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Research Article

Parseval Relationship of Samples in the Fractional Fourier Transform Domain

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This paper investigates the Parseval relationship of samples associated with the fractional Fourier transform. Firstly, the Parseval relationship for uniform samples of band-limited signal is obtained. Then, the relationship is extended to a general set of nonuniform samples of band-limited signal associated with the fractional Fourier transform. Finally, the two dimensional case is investigated in detail, it is also shown that the derived results can be regarded as the generalization of the classical ones in the Fourier domain to the fractional Fourier transform domain.

1. Introduction

As a generalization of the classical Fourier transform, the fractional Fourier transform (FrFT) has received much attention in recent years [1–5]. It has been shown that the FrFT can be applied to various applications, including optics, radar and sonar, communication signals and underwater signal processing, and so forth, [1–5]. The relationship between the Fourier transform and the FrFT is derived in [6–8]. The discretization and fast computation of FrFT have been proposed by researchers from different perspectives [9–14]. The generalization of the sampling formulae in the traditional Fourier domain to the FrFT domain has been deduced in [7, 8] and [15, 16]. The properties and advantages of the FrFT in signal processing community have been discussed in [17, 18]. For further properties and applications of FrFT in optics and signal processing community, one can refer to [1, 2].

The well-known operations and relations (such as Hilbert transform [19], convolution and product operations [20, 21], uncertainty principle [22], and Poisson summation formula [23]) in traditional Fourier domain have been extended to the fractional Fourier domain by different authors. The spectral analysis and reconstruction for periodic nonuniform samples is investigated in [24], and the short-time FrFT and its applications are studied in [25].

Recently, Lima and Campello De Souza give the definition and properties of FrFT over finite fields [26], Irarrazaval et al. investigates the application of the FrFT in quadratic field magnetic resonance imaging [27]. The relationship between the FrFT and the fractional calculus operators is studied and given in [28]. But, so far none of the research papers throw light on the extension of the traditional Parseval's relationship for band-limited signals associated with the fractional Fourier domain. It is, therefore, worthwhile and interesting to investigate the extension of the Parseval's relationship of band-limited signals in the FrFT domain.

Parseval relationship plays an important role in the Fourier transform domain [29–31], it relates the energy (or power) in the uniformly spaced sample values of a band-limited signal and the energy in the corresponding analog signal. Based on the relationship between the Fourier transform and the FrFT, this paper investigates the generalization of the traditional Parseval relationship of the Fourier domain to the FrFT domain.

The paper is organized as follows: the preliminaries are presented in Section 2, the main results of the paper are obtained in Section 3, and the conclusion and future working directions are given in Section 4.

2. Preliminaries

2.1. The Fractional Fourier Transform

The ordinary Fourier transform plays an important role in modern signal processing community, little need be said of the importance and ubiquity of the ordinary Fourier transform, and frequency domain concepts and techniques in many diverse areas of science and engineering. The Fourier transform of a signal $f(t)$ is defined as

$$F(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t) e^{-jut} dt. \tag{2.1}$$

The FrFT can be viewed as the generalization of the Fourier transform with an order parameter α, and the FrFT of a signal $f(t)$ is given by [1, 2] as

$$F_\alpha(u) = \begin{cases} \sqrt{\dfrac{1 - j\cot\alpha}{2\pi}} \displaystyle\int_{-\infty}^{+\infty} f(t) K_\alpha(u,t) dt, & \alpha \neq n\pi, \\ f(u), & \alpha = 2n\pi, \\ f(-u), & \alpha = (2n+1)\pi, \end{cases} \tag{2.2}$$

where $K_\alpha(u,t) = \exp\{j(1/2)[\cot\alpha t^2 - 2\csc\alpha tu + \cot\alpha u^2]\}$. The original signal $f(t)$ can be derived by the inverse FrFT transform of F_α as

$$f(t) = \begin{cases} \sqrt{\dfrac{1 + j\cot\alpha}{2\pi}} \displaystyle\int_{-\infty}^{+\infty} F_\alpha(u) K_{-\alpha}(t,u) du, & \alpha \neq n\pi, \\ F_\alpha(t), & \alpha = 2n\pi, \\ F_\alpha(-t), & \alpha = (2n+1)\pi. \end{cases} \tag{2.3}$$

It is easy to show that the FrFT reduces to the ordinary Fourier transform when $\alpha = \pi/2$. In order to obtain new results, this paper deals with the case of $\alpha \neq n\pi$.

A signal $f(t)$ is said to be band-limited with respect to Ω_α in FrFT domain with order α, if

$$F_\alpha(u) = 0, \quad |u| > \Omega_\alpha. \tag{2.4}$$

For a signal $f(t)$ bandlimited in the LCT domain, the following lemma reflects the relationship between the band-limited signals in Fourier domain and the FrFT domain.

Lemma 2.1. *Suppose that a signal $f(t)$ is band-limited with respect to Ω_α in FrFT domain with order α, and let*

$$g(t) = \int_{-\Omega_\alpha}^{\Omega_\alpha} F_\alpha(u) \exp\left[-j\frac{1}{2} \cot \alpha u^2 + j \csc \alpha u t\right] du, \tag{2.5}$$

then the Fourier transform of signal $g(t)$ can be represented by the FrFT of signal $f(t)$ as

$$G(u) = \frac{1}{2\pi} F_\alpha(\sin \alpha u) e^{-j(1/4)\sin 2\alpha u^2}, \tag{2.6}$$

and $g(t)$ is a $|\csc\alpha|\Omega_\alpha$ band-limited signal in the ordinary Fourier transform domain.

Proof. Performing the Fourier transform to (2.5), we obtain that

$$\begin{aligned}
G(w) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(t) e^{-jwt} dt \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \int_{-\Omega_\alpha}^{\Omega_\alpha} F_\alpha(u) \exp\left[-j\frac{1}{2}\cot \alpha u^2 + j\csc \alpha u t\right] du e^{-jwt} du\, dt \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\Omega_\alpha}^{\Omega_\alpha} F_\alpha(u) \exp\left[-j\frac{1}{2}\cot \alpha u^2\right] \int_{-\infty}^{+\infty} e^{j(\csc \alpha u - w)t} du\, dt \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\Omega_\alpha}^{\Omega_\alpha} F_\alpha(u) \exp\left[-j\frac{1}{2}\cot \alpha u^2\right] \frac{1}{\sqrt{2\pi}} \delta(\csc \alpha u - w) du \\
&= \frac{1}{2\pi} F_\alpha(\sin \alpha w) e^{-j(1/4)\sin 2\alpha w^2}.
\end{aligned} \tag{2.7}$$

This proves the relationship between the Fourier transform of $g(t)$ and the FrFT of $f(t)$. Because $f(t)$ is band-limited with respect to Ω_α in FrFT domain with order α, so it is easy to show that signal $g(t)$ is a $|\csc\alpha|\Omega_\alpha$ band-limited in the ordinary Fourier transform domain. □

From the definition of signal $f(t)$ and $g(t)$, the relationship between the signal $f(t)$ and $g(t)$ can be derived as

$$g(t) = f(t)\sqrt{\frac{2\pi}{1 + j\cot\alpha}}\exp\left(j\frac{1}{2}\cot\alpha t^2\right). \tag{2.8}$$

2.2. The Two Dimensional FrFT

In [32], the-two dimensional FrFT of a signal $f(x, y)$ is defined as

$$F_{\alpha,\beta}(u, v) = \iint_{-\infty}^{+\infty} f(x, y)K_{\alpha,\beta}(x, y, u, v)dx\,dy, \tag{2.9}$$

where the FrFT kernel $K_{\alpha,\beta}(x, y, u, v)$ can be written as

$$K_{\alpha,\beta}(x, y, u, v) = \frac{\sqrt{1 - j\cot\alpha}\sqrt{1 - j\cot\beta}}{2\pi}e^{j((x^2+u^2)/2)\cot\alpha - jux\csc\alpha}e^{j((y^2+v^2)/2)\cot\beta - jyv\csc\beta}. \tag{2.10}$$

The original signal $f(x, y)$ can be recovered by a two-dimensional FrFT with backward angles $(-\alpha, -\beta)$ as follows:

$$f(x, y) = \iint_{-\infty}^{+\infty} F_{\alpha,\beta}(u, v)K_{-\alpha,-\beta}(u, v, x, y)du\,dv. \tag{2.11}$$

The definition of bandlimited two-dimensional signals can be similarly defined as the one-dimensional signal; following the prove of Lemma 2.1, the two-dimensional cases can be summarized as Lemma 2.2.

Lemma 2.2. *Suppose that a signal $f(x, y)$ is band-limited with respect to $(\Omega_\alpha, \Omega_\beta)$ in FrFT domain with order α and β, and let*

$$g(x, y) = \int_{-\Omega_\alpha}^{\Omega_\alpha}\int_{-\Omega_\beta}^{\Omega_\beta} F_{\alpha,\beta}(u, v)e^{-j(1/2)\cot\alpha u^2 + j\csc\alpha ux}e^{-j(1/2)\cot\beta v^2 + j\csc\beta vy}du\,dv, \tag{2.12}$$

then the Fourier transform of signal $g(x, y)$ can be represented by the FrFT of signal $f(x, y)$ as

$$G(u, v) = \frac{1}{(2\pi)^2}F_\alpha(\sin\alpha u, \sin\beta v)e^{-j(1/4)\sin 2\alpha u^2}e^{-j(1/4)\sin 2\beta v^2}, \tag{2.13}$$

and $g(x, y)$ is a $(|\csc\alpha|\Omega_\alpha, |\csc\beta|\Omega_\beta)$ band-limited signal in the ordinary Fourier transform domain.

Proof. Similar with the proof of Lemma 2.1, the results can derived easily. □

From (2.12) and the definition of the two-dimensional FrFT, the relationship between the signal $f(x,y)$ and $g(x,y)$ is

$$g(x,y) = \sqrt{\frac{2\pi}{1+j\cot\alpha}}\sqrt{\frac{2\pi}{1+j\cot\beta}} f(x,y) e^{j(1/2)\cot\alpha x^2} e^{j(1/2)\cot\beta y^2}. \tag{2.14}$$

2.3. The Parseval Relationship

The Parseval's relation states that the energy in time domain is the same as the energy in frequency domain, which can be expressed as follows [29]:

$$\int_{-\infty}^{+\infty} f(t) g^*(t) dt = \int_{-\infty}^{+\infty} F(u) G^*(u) du, \tag{2.15}$$

where $F(u)$ and $G(u)$ are Fourier transforms of $f(t)$ and $g(t)$, respectively. This formula is called Parseval's relation and holds for all members of the Fourier transform family.

The FrFT can be regarded as the generalization of the Fourier transform, and the similar relation of (2.15) in the FrFT sense can be obtained as [1, 2]

$$\int_{-\infty}^{+\infty} f(t) g^*(t) dt = \int_{-\infty}^{+\infty} F_\alpha(u) G_\alpha^*(u) du, \tag{2.16}$$

where $F_\alpha(u)$ and $G_\alpha(u)$ are FrFT of $f(t)$ and $g(t)$ with order α, respectively. When $f(t) = g(t)$, the relation of (2.16) can be written as

$$\int_{-\infty}^{+\infty} |f(t)|^2 dt = \int_{-\infty}^{+\infty} |F_\alpha(u)|^2 du. \tag{2.17}$$

Equations (2.15)–(2.17) are the Parseval's relationship between the continuous signal and its fractional Fourier transform (or Fourier transform) and can be derived by the Parseval theorem for L^2 signals.

In practical situations, we often encounter the calculation of Parseval relations between the discrete signal and the analog signal. Marvasti and Chuande in [30], and Luthra in [31] investigate the Parseval relations of band-limited signal in the traditional Fourier transform domain. The Parseval relation for band-limited discrete uniformly sampled signal $f(t)$ in the Fourier domain is [30, 31]

$$\sum_{n=-\infty}^{+\infty} |f(nT)|^2 = \frac{1}{T}\int_{-\infty}^{+\infty} |f(t)|^2 dt = \frac{1}{T}\int_{-W}^{+W} |F(u)|^2 du, \tag{2.18}$$

where $F(u)$ is the ordinary Fourier transform of $f(t)$, and T is the sampling interval that satisfies $1/T \geq W/\pi$, and $f(t)$ is band-limited to $(-W, W)$ in the ordinary Fourier transform domain. Similarly, the Parseval relationship for bandlimited two-dimensional signal $f(x,y)$ associated with the Fourier transform can be written as follow:

$$\begin{aligned}\sum_{n=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty}\left|f(nT_1,mT_2)\right|^2 &= \frac{1}{T_1T_2}\iint_{-\infty}^{+\infty}\left|f(x,y)\right|^2dt \\ &= \frac{1}{T_1T_2}\int_{-W_1}^{+W_1}\int_{-W_2}^{+W_2}|F(u,v)|^2du\,dv.\end{aligned} \tag{2.19}$$

It is proved in [30] that if a set of samples $t_n(n=\ldots,-1,0,1,\ldots)$ is a sampling set, then the associated Parseval relation for the nonuniformly sampled signals can be written as

$$\sum_{n=-\infty}^{+\infty}\left|f(t_n)\right|^2 = \frac{1}{T}\int_{-\infty}^{+\infty} f(t)f_{lp}^*(t)dt = \frac{1}{T}\int_{-W}^{+W} F(w)F_{lp}^*(w)dw, \tag{2.20}$$

where $f_{lp}(t)$ is the low-pass filtered version of the nonuniformly samples, and $T = W/\pi$. $F(w)$ and $F_{lp}(w)$ are the corresponding Fourier transforms of $f(t)$ and $f_{lp}(t)$.

The objective of this paper is to obtain the corresponding Parseval relationship for a set of uniform and nonuniform samples of a band-limited signal in the FrFT domain. It is shown that the derived results can be seen as the generalization of the classical results in the Fourier domain.

3. The Main Results

Suppose that a signal $f(t)$ is band-limited to $(-\Omega_\alpha,\Omega_\alpha)$ in the FrFT domain for order α, and T_α is the sampling interval that satisfies the uniform sampling theorem of signal in the FrFT domain [1, 2]; for example, $1/T_\alpha \geq \Omega_\alpha|\csc\alpha|/\pi$. The objective of this section is to investigate the Parseval relationship for uniform and nonuniform samples of signal $f(t)$ in the FrFT domain.

3.1. The Parseval Relationship for Uniform Samples

Theorem 3.1. *Suppose that a signal $f(t)$ is Ω_α band-limited in the FrFT domain with order α, then the Parseval relationship associated with the signal $f(t)$ in the FrFT domain can be expressed as*

$$\sum_{n=-\infty}^{+\infty}\left|f(nT_\alpha)\right|^2 = \frac{1}{T_\alpha}\int_{-\infty}^{+\infty}\left|f(t)\right|^2dt = \frac{|\csc\alpha|}{T_\alpha 2\pi}\int_{-|\csc\alpha|\Omega_\alpha}^{|\csc\alpha|\Omega_\alpha}|F_\alpha(\sin\alpha u)|^2du, \tag{3.1}$$

where T_α is sampling interval, and $F_\alpha(u)$ is the FrFT of signal $f(t)$ with order α.

Proof. Let

$$g(t) = \int_{-\Omega_\alpha}^{\Omega_\alpha} F_\alpha(u)\exp\left[-j\frac{1}{2}\cot\alpha u^2 + j\csc\alpha ut\right]du, \tag{3.2}$$

then $g(t)$ is a band-limited signal in the traditional Fourier domain. Applying (2.18) to signal $g(t)$, we obtain that

$$\sum_{n=-\infty}^{+\infty} \left|g(nT_\alpha)\right|^2 = \frac{1}{T_\alpha}\int_{-\infty}^{+\infty} \left|g(t)\right|^2 dt = \frac{1}{T_\alpha}\int_{-|\csc\alpha|\Omega_\alpha}^{+|\csc\alpha|\Omega_\alpha} |G(u)|^2 du. \tag{3.3}$$

Substituting (2.8) into (3.3), we obtain that

$$\begin{aligned}\sum_{n=-\infty}^{+\infty} &\left| f(nT_\alpha)\sqrt{\frac{2\pi}{1+j\cot\alpha}}\exp\left(j\frac{1}{2}\cot\alpha(nT_\alpha)^2\right)\right|^2 \\ &= \frac{1}{T_\alpha}\int_{-\infty}^{+\infty}\left| f(t)\sqrt{\frac{2\pi}{1+j\cot\alpha}}\exp\left(j\frac{1}{2}\cot\alpha t^2\right)\right|^2 dt \\ &= \frac{1}{T_\alpha}\int_{-|\csc\alpha|\Omega_\alpha}^{+|\csc\alpha|\Omega_\alpha}\left|F_\alpha(\sin\alpha u)e^{-j(1/4)\sin 2\alpha u^2}\right|^2 du.\end{aligned} \tag{3.4}$$

It is easy to verify that, $|\sqrt{2\pi/(1+j\cot\alpha)}|^2 = 2\pi|\sqrt{(1-j\cot\alpha)/(1+\cot^2\alpha)}|^2 = 2\pi\sqrt{|1-j\cot\alpha|/(1+\cot^2\alpha)}^2 = 2\pi/|\csc\alpha|$, and the magnitude of exponential function is

$$\left|\exp\left(j\frac{1}{2}\cot\alpha(nT_\alpha)^2\right)\right| = 1, \qquad \left|\exp\left(j\frac{1}{2}\cot\alpha(t)^2\right)\right| = 1. \tag{3.5}$$

Substituting these results in (3.4), we obtain the final result as follows:

$$\sum_{n=-\infty}^{+\infty} \left|f(nT_\alpha)\right|^2 = \frac{1}{T_\alpha}\int_{-\infty}^{+\infty} \left|f(t)\right|^2 dt = \frac{|\csc\alpha|}{T_\alpha 2\pi}\int_{-|\csc\alpha|\Omega_\alpha}^{|\csc\alpha|\Omega_\alpha} |F_\alpha(\sin\alpha u)|^2 du. \tag{3.6}$$

□

Equation (3.1) can be seen as the generalization of the Parseval relations for the uniformly sampled signals associated with the FrFT. The next subsection focus on the generalization of the Parseval relations for the nonuniform sampling sets in the FrFT domain.

3.2. The Parseval Relationship of Nonuniform Samples

Suppose that a general nonuniform sampling set $\{t_n, n = \ldots, -1, 0, 1, \ldots\}$ is obtained from the Ω_α bandlimited signal $f(t)$ in the FrFT domain. If this sampling set satisfies the condition proposed in [30], then the Parseval relationship for this nonuniform sampling set can be derived as the following Theorem 3.2.

Theorem 3.2. *The Parseval relationship of nonuniform samples can be written as*

$$\sum_{n=-\infty}^{+\infty} |f(t_n)|^2 = \frac{1}{T_\alpha}\int_{-\infty}^{+\infty} f(t) f_{lp}^*(t)dt$$
$$= \frac{|\csc\alpha|}{T_\alpha(2\pi)^2}\int_{-|\csc\alpha|\Omega_\alpha}^{+|\csc\alpha|\Omega_\alpha} e^{-j(1/4)\sin 2\alpha u^2} F_\alpha(u\sin\alpha) G_{lp}^*(u)du, \tag{3.7}$$

where $f_{lp}(t) = \sum_{n=-\infty}^{+\infty} f(t_n)\exp(j(1/2)\cot\alpha(t^2 - t_n^2))(\sin c[\csc\alpha\Omega_\alpha(t-t_n)])$, $F_\alpha(u)$ *is the FrFT of* $f(t)$, *and* $G_{lp}(u)$ *is the Fourier transform of* $g_{lp}(t)$.

Proof. Let

$$g(t) = \int_{-\Omega_\alpha}^{\Omega_\alpha} F_\alpha(u)\exp\left[-j\frac{1}{2}\cot\alpha u^2 + j\csc\alpha ut\right]du \tag{3.8}$$

then $g(t)$ is a $|\csc\alpha|\Omega_\alpha$ bandlimited signal in the Fourier domain. Applying the classical Parseval relationship of (2.20) for the bandlimited signals in the Fourier domain to signal $g(t)$, we obtain

$$\sum_{n=-\infty}^{+\infty} |g(t_n)|^2 = \frac{1}{T_\alpha}\int_{-\infty}^{+\infty} g(t) g_{lp}^*(t)dt = \frac{1}{T_\alpha}\int_{-|\csc\alpha|\Omega_\alpha}^{+|\csc\alpha|\Omega_\alpha} G(u)G_{lp}^*(u)du, \tag{3.9}$$

where g_{lp} is the signal obtained after low-pass filtering of the sampled signal

$$g_{lp}(t) = \sum_{n=-\infty}^{+\infty} g(t_n)\sin c[\csc\alpha\Omega_\alpha(t-t_n)]$$
$$= \sum_{n=-\infty}^{+\infty} f(t_n)\sqrt{\frac{2\pi}{1+j\cot\alpha}}\exp\left(j\frac{1}{2}\cot\alpha t_n^2\right)\sin c[\csc\alpha\Omega_\alpha(t-t_n)]. \tag{3.10}$$

From the relationship between $g(t)$ and $f(t)$, the following relations can be obtained:

$$\sum_{n=-\infty}^{+\infty} |g(t_n)|^2 = \sum_{n=-\infty}^{+\infty}\left|f(t_n)\sqrt{\frac{2\pi}{1+j\cot\alpha}}\exp\left(j\frac{1}{2}\cot\alpha t_n^2\right)\right|^2$$
$$= \frac{2\pi}{|\csc\alpha|}\sum_{n=-\infty}^{+\infty}|f(t_n)|^2,$$
$$\frac{1}{T_\alpha}\int_{-\infty}^{+\infty} g(t)g_{lp}^*(t)dt = \frac{2\pi}{|\csc\alpha|T_\alpha}\int_{-\infty}^{+\infty} f(t)\sum_{n=-\infty}^{+\infty} f^*(t_n)\exp\left(j\frac{1}{2}\cot\alpha\left(t^2-t_n^2\right)\right)$$
$$\times(\sin c[\csc\alpha\Omega_\alpha(t-t_n)])dt$$
$$= \frac{2\pi}{|\csc\alpha|T_\alpha}\int_{-\infty}^{+\infty} f(t) f_{lp}^*(t)dt. \tag{3.11}$$

From (3.11), the first part of (3.9) can be rewritten as

$$\sum_{n=-\infty}^{+\infty} \left| f(t_n) \right|^2 = \frac{1}{T_\alpha} \int_{-\infty}^{+\infty} f(t) f_{lp}^*(t) dt. \tag{3.12}$$

From Lemma 2.1, the following relationship holds for $G(u)$ and $F_\alpha(u)$:

$$G(u) = \frac{1}{2\pi} F_\alpha(\sin \alpha u) e^{-j(1/4)\sin 2\alpha u^2}. \tag{3.13}$$

Substitute (3.13) in to the final part of (3.9), we obtain that

$$\frac{1}{T_\alpha} \int_{-|\csc \alpha|\Omega_\alpha}^{+|\csc \alpha|\Omega_\alpha} G(u) G_{lp}^*(u) du = \frac{1}{2\pi T_\alpha} \int_{-|\csc \alpha|\Omega_\alpha}^{+|\csc \alpha|\Omega_\alpha} e^{-j(1/4)\sin 2\alpha u^2} F_\alpha(u \sin \alpha) G_{lp}^*(u) du. \tag{3.14}$$

The final result follows from (3.11) and (3.14). □

3.3. *The Parseval Relationship for Two-Dimensional Case*

Based on the definitions of two-dimensional FrFT and bandlimited signals, the Parseval relationship of the one-dimensional cases can be generalized to 2-D signals based on the Lemma 2.2. We would like to give the following Theorem 3.3.

Theorem 3.3. *Suppose that a signal $f(x,y)$ is $(\Omega_\alpha, \Omega_\beta)$ band-limited in the FrFT domain with order α and β, and then the Parseval relationship associated with the signal $f(x,y)$ in the FrFT domain can be expressed as*

$$\begin{aligned} \sum_{n=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} \left| f(nT_1, mT_2) \right|^2 &= \frac{1}{T_\alpha T_\beta} \iint_{-\infty}^{+\infty} \left| f(x,y) \right|^2 dt \\ &= \frac{1}{T_\alpha T_\beta} \int_{-\Omega_\alpha}^{+\Omega_\alpha} \int_{-\Omega_\beta}^{+\Omega_\beta} \left| F_{\alpha,\beta}(\sin \alpha u, \sin \beta v) \right|^2 du\, dv, \end{aligned} \tag{3.15}$$

where T_α and T_β are sampling interval, and $F_{\alpha,\beta}(u,v)$ is the two-dimensional FrFT of signal $f(x,y)$ with order α and β.

Proof. Similar with the proof of Theorem 3.1, let

$$g(x,y) = \sqrt{\frac{2\pi}{1 + j\cot \alpha}} \sqrt{\frac{2\pi}{1 + j\cot \beta}} f(x,y) e^{j(1/2)\cot\alpha x^2} e^{j(1/2)\cot\beta y^2}. \tag{3.16}$$

Then, from Lemma 2.2 $g(x,y)$ is a $(|\csc \alpha|\Omega_\alpha, |\csc \beta|\Omega_\beta)$ band-limited signal in the ordinary Fourier transform domain. By applying the classical two-dimensional Parseval relationship of (2.19) to signal $g(x,y)$, we can obtain the final result. □

4. Conclusions

Based on the relationship between the Fourier transform and the FrFT, this paper investigates the Parseval's relationship of sampled signals in the FrFT domain. We firstly investigate the Parseval relationship for the uniformly samples of bandlimited signal associated with the FrFT. Then, we extend this relationship to a general set of nonuniform samples of band-limited signal in the FrFT domain. Finally, we studied the Parseval relations for uniformly sampled bandlimited two-dimensional signals, and it is also shown that the derived results can be seen as the generalization of the classical results in the Fourier domain to the FrFT domain. Future works includes the derivation of the Parseval's relations in the linear canonical transform domain for one- and two-dimensional uniformly (nonuniformly) sampled signals, and the applications of the derived results in the sampling theories and other related areas.

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