof of this fact can be done by using the following identity [2].

If  $f : [a, b] \rightarrow \mathbb{R}$  is differentiable on  $[a, b]$  with the first derivative  $f'$  integrable on  $[a, b]$ , then Montgomery identity holds

$$f(x) = \frac{1}{b-a} \int_a^b f(t) dt + \int_a^b P_1(x, t) f'(t) dt, \quad (1.2)$$

where  $P_1(x, t)$  is the Peano kernel defined by

$$P_1(x, t) := \begin{cases} \frac{t-a}{b-a}, & a \leq t < x, \\ \frac{t-b}{b-a}, & x \leq t \leq b. \end{cases} \quad (1.3)$$

Recently, several generalizations of the Ostrowski integral inequality are considered by many authors; for instance, covering the following concepts: functions of bounded variation, Lipschitzian, monotonic, absolutely continuous, and  $n$ -times differentiable mappings with error estimates with some special means together with some numerical quadrature rules. For recent results and generalizations concerning Ostrowski's inequality, we refer the reader to the recent papers [3–10].

In this paper, we extend the Montgomery identities for the Riemann-Liouville fractional integrals. We also use these Montgomery identities to establish some new integral inequalities of Ostrowski's type. Finally, we develop some integral inequalities for the fractional integral using differentiable convex functions. Later, we develop some integral inequalities for the fractional integral using differentiable convex functions. From our results, the weighted and the classical Ostrowski's inequalities can be deduced as some special cases.

## 2. Fractional Calculus

Firstly, we give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used further in this paper. For more details, one can consult [11, 12].

*Definition 2.1.* The Riemann-Liouville fractional integral operator of order  $\alpha \geq 0$  with  $a \geq 0$  is defined as

$$J_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad (2.1)$$

$$J_a^0 f(x) = f(x).$$

Recently, many authors have studied a number of inequalities by using the Riemann-Liouville fractional integrals, see [13–16] and the references cited therein.

## 3. Main Results

In order to prove some of our results, by using a different method of proof, we give the following identities, which are proved in [13]. Later, we will generalize the Montgomery identities in the next theorem.

**Lemma 3.1.** *Let  $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$  be a differentiable function on  $I^\circ$  with  $a, b \in I (a < b)$  and  $f' \in L_1[a, b]$ , then*

$$f(x) = \frac{\Gamma(\alpha)}{b-a} (b-x)^{1-\alpha} J_a^\alpha f(b) - J_a^{\alpha-1} (P_2(x, b) f(b)) + J_a^\alpha (P_2(x, b) f'(b)), \quad \alpha \geq 1, \quad (3.1)$$

where  $P_2(x, t)$  is the fractional Peano kernel defined by

$$P_2(x, t) := \begin{cases} \frac{t-a}{b-a}(b-x)^{1-\alpha}\Gamma(\alpha), & a \leq t < x, \\ \frac{t-b}{b-a}(b-x)^{1-\alpha}\Gamma(\alpha), & x \leq t \leq b. \end{cases} \quad (3.2)$$

*Proof.* By definition of  $P_1(x, t)$ , we have

$$\begin{aligned} \Gamma(\alpha)J_a^\alpha(P_1(x, b)f'(b)) &= \int_a^b (b-t)^{\alpha-1}P_1(x, t)f'(t)dt \\ &= \int_a^x (b-t)^{\alpha-1}\left(\frac{t-a}{b-a}\right)f'(t)dt + \int_x^b (b-t)^{\alpha-1}\left(\frac{t-b}{b-a}\right)f'(t)dt \\ &= \frac{1}{b-a}\left[\int_a^x (b-t)^{\alpha-1}(t-a)f'(t)dt - \int_x^b (b-t)^\alpha f'(t)dt\right] \\ &= \frac{1}{b-a}(I_1 + I_2). \end{aligned} \quad (3.3)$$

Integrating by parts, we can state

$$\begin{aligned} I_1 &= (b-t)^{\alpha-1}(t-a)f(t)\Big|_a^x - \int_a^x [-(\alpha-1)(b-t)^{\alpha-2}(t-a) + (b-t)^{\alpha-1}]f(t)dt \\ &= (b-x)^{\alpha-1}(x-a)f(x) + (\alpha-1)\int_a^x (b-t)^{\alpha-2}(t-a)f(t)dt - \int_a^x (b-t)^{\alpha-1}f(t)dt, \end{aligned} \quad (3.4)$$

and similarly,

$$I_2 = -(b-t)^\alpha f(t)\Big|_x^b - \alpha \int_x^b (b-t)^{\alpha-1}f(t)dt = (b-x)^\alpha f(x) - \alpha \int_x^b (b-t)^{\alpha-1}f(t)dt. \quad (3.5)$$

Adding (3.4) and (3.5), we get

$$\begin{aligned} \Gamma(\alpha)J_a^\alpha(P_1(x, b)f'(b)) &= \frac{1}{b-a}\left\{(b-a)(b-x)^{\alpha-1}f(x) + (\alpha-1)\int_a^x (b-t)^{\alpha-2}(t-a)f(t)dt \right. \\ &\quad \left. - \alpha \int_x^b (b-t)^{\alpha-1}f(t)dt - \int_a^x (b-t)^{\alpha-1}f(t)dt\right\}. \end{aligned} \quad (3.6)$$

If we add and subtract the integral  $(\alpha - 1) \int_x^b (b-t)^{\alpha-2} (t-b) f(t) dt$  to the right-hand side of the equation above, then we have

$$\begin{aligned}
 \Gamma(\alpha) J_a^\alpha (P_1(x, b) f'(b)) &= \frac{1}{b-a} \left\{ (b-a)(b-x)^{\alpha-1} f(x) \right. \\
 &\quad + (b-a)(\alpha-1) \int_a^b (b-t)^{\alpha-2} P_1(x, t) f(t) dt \\
 &\quad \left. - \int_a^b (b-t)^{\alpha-1} f(t) dt \right\} \\
 &= (b-x)^{\alpha-1} f(x) + (\alpha-1) \int_a^b (b-t)^{\alpha-2} P_1(x, t) f(t) dt \\
 &\quad - \frac{1}{b-a} \int_a^b (b-t)^{\alpha-1} f(t) dt \\
 &= (b-x)^{\alpha-1} f(x) - \frac{\Gamma(\alpha)}{b-a} J_a^\alpha f(b) + \Gamma(\alpha) J_a^{\alpha-1} (P_1(x, b) f(b)).
 \end{aligned} \tag{3.7}$$

Multiplying both sides by  $(b-x)^{1-\alpha}$ , we obtain

$$J_a^\alpha (P_2(x, b) f'(b)) = f(x) - \frac{\Gamma(\alpha)}{b-a} (b-x)^{1-\alpha} J_a^\alpha f(b) + J_a^{\alpha-1} (P_2(x, b) f(b)), \tag{3.8}$$

and so

$$f(x) = \frac{\Gamma(\alpha)}{b-a} (b-x)^{1-\alpha} J_a^\alpha f(b) - J_a^{\alpha-1} (P_2(x, b) f(b)) + J_a^\alpha (P_2(x, b) f'(b)). \tag{3.9}$$

This completes the proof. □

Now, we extend Lemma 3.1 as follows.

**Theorem 3.2.** *Let  $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$  be a differentiable function on  $I^\circ$  with  $f' \in L_1[a, b]$ , then the following identity holds:*

$$\begin{aligned}
 (1-2\lambda) f(x) &= \frac{\Gamma(\alpha)}{b-a} (b-x)^{1-\alpha} J_a^\alpha f(b) - \lambda \left( \frac{b-a}{b-x} \right)^{\alpha-1} f(a) \\
 &\quad - J_a^{\alpha-1} (P_3(x, b) f(b)) + J_a^\alpha (P_3(x, b) f'(b)), \quad \alpha \geq 1,
 \end{aligned} \tag{3.10}$$

where  $P_3(x, t)$  is the fractional Peano kernel defined by

$$P_3(x, t) := \begin{cases} \frac{t - (1 - \lambda)a - \lambda b}{b - a} (b - x)^{1-\alpha} \Gamma(\alpha), & a \leq t < x, \\ \frac{t - (1 - \lambda)b - \lambda a}{b - a} (b - x)^{1-\alpha} \Gamma(\alpha), & x \leq t \leq b, \end{cases} \quad (3.11)$$

for  $0 \leq \lambda \leq 1$ .

*Proof.* By similar way in proof of Lemma 3.1, we have

$$\begin{aligned} \Gamma(\alpha) J_a^\alpha (P_3(x, b) f'(b)) &= \int_a^b (b - t)^{\alpha-1} P_3(x, t) f'(t) dt \\ &= \frac{\Gamma(\alpha)(b - x)^{1-\alpha}}{b - a} \left[ \int_a^x (b - t)^{\alpha-1} (t - (1 - \lambda)a - \lambda b) f'(t) dt \right. \\ &\quad \left. + \int_x^b (b - t)^{\alpha-1} (t - (1 - \lambda)b - \lambda a) f'(t) dt \right] \\ &= \frac{\Gamma(\alpha)(b - x)^{1-\alpha}}{b - a} (J_1 + J_2). \end{aligned} \quad (3.12)$$

Integrating by parts, we can state

$$\begin{aligned} J_1 &= (b - x)^{\alpha-1} (x - (1 - \lambda)a - \lambda b) f(x) + (b - a)^\alpha f(a) \\ &\quad + (\alpha - 1) \int_a^x (b - t)^{\alpha-2} (t - (1 - \lambda)a - \lambda b) f(t) dt - \int_a^x (b - t)^{\alpha-1} f(t) dt, \end{aligned} \quad (3.13)$$

and similarly,

$$\begin{aligned} J_2 &= -(b - x)^\alpha (x - (1 - \lambda)b - \lambda a) f(x) \\ &\quad + (\alpha - 1) \int_x^b (b - t)^{\alpha-2} (t - (1 - \lambda)a - \lambda b) f(t) dt - \int_x^b (b - t)^{\alpha-1} f(t) dt. \end{aligned} \quad (3.14)$$

Thus, by using  $J_1$  and  $J_2$  in (3.12), we get (3.10) which completes the proof.  $\square$

*Remark 3.3.* We note that in the special cases, if we take  $\lambda = 0$  in Theorem 3.2, then we get (3.1) with the kernel  $P_2(x, t)$ .

**Theorem 3.4.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be a differentiable on  $(a, b)$  such that  $f' \in L_1[a, b]$ , where  $a < b$ . If  $|f'(x)| \leq M$  for every  $x \in [a, b]$  and  $\alpha \geq 1$ , then the following inequality holds:

$$\begin{aligned} & \left| (1-2\lambda)f(x) - \frac{\Gamma(\alpha)}{b-a}(b-x)^{1-\alpha}J_a^\alpha f(b) + \lambda\left(\frac{b-a}{b-x}\right)^{\alpha-1}f(a) + J_a^{\alpha-1}(P_3(x,b)f(b)) \right| \\ & \leq \frac{M}{\alpha(\alpha+1)} \left\{ (b-a)^\alpha(b-x)^{1-\alpha} \left[ 2\lambda^{\alpha+1} + 2(1-\lambda)^{\alpha+1} + \lambda(b-a) - 1 \right] \right. \\ & \quad \left. + (b-x) \left[ 2\alpha\frac{b-x}{b-a} - (\alpha+1) \right] \right\}. \end{aligned} \quad (3.15)$$

*Proof.* From Theorem 3.2, we get

$$\begin{aligned} & \left| (1-2\lambda)f(x) - \frac{\Gamma(\alpha)}{b-a}(b-x)^{1-\alpha}J_a^\alpha f(b) + \lambda\left(\frac{b-a}{b-x}\right)^{\alpha-1}f(a) + J_a^{\alpha-1}(P_3(x,b)f(b)) \right| \\ & \leq \frac{1}{\Gamma(\alpha)} \int_a^b (b-t)^{\alpha-1}|P_3(x,t)||f'(t)|dt \\ & = \frac{(b-x)^{1-\alpha}}{b-a} \left[ \int_a^x (b-t)^{\alpha-1}|t-(1-\lambda)a-\lambda b||f'(t)|dt \right. \\ & \quad \left. + \int_x^b (b-t)^{\alpha-1}|t-(1-\lambda)b-\lambda a||f'(t)|dt \right] \\ & \leq \frac{M(b-x)^{1-\alpha}}{b-a} \left\{ \int_a^x (b-t)^{\alpha-1}|t-(1-\lambda)a-\lambda b|dt + \int_x^b (b-t)^{\alpha-1}|t-(1-\lambda)b-\lambda a|dt \right\} \\ & = \frac{M(b-x)^{1-\alpha}}{b-a} \{J_3 + J_4\}. \end{aligned} \quad (3.16)$$

By simple computation, we obtain

$$\begin{aligned} J_3 & = \int_a^x (b-t)^{\alpha-1}|t-(1-\lambda)a-\lambda b|dt \\ & = \int_a^{\lambda b+(1-\lambda)a} (b-t)^{\alpha-1}(\lambda b+(1-\lambda)a-t)dt + \int_{\lambda b+(1-\lambda)a}^x (b-t)^{\alpha-1}(t-\lambda b-(1-\lambda)a)dt \\ & = \frac{(b-a)^{\alpha+1}}{\alpha(\alpha+1)} \left[ 2(1-\lambda)^{\alpha+1} + \lambda(b-a) - 1 \right] + \frac{(b-x)^\alpha}{\alpha(\alpha+1)} [\alpha(b-x) - (1-\lambda)(b-a)(\alpha+1)], \end{aligned} \quad (3.17)$$

and similarly

$$\begin{aligned}
 J_4 &= \int_x^b (b-t)^{\alpha-1} |t - (1-\lambda)b - \lambda a| dt \\
 &= \int_x^{\lambda a + (1-\lambda)b} (b-t)^{\alpha-1} (\lambda a + (1-\lambda)b - t) dt + \int_{\lambda a + (1-\lambda)b}^b (b-t)^{\alpha-1} (t - \lambda a - (1-\lambda)b) dt \quad (3.18) \\
 &= \frac{2\lambda^{\alpha+1}(b-a)^{\alpha+1}}{\alpha(\alpha+1)} + \frac{(b-x)^\alpha}{\alpha(\alpha+1)} [\alpha(b-x) - \lambda(b-a)(\alpha+1)].
 \end{aligned}$$

By using  $J_3$  and  $J_4$  in (3.16), we obtain (3.15). □

*Remark 3.5.* If we take  $\lambda = 0$  in Theorem 3.4, then it reduces Theorem 4.1 proved by Anastassiou et al. [13]. So, our results are generalizations of the corresponding results of Anastassiou et al. [13].

**Theorem 3.6.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be a differentiable convex function on  $(a, b)$  and  $f' \in L_1[a, b]$ . Then for any  $x \in (a, b)$ , the following inequality holds:

$$\begin{aligned}
 &\frac{1}{\alpha(\alpha+1)} \left[ \alpha \frac{(b-x)^2}{b-a} f'_+(x) - \left( (b-a)^\alpha (b-x)^{1-\alpha} + \alpha \frac{(b-x)^2}{b-a} - (\alpha+1)(b-x) \right) f'_-(x) \right] \quad (3.19) \\
 &\leq \frac{\Gamma(\alpha)}{b-a} (b-x)^{1-\alpha} J_a^\alpha f(b) - J_a^{\alpha-1} (P_2(x, b) f(b)) - f(x), \quad \alpha \geq 1.
 \end{aligned}$$

*Proof.* Similarly to the proof of Lemma 3.1, we have

$$\begin{aligned}
 f(x) &- \frac{\Gamma(\alpha)}{b-a} (b-x)^{1-\alpha} J_a^\alpha f(b) + J_a^{\alpha-1} (P_2(x, b) f(b)) \\
 &= \frac{(b-x)^{1-\alpha}}{b-a} \left[ \int_a^x (b-t)^{\alpha-1} (t-a) f'(t) dt - \int_x^b (b-t)^\alpha f'(t) dt \right]. \quad (3.20)
 \end{aligned}$$

Since  $f$  is convex, then for any  $x \in (a, b)$  we have the following inequalities:

$$f'(t) \leq f'_-(x) \quad \text{for a.e. } t \in [a, x], \quad (3.21)$$

$$f'(t) \geq f'_+(x) \quad \text{for a.e. } t \in [x, b]. \quad (3.22)$$

If we multiply (3.21) by  $(b-t)^{\alpha-1}(t-a) \geq 0, t \in [a, x], \alpha \geq 1$  and integrate on  $[a, x]$ , we get

$$\begin{aligned}
 \int_a^x (b-t)^{\alpha-1} (t-a) f'(t) dt &\leq \int_a^x (b-t)^{\alpha-1} (t-a) f'_-(x) dt \\
 &= \frac{1}{\alpha(\alpha+1)} \left[ (b-a)^{\alpha+1} + (b-x)^\alpha [\alpha(b-x) - (\alpha+1)(b-a)] \right] f'_-(x), \quad (3.23)
 \end{aligned}$$

and if we multiply (3.22) by  $(b-t)^\alpha \geq 0$ ,  $t \in [x, b]$ ,  $\alpha \geq 1$  and integrate on  $[x, b]$ , we also get

$$\int_x^b (b-t)^\alpha f'(t) dt \geq \int_x^b (b-t)^\alpha f'_+(x) dt = \frac{(b-x)^{\alpha+1}}{\alpha+1} f'_+(x). \quad (3.24)$$

Finally, if we subtract (3.24) from (3.23) and use the representation (3.20) we deduce the desired inequality (3.19).  $\square$

**Corollary 3.7.** *Under the assumptions Theorem 3.6 with  $\alpha = 1$ , one has*

$$\frac{1}{2} \left[ (b-x)^2 f'_+(x) - (a-x)^2 f'_-(x) \right] \leq \int_a^b f(t) dt - (b-a)f(x). \quad (3.25)$$

The proof of Corollary 3.7 is proved by Dragomir in [6]. Hence, our results in Theorem 3.6 are generalizations of the corresponding results of Dragomir [6].

*Remark 3.8.* If we take  $x = (a+b)/2$  in Corollary 3.7, we get

$$0 \leq \frac{b-a}{8} \left[ f'_+\left(\frac{a+b}{2}\right) - f'_-\left(\frac{a+b}{2}\right) \right] \leq \frac{1}{b-a} \int_a^b f(t) dt - f\left(\frac{a+b}{2}\right). \quad (3.26)$$

**Theorem 3.9.** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be a differentiable convex function on  $(a, b)$  and  $f' \in L_1[a, b]$ . Then for any  $x \in [a, b]$ , the following inequality holds:*

$$\begin{aligned} & \frac{\Gamma(\alpha)}{b-a} (b-x)^{1-\alpha} J_a^\alpha f(b) - J_a^{\alpha-1}(P_2(x, b)f(b)) - f(x) \\ & \leq \frac{1}{\alpha(\alpha+1)} \left[ \alpha \frac{(b-x)^2}{b-a} f'_-(b) \right. \\ & \quad \left. - \left( (b-a)^\alpha (b-x)^{1-\alpha} + \alpha \frac{(b-x)^2}{b-a} - (\alpha+1)(b-x) \right) \right], \quad \alpha \geq 1. \end{aligned} \quad (3.27)$$

*Proof.* Assume that  $f'_+(a)$  and  $f'_-(b)$  are finite. Since  $f$  is convex on  $[a, b]$ , then we have the following inequalities:

$$f'(t) \geq f'_+(a) \quad \text{for a.e. } t \in [a, x], \quad (3.28)$$

$$f'(t) \leq f'_-(b) \quad \text{for a.e. } t \in [x, b]. \quad (3.29)$$



If we multiply (3.28) by  $(b-t)^{\alpha-1}(t-a) \geq 0$ ,  $t \in [a, x]$ ,  $\alpha \geq 1$  and integrate on  $[a, x]$ , we have

$$\begin{aligned} \int_a^x (b-t)^{\alpha-1}(t-a)f'(t)dt &\geq \int_a^x (b-t)^{\alpha-1}(t-a)f'_+(a)dt \\ &= \frac{1}{\alpha(\alpha+1)} \left[ (b-a)^{\alpha+1} + (b-x)^\alpha [\alpha(b-x) - (\alpha+1)(b-a)] \right] f'_+(a), \end{aligned} \quad (3.30)$$

and if we multiply (3.29) by  $(b-t)^\alpha \geq 0$ ,  $t \in [x, b]$ ,  $\alpha \geq 1$  and integrate on  $[x, b]$ , we also have

$$\int_x^b (b-t)^\alpha f'(t)dt \leq \int_x^b (b-t)^\alpha f'_-(b)dt = \frac{(b-x)^{\alpha+1}}{\alpha+1} f'_-(b). \quad (3.31)$$

Finally, if we subtract (3.30) from (3.31) and use the representation (3.20) we deduce the desired inequality (3.27).  $\square$

**Corollary 3.10.** *Under the assumptions Theorem 3.9 with  $\alpha = 1$ , one*

$$\int_a^b f(t)dt - (b-a)f(x) \leq \frac{1}{2} \left[ (b-x)^2 f'_-(b) - (a-x)^2 f'_+(a) \right]. \quad (3.32)$$

The proof of Corollary 3.10 is proved by Dragomir in [6]. So, our results in Theorem 3.9 are generalizations of the corresponding results of Dragomir [6].

*Remark 3.11.* If we take  $x = (a+b)/2$  in Corollary 3.10, we get

$$0 \leq \frac{1}{b-a} \int_a^b f(t)dt - f\left(\frac{a+b}{2}\right) \leq \frac{b-a}{8} [f'_-(b) - f'_+(a)]. \quad (3.33)$$

## References

- [1] A. M. Ostrowski, "Über die absolutabweichung einer differentiebaren funktion von ihrem integralmittelwert," *Commentarii Mathematici Helvetici*, vol. 10, pp. 226–227, 1938.
- [2] D. S. Mitrinović, J. E. Pečarić, and A. M. Fink, *Inequalities Involving Functions and Their Integrals and Derivatives*, vol. 53, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1991.
- [3] P. Cerone and S. S. Dragomir, "Trapezoidal-type rules from an inequalities point of view," in *Handbook of Analytic-Computational Methods in Applied Mathematics*, pp. 65–134, Chapman & Hall/CRC, Boca Raton, Fla, USA, 2000.
- [4] J. Duoandikoetxea, "A unified approach to several inequalities involving functions and derivatives," *Czechoslovak Mathematical Journal*, vol. 51, no. 126, pp. 363–376, 2001.
- [5] S. S. Dragomir and N. S. Barnett, "An Ostrowski type inequality for mappings whose second derivatives are bounded and applications," *RGMI Research Report Collection*, vol. 1, pp. 67–76, 1999.
- [6] S. S. Dragomir, "An Ostrowski type inequality for convex functions," *Univerzitet u Beogradu. Publikacije Elektrotehničkog Fakulteta. Serija Matematika*, vol. 16, pp. 12–25, 2005.
- [7] Z. Liu, "Some companions of an Ostrowski type inequality and applications," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 10, no. 2, article 52, 12 pages, 2009.
- [8] M. Z. Sarikaya, "On the Ostrowski type integral inequality," *Acta Mathematica Universitatis Comenianae*, vol. 79, no. 1, pp. 129–134, 2010.

- [9] M. Z. Sarikaya, "On the Ostrowski type integral inequality for double integrals," *Demonstratio Mathematica*, vol. 45, no. 3, pp. 533–540, 2012.
- [10] M. Z. Sarikaya and H. Ogunmez, "On the weighted Ostrowski-type integral inequality for double integrals," *Arabian Journal for Science and Engineering*, vol. 36, no. 6, pp. 1153–1160, 2011.
- [11] R. Gorenflo and F. Mainardi, *Fractional Calculus: Integral and Differentiable Equations of Fractional Order*, Springer, Wien, Austria, 1997.
- [12] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integrals and Derivatives Theory and Application*, Gordon and Breach Science, New York, NY, USA, 1993.
- [13] G. Anastassiou, M. R. Hooshmandasl, A. Ghasemi, and F. Moftakharzadeh, "Montgomery identities for fractional integrals and related fractional inequalities," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 10, no. 4, article 97, 6 pages, 2009.
- [14] S. Belarbi and Z. Dahmani, "On some new fractional integral inequalities," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 10, no. 3, article 86, 5 pages, 2009.
- [15] Z. Dahmani, L. Tabharit, and S. Taf, "Some fractional integral inequalities," *Nonlinear Science Letters*, vol. 2, no. 1, pp. 155–160, 2010.
- [16] Z. Dahmani, L. Tabharit, and S. Taf, "New inequalities via Riemann-Liouville fractional integration," *Journal of Advanced Research in Scientific Computing*, vol. 2, no. 1, pp. 40–45, 2010.