Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2012, Article ID 427479, 30 pages
doi:10.1155/2012/427479

Research Article

The Modified Rational Jacobi Elliptic Functions Method for Nonlinear Differential Difference Equations

Khaled A. Gepreel,[1,2] Taher A. Nofal,[2,3] and Ali A. Al-Thobaiti[2]

[1] *Mathematics Department, Faculty of Science, Zagazig University, Zagazig, Egypt*
[2] *Mathematics Department, Faculty of Science, Taif University, Saudi Arabia*
[3] *Mathematics Department, Faculty of Science, El-Minia University, El-Minia, Egypt*

Correspondence should be addressed to Ali A. Al-Thobaiti, ali22c@hotmail.com

Received 21 March 2012; Accepted 30 May 2012

Academic Editor: Jianke Yang

We modified the rational Jacobi elliptic functions method to construct some new exact solutions for nonlinear differential difference equations in mathematical physics via the lattice equation, the discrete nonlinear Schrodinger equation with a saturable nonlinearity, the discrete nonlinear Klein-Gordon equation, and the quintic discrete nonlinear Schrodinger equation. Some new types of the Jacobi elliptic solutions are obtained for some nonlinear differential difference equations in mathematical physics. The proposed method is more effective and powerful to obtain the exact solutions for nonlinear differential difference equations.

1. Introduction

In recent years, the study of difference equations has acquired a new significance, due in large part to their use in the formulation and analysis of discrete-time systems, the numerical integration of differential equation by finite-difference schemes and the study of deterministic chaos.

Since the work of Fermi et al. in the 1960s [1], DDEs have been the focus of many nonlinear studies. On the other hand, a considerable number of well-known analytic methods are successfully extended to nonlinear DDEs by researchers [2–17]. However, no method obeys the strength and the flexibility for finding all solutions to all types of nonlinear DDEs. Zhang et al. [18] and Aslan [19] used the (G'/G)-expansion function method to some physically important nonlinear DDEs. Qiong and Bin [12] constructed the Jacobi elliptic solutions for nonlinear DDEs. Recently, Zhang [20] and Gepreel [21] and Gepreel and

Shehata [22] have used the Jacobi elliptic function method for constructing new and more general Jacobi elliptic function solutions of the nonlinear differential difference equations. The main objective of this paper is to modify the rational Jacobi elliptic functions method to obtain the exact wave solutions for nonlinear DDEs. We use this method to calculate the exact wave solutions for some nonlinear DDEs in mathematical physics via the lattice equation, the discrete nonlinear Schrodinger equation with a saturable nonlinearity, the discrete nonlinear Klein-Gordon equation, and the quintic discrete nonlinear Schrodinger equation.

2. Description of the Modified Rational Jacobi Elliptic Functions Method

In this section, we would like to outline an algorithm for using the modified rational Jacobi elliptic functions method to solve nonlinear DDEs. For a given nonlinear DDEs,

$$\begin{aligned}\Delta\Big(&u_{n+p_1}(x),\dots,u_{n+p_k}(x),u'_{n+p_1}(x),\dots,u'_{n+p_k}(x),\dots,u^{(r)}_{n+p_1}(x),\dots,u^{(r)}_{n+p_k}(x),\\&v_{n+p_1}(x),\dots,v_{n+p_k}(x),v'_{n+p_1}(x),\dots,v'_{n+p_k}(x),\dots,v^{(r)}_{n+p_1}(x),\dots,v^{(r)}_{n+p_k}(x),\dots\Big)=0,\end{aligned} \tag{2.1}$$

where $\Delta = (\Delta_1, ..., \Delta_g)$, $x = (x_1, x_2, ..., x_m)$, $n = (n_1, ..., n_Q)$, and $g, m, Q, p_1, ..., p_k$ are integers, $u_i^{(r)}$, $v_i^{(r)}$ denotes the set of all rth order derivatives of u_i, v_i with respect to x.

The main steps of the algorithm for the modified rational Jacobi elliptic functions method to nonlinear DDEs are outlined as follows.

Step 1. We seek the traveling wave solutions of the following form:

$$u_n(x) = U(\xi_n), \qquad v_n(x) = V(\xi_n), \dots, \tag{2.2}$$

where

$$\xi_n = \sum_{i=1}^{Q} d_i n_i + \sum_{j=1}^{m} c_j x_j + \xi_0, \tag{2.3}$$

where d_i $(i = 1, \dots, Q)$, c_j $(j = 1, \dots, m)$ and the phase ξ_0 are constants to be determined later. The transformations (2.2) are reduced (2.1) to the following ordinary differential difference equations:

$$\begin{aligned}\Omega\Big(&U(\xi_{n+p_1}),\dots,U(\xi_{n+p_k}),U'(\xi_{n+p_1}),\dots,U'(\xi_{n+p_k}),\dots,U^{(r)}(\xi_{n+p_1}),\dots,U^{(r)}(\xi_{n+p_k}),\\&V(\xi_{n+p_1}),\dots,V(\xi_{n+p_k}),V'(\xi_{n+p_1}),\dots,V'(\xi_{n+p_k}),\dots,V^{(r)}_{n+p_1}(\xi_{n+p_1}),\dots,V^{(r)}_{n+p_k}(\xi_{n+p_k}),\dots\Big)=0,\end{aligned} \tag{2.4}$$

where $\Omega = (\Omega_1, \dots, \Omega_g)$.

Step 2. We suppose the modified rational Jacobi elliptic functions solutions of (2.4) in the following form:

$$U(\xi_n) = \sum_{i=0}^{K} \alpha_i \left(\frac{F'(\xi_n)}{F(\xi_n)}\right)^i + \sum_{i=1}^{K} \beta_i \left(\frac{F(\xi_n)}{F'(\xi_n)}\right)^i,$$
$$V(\xi_n) = \sum_{j=0}^{L} \gamma_j \left(\frac{F'(\xi_n)}{F(\xi_n)}\right)^j + \sum_{j=1}^{L} \lambda_j \left(\frac{F(\xi_n)}{F'(\xi_n)}\right)^j, \ldots, \tag{2.5}$$

where α_i, β_i $(i = 1, 2, \ldots, K)$, γ_j, λ_j $(j = 1, 2, \ldots, L)$ are constants to be determined later while $F(\xi_n)$ satisfies a discrete Jacobi elliptic differential equation:

$$F'^2(\xi_n) = e_0 + e_1 F^2(\xi_n) + e_2 F^4(\xi_n), \tag{2.6}$$

and e_0, e_1, e_2 are arbitrary constants.

Step 3. Since the general solution of the proposed equation (2.6) is difficult to obtain and so the iteration relations corresponding to the general exact solutions we discuss the solutions of the proposed discrete Jacobi elliptic differential equation (2.6) at some special cases to e_0, e_1, and e_2, to cover all the Jacobi elliptic functions as follows.

Type 1. If $e_0 = 1$, $e_1 = -(1 + m^2)$, $e_2 = m^2$. In this case, (2.6) has the solution $F(\xi_n) = sn(\xi_n, m)$, where $sn(\xi_n, m)$ is the Jacobi elliptic sine function, and m is the modulus. In this case from using the properties of Jacobi elliptic functions (see [22]), the series expansion solutions (2.5) take the following form:

$$U(\xi_n) = \sum_{i=0}^{K} \alpha_i \left(\frac{cn(\xi_n, m)dn(\xi_n, m)}{sn(\xi_n, m)}\right)^i + \sum_{i=1}^{K} \beta_i \left(\frac{sn(\xi_n, m)}{cn(\xi_n, m)dn(\xi_n, m)}\right),$$
$$V(\xi_n) = \sum_{i=0}^{L} \gamma_i \left(\frac{cn(\xi_n, m)dn(\xi_n, m)}{sn(\xi_n, m)}\right)^i + \sum_{i=1}^{L} \lambda_i \left(\frac{sn(\xi_n, m)}{cn(\xi_n, m)dn(\xi_n, m)}\right), \ldots. \tag{2.7}$$

Further using the properties of Jacobi elliptic functions, the iterative relations can be written in the following form:

$$U(\xi_{n\pm p}) = \sum_{i=0}^{K} \alpha_i \left(\frac{F'(\xi_{n\pm p})}{F(\xi_{n\pm p})}\right)^i + \sum_{i=1}^{K} \beta_i \left(\frac{F(\xi_{n\pm p})}{F'(\xi_{n\pm p})}\right)^i,$$
$$V(\xi_{n\pm p}) = \sum_{i=0}^{L} \gamma_i \left(\frac{F'(\xi_{n\pm p})}{F(\xi_{n\pm p})}\right)^i + \sum_{i=1}^{L} \lambda_i \left(\frac{F(\xi_{n\pm p})}{F'(\xi_{n\pm p})}\right)^i, \ldots, \tag{2.8}$$

where

$$\begin{aligned}\frac{F'(\xi_{n\pm p})}{F(\xi_{n\pm p})} &= \frac{1}{M_1}\Big\{\pm cn(d,m)cn(\xi_n,m)dn(\xi_n,m)dn(d,m) \pm m^2 sn(d,m)sn(\xi_n,m)\\
&\quad \mp 2m^2 sn(d,m)sn^3(\xi_n,m) \mp 2m^2 sn^3(d,m)sn(\xi_n,m) \pm m^2 sn^3(d,m)sn^3(\xi_n,m)\\
&\quad + sn(d,m)sn(\xi_n,m) \pm m^4 sn^3(d,m)\mathrm{sn}^3(\xi_n,m)\\
&\quad \mp m^2 sn^2(d,m)sn^2(\xi_n,m)dn(\xi_n,m)dn(d,m)cn(d,m)cn(\xi_n,m)\Big\},\\
M_1 &= -cn(\phi,m)dn(\phi,m)sn(\xi_n,m) \mp sn(\phi,m)dn(\xi_n,m)(\phi)cn(\xi_n,m)\\
&\quad + m^2 sn^3(\xi_n,m)sn^2(\phi,m)cn(\phi,m)dn(\phi,m)\\
&\quad \pm m^2 sn^2(\xi_n,m)sn^3(\phi,m)cn(\xi_n,m)dn(\xi_n,m),
\end{aligned} \tag{2.9}$$

$d = p_{s1}d_1 + p_{s2}d_2 + \cdots + p_{sQ}d_Q$, p_{sj} is the jth component of shift vector p_s.

Type 2. If $e_0 = 1 - m^2$, $e_1 = 2m^2 - 1$, and $e_2 = -m^2$. In this case, (2.6) has the solution $F(\xi_n) = cn(\xi_n, m)$. From using the properties of Jacobi elliptic functions, the series expansion solutions (2.5) take the following form:

$$\begin{aligned}U(\xi_n) &= \sum_{i=0}^{K}\alpha_i\left(-\frac{sn(\xi_n,m)dn(\xi_n,m)}{cn(\xi_n,m)}\right)^i + \sum_{i=1}^{K}\beta_i\left(-\frac{cn(\xi_n,m)}{sn(\xi_n,m)dn(\xi_n,m)}\right)^i,\\
V(\xi_n) &= \sum_{i=0}^{L}\gamma_i\left(-\frac{sn(\xi_n,m)dn(\xi_n,m)}{cn(\xi_n,m)}\right)^i + \sum_{i=1}^{L}\lambda_i\left(-\frac{cn(\xi_n,m)}{sn(\xi_n,m)dn(\xi_n,m)}\right)^i,\ldots.
\end{aligned} \tag{2.10}$$

Type 3. If $e_0 = m^2 - 1$, $e_1 = 2 - m^2$, and $e_2 = -1$. In this case, (2.6) has the solution $F(\xi_n) = dn(\xi_n, m)$. From using the properties of Jacobi elliptic functions, the series expansion solutions (2.5) take the following form:

$$\begin{aligned}U(\xi_n) &= \sum_{i=0}^{K}\alpha_i\left(-\frac{m^2 sn(\xi_n,m)cn(\xi_n,m)}{dn(\xi_n,m)}\right)^i + \sum_{i=1}^{K}\beta_i\left(-\frac{dn(\xi_n,m)}{m^2 sn(\xi_n,m)cn(\xi_n,m)}\right)^i,\\
V(\xi_n) &= \sum_{i=0}^{L}\gamma_i\left(-\frac{m^2 sn(\xi_n,m)cn(\xi_n,m)}{dn(\xi_n,m)}\right)^i + \sum_{i=1}^{L}\lambda_i\left(-\frac{dn(\xi_n,m)}{m^2 sn(\xi_n,m)cn(\xi_n,m)}\right)^i,\ldots.
\end{aligned} \tag{2.11}$$

From the properties of the Jacobi elliptic functions, we can deduce the iterative relation to the above kind of solutions from Types 2 and 3 as we show in Type 1.

Step 4. Determining the degree $K, L, \ldots$ of (2.5) by balancing the nonlinear term(s) and the highest order derivatives of $U(\xi_n), V(\xi_n), \ldots$ in (2.4). It should be noted that the leading terms $U(\xi_{n\pm p}), V(\xi_{n\pm p}), \ldots, p \neq 0$ will not affect the balance because we are interested in balancing the terms of $F'(\xi_n)/F(\xi_n)$.

Step 5. Substituting $U(\xi_n), V(\xi_n), \ldots$ in each type form (2.1)–(2.4) and the given values of $K, L, \ldots$ into (2.4). Cleaning the denominator and collecting all terms with the same degree of $sn(\xi_n, m)$, $cn(\xi_n, m)$, $dn(\xi_n, m)$ together, the left-hand side of (2.4) is converted into a polynomial in $sn(\xi_n, m)$, $cn(\xi_n, m)$, $dn(\xi_n, m)$. Setting each coefficient of this polynomial to zero, we derive a set of algebraic equations for α_i, β_i, d_i, γ_i, λ_i, and c_i.

Step 6. Solving the over determined system of nonlinear algebraic equations by using Maple or Mathematica. We end up with explicit expressions for α_i, β_i, d_i, γ_i, λ_i, and c_j.

Step 7. Substituting α_i, β_i, d_i, γ_i, λ_i, and c_i into $U(\xi_n), V(\xi_n), \ldots$ in the corresponding type from (2.1)–(2.4), we can finally obtain exact solutions for (2.1).

3. Applications

In this section, we use the proposed method to construct the rational Jacobi elliptic wave solutions for the nonlinear DDEs via the lattice equation, the discrete nonlinear Schrodinger equation with a saturable nonlinearity, the discrete nonlinear Klein-Gordon equation, and the quintic discrete nonlinear Schrodinger equation, which are very important in mathematical physics and have been paid attention to by many researchers.

3.1. Example 1. The Lattice Equation

In this section, we study the lattice equation which takes the following form [23–26]:

$$\frac{du_n(t)}{dt} = \left(\alpha + \beta u_n + \gamma u_n^2\right)(u_{n-1} - u_{n+1}), \tag{3.1}$$

where α, β, and γ are nonzero constants. This equation contains hybrid lattice equation, mKdV lattice equation, modified Volterra lattice equation, and Langmuir chain equation for some special values of α, β, γ. According to the above steps, to seek traveling wave solutions of (3.1), we construct the transformation

$$u_n(t) = U(\xi_n), \qquad \xi_n = dn + c_1 t + \xi_0, \tag{3.2}$$

where d, c_1, and ξ_0 are constants. The transformation (3.2) permits us converting equation (3.1) into the following form:

$$c_1 U'(\xi_n) = \left[\alpha + \beta U(\xi_n) + \gamma U^2(\xi_n)\right][U(\xi_n - d) - U(\xi_n + d)], \tag{3.3}$$

where $' = d/d\xi_n$. Considering the homogeneous balance between the highest order derivative and the nonlinear term in (3.3), we get $K = 1$. Thus, the solution of (3.3) has the following form:

$$U(\xi_n) = \alpha_0 + \alpha_1\left(\frac{F'(\xi_n)}{F(\xi_n)}\right) + \beta_1\left(\frac{F(\xi_n)}{F'(\xi_n)}\right), \tag{3.4}$$

where α_0, α_1, and β_1 are constants to be determined later and $F(\xi_n)$ satisfies a discrete Jacobi elliptic ordinary differential equation (2.6). When we discuss the solutions of (2.6), we get the following types.

Type 1. If $e_0 = 1$, $e_1 = -(1 + m^2)$, and $e_2 = m^2$. In this case, the series expansion solution of (3.3) has the form:

$$U(\xi_n) = \alpha_0 + \frac{\alpha_1 cn(\xi_n, m) dn(\xi_n, m)}{sn(\xi_n, m)} + \frac{\beta_1 sn(\xi_n, m)}{cn(\xi_n, m) dn(\xi_n, m)}. \tag{3.5}$$

With the help of Maple, we substitute (3.5) and (2.8) into (3.3). Cleaning the denominator and collecting all terms with the same degree of $sn(\xi_n, m)$, $cn(\xi_n, m)$, $dn(\xi_n, m)$ together, the left-hand side of (3.3) is converted into polynomial in $sn(\xi_n, m)$, $cn(\xi_n, m)$, $dn(\xi_n, m)$. Setting each coefficient of this polynomial to be zero, we derive a set of algebraic equations for α_0, α_1, d, β_1, γ, β, α, and c_1. Solving the set of algebraic equations by using Maple or Mathematica software package, we have the following.

Family 1.

$$\begin{gathered} \alpha_0 = \frac{-\beta}{2\gamma}, \qquad \beta_1 = \left(m^2 - 1\right)\alpha_1, \qquad c_1 = \frac{2\gamma\alpha_1^2\left[cn^2(d, m) - sn^2(d, m) dn^2(d, m)\right]}{sn(d, m) cn(d, m) dn(d, m)}, \\ \alpha = \frac{\beta^2}{4\gamma} + \frac{\alpha_1^2\gamma\left[-m^4 sn^8(d, m) + 4m^2 sn^6(d, m) - \left(4 + 2m^2\right) sn^4(d, m) + 4 sn^2(d, m) - 1\right]}{sn^2(d, m) cn^2(d, m) dn^2(d, m)}, \end{gathered} \tag{3.6}$$

where α_1, d, γ, β, and m are arbitrary constants.

From (3.5) and (3.6), the solution of (3.3) takes the following form:

$$U(\xi_n) = -\frac{\beta}{2\gamma} + \frac{\alpha_1 cn(\xi_n, m) dn(\xi_n, m)}{sn(\xi_n, m)} + \frac{\alpha_1\left(m^2 - 1\right) sn(\xi_n, m)}{cn(\xi_n, m) dn(\xi_n, m)}, \tag{3.7}$$

where $\xi_n = dn + (2\gamma\alpha_1^2[cn^2(d, m) - sn^2(d, m) dn^2(d, m)]/sn(d, m) cn(d, m) dn(d, m))t + \xi_0$. Figure 1 illustrates the behavior of the exact solution (3.7).

Family 2.

$$\begin{gathered} \alpha_0 = \frac{-\beta}{2\gamma}, \qquad \beta_1 = -\left(m^2 - 1\right)\alpha_1, \qquad c_1 = \frac{2\gamma\alpha_1^2\left[dn^2(d, m) - m^2 sn^2(d, m) cn^2(d, m)\right]}{sn(d, m) cn(d, m) dn(d, m)}, \\ \alpha = \frac{\beta^2}{4\gamma} + \frac{\alpha_1^2\gamma\left[-m^4 sn^8(d, m) + 4m^4 sn^6(d, m) - \left(4m^4 + 2m^2\right) sn^4(d, m) + 4m^2 sn^2(d, m) - 1\right]}{sn^2(d, m) cn^2(d, m) dn^2(d, m)}, \end{gathered} \tag{3.8}$$

where α_1, d, γ, β, and m are arbitrary constants.

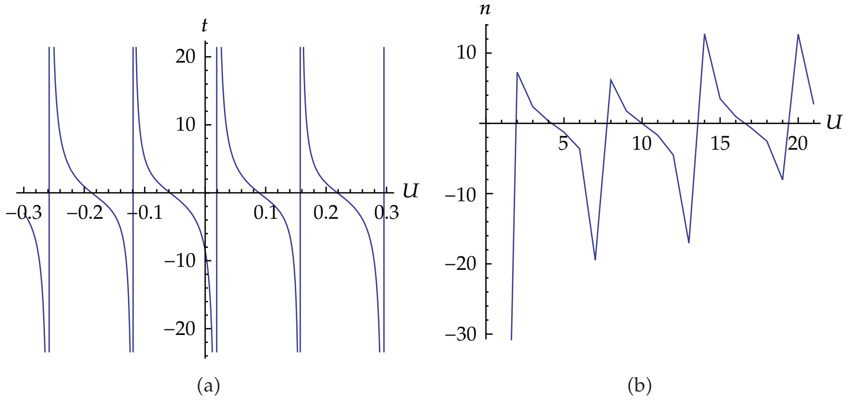

Figure 1: (a) represents the Jacobi elliptic solution (3.7) when $\beta = m = 0.5$, $\gamma = 1$, $\alpha_1 = 1.5$, $d = 0.3$, $n = 2$. (b) represents the Jacobi elliptic solution (3.7) when $\beta = m = 0.5$, $\gamma = 1$, $\alpha_1 = 1.5$, $d = 0.3$, $t = 2$.

From (3.5) and (3.8), the solution of (3.3) has the following form:

$$U(\xi_n) = -\frac{\beta}{2\gamma} + \frac{\alpha_1 cn(\xi_n, m) dn(\xi_n, m)}{sn(\xi_n, m)} - \frac{\alpha_1 (m^2 - 1) sn(\xi_n, m)}{cn(\xi_n, m) dn(\xi_n, m)}, \tag{3.9}$$

where $\xi_n = dn + (2\gamma\alpha_1^2 [dn^2(d,m) - m^2 sn^2(d,m) cn^2(d,m)] / sn(d,m) cn(d,m) dn(d,m)) t + \xi_0$.

Type 2. If $e_0 = 1 - m^2$, $e_1 = 2m^2 - 1$, and $e_2 = -m^2$. In this case, the solution of (3.3) has the form:

$$U(\xi_n) = \alpha_0 - \alpha_1 \frac{sn(\xi_n, m) dn(\xi_n, m)}{cn(\xi_n, m)} - \beta_1 \frac{cn(\xi_n, m)}{sn(\xi_n, m) dn(\xi_n, m)}. \tag{3.10}$$

With the help of Maple, we substitute (3.10) into (3.3). Cleaning the denominator and collecting all terms with the same degree of $sn(\xi_n, m)$, $cn(\xi_n, m)$, $dn(\xi_n, m)$ together, the left-hand side of (3.3) is converted into polynomial in $sn(\xi_n, m)$, $cn(\xi_n, m)$, $dn(\xi_n, m)$. Setting each coefficient of this polynomial to zero, we derive a set of algebraic equations in α_0, α_1, d, β_1, γ, β, α, and c_1. Solving the set of algebraic equations by using Maple or Mathematica software package, we get the following.

Family 1.

$$\begin{gathered} \alpha_0 = \frac{-\beta}{2\gamma}, \qquad \beta_1 = -\alpha_1, \qquad c_1 = \frac{2\gamma\alpha_1^2 [cn^2(d,m) - sn^2(d,m) dn^2(d,m)]}{sn(d,m) cn(d,m) dn(d,m)}, \\ \alpha = \frac{\beta^2}{4\gamma} + \frac{\alpha_1^2 \gamma [-m^4 sn^8(d,m) + 4m^2 sn^6(d,m) - (4 + 2m^2) sn^4(d,m) + 4 sn^2(d,m) - 1]}{sn^2(d,m) cn^2(d,m) dn^2(d,m)}, \end{gathered} \tag{3.11}$$

where α_1, d, γ, β, and m are arbitrary constants.

In this case, the solution of (3.3) takes the following form:

$$U(\xi_n) = -\frac{\beta}{2\gamma} - \frac{\alpha_1 sn(\xi_n, m)dn(\xi_n, m)}{cn(\xi_n, m)} + \frac{\alpha_1 cn(\xi_n, m)}{sn(\xi_n, m)dn(\xi_n, m)}, \tag{3.12}$$

where $\xi_n = dn + (2\gamma\alpha_1^2[cn^2(d,m) - sn^2(d,m)dn^2(d,m)]/sn(d,m)cn(d,m)dn(d,m))t + \xi_0$.

The Jacobi elliptic functions could be generated into the hyperbolic functions when m tends to one in the other hand, they are generated into trigonometrical functions when m tends to zero.

When $m = 0$, the trigonometrical solution (3.12) takes the following form:

$$U(\xi_n) = -\frac{\beta}{2\gamma} + 2\alpha_1 \cot(2\xi_n), \quad \text{where } \xi_n = dn + 4\gamma\alpha_1^2 \cot(2d)t + \xi_0. \tag{3.13}$$

Also if $m = 1$, the hyperbolic solution (3.12) takes the following form:

$$U(\xi_n) = -\frac{\beta}{2\gamma} + \frac{2\alpha_1}{\sinh(2\xi_n)}, \quad \text{where } \xi_n = dn + \frac{4\gamma\alpha_1^2}{\sinh(2d)}t + \xi_0. \tag{3.14}$$

Family 2.

$$\begin{gathered}\alpha_0 = \frac{-\beta}{2\gamma}, \qquad \beta_1 = \alpha_1, \qquad c_1 = \frac{-2\gamma\alpha_1^2\left[m^2 sn^4(d,m) - 1\right]}{sn(d,m)cn(d,m)dn(d,m)}, \\ \alpha = \frac{\beta^2}{4\gamma} + \frac{\alpha_1^2\gamma\left[-m^4 sn^8(d,m) + 2m^2 sn^4(d,m) - 1\right]}{sn^2(d,m)cn^2(d,m)dn^2(d,m)},\end{gathered} \tag{3.15}$$

where α_1, d, γ, β, and m are arbitrary constants.

In this case, the solution of (3.3) takes the following form:

$$U(\xi_n) = -\frac{\beta}{2\gamma} - \frac{\alpha_1 sn(\xi_n, m)dn(\xi_n, m)}{cn(\xi_n, m)} - \frac{\alpha_1 cn(\xi_n, m)}{sn(\xi_n, m)dn(\xi_n, m)}, \tag{3.16}$$

where $\xi_n = dn - (2\gamma\alpha_1^2[m^2 sn^4(d,m) - 1]/sn(d,m)cn(d,m)dn(d,m))t + \xi_0$.

When $m = 0$, the trigonometrical solution (3.16) takes the following form:

$$U(\xi_n) = -\frac{\beta}{2\gamma} - \frac{2\alpha_1}{\sin(2\xi_n)}, \quad \text{where } \xi_n = dn + \frac{4\gamma\alpha_1^2}{\sin(2d)}t + \xi_0. \tag{3.17}$$

When $m = 1$, the hyperbolic solution (3.16) takes the following form:

$$U(\xi_n) = -\frac{\beta}{2\gamma} - 2\alpha_1 \coth(2\xi_n), \quad \text{where } \xi_n = dn + 4\gamma\alpha_1^2 \coth(2d)t + \xi_0. \tag{3.18}$$

Type 3. If $e_0 = m^2 - 1$, $e_1 = 2 - m^2$, and $e_2 = -1$. In this case, the series expansion solution of (3.3) has the form:

$$U(\xi_n) = \alpha_0 - \frac{m^2\alpha_1 sn(\xi_n, m)cn(\xi_n, m)}{dn(\xi_n, m)} - \frac{\beta_1 dn(\xi_n, m)}{m^2 sn(\xi_n, m)cn(\xi_n, m)}. \tag{3.19}$$

Consequently, using the Maple or Mathematica we get the following results.

Family 1.

$$\alpha_0 = \frac{-\beta}{2\gamma}, \qquad \beta_1 = -m^2\alpha_1, \qquad c_1 = \frac{2\gamma\alpha_1^2\left[dn^2(d,m) - m^2 sn^2(d,m)cn^2(d,m)\right]}{sn(d,m)cn(d,m)dn(d,m)},$$
$$\alpha = \frac{\beta^2}{4\gamma} + \frac{\alpha_1^2\gamma\left[-m^4 sn^8(d,m) + 4m^4 sn^6(d,m) - \left(4m^4 + 2m^2\right)sn^4(d,m) + 4m^2 sn^2(d,m) - 1\right]}{sn^2(d,m)cn^2(d,m)dn^2(d,m)}, \tag{3.20}$$

where α_1, d, γ, β, and m are arbitrary constants.

In this case, the solution of (3.3) takes the following form:

$$U(\xi_n) = -\frac{\beta}{2\gamma} - \frac{\alpha_1 m^2 sn(\xi_n, m)cn(\xi_n, m)}{dn(\xi_n, m)} + \frac{\alpha_1 dn(\xi_n, m)}{sn(\xi_n, m)cn(\xi_n, m)}, \tag{3.21}$$

where $\xi_n = dn + (2\gamma\alpha_1^2[dn^2(d,m) - m^2 sn^2(d,m)cn^2(d,m)]/sn(d,m)cn(d,m)dn(d,m))t + \xi_0$.

Family 2.

$$\alpha_0 = \frac{-\beta}{2\gamma}, \qquad \beta_1 = \alpha_1 m^2, \qquad c_1 = \frac{-2\gamma\alpha_1^2\left[m^2 sn^4(d,m) - 1\right]}{sn(d,m)cn(d,m)dn(d,m)},$$
$$\alpha = \frac{\beta^2}{4\gamma} + \frac{\alpha_1^2\gamma\left[-m^4 sn^8(d,m) + 2m^2 sn^4(d,m) - 1\right]}{sn^2(d,m)cn^2(d,m)dn^2(d,m)}, \tag{3.22}$$

where α_1, d, γ, β, and m are arbitrary constants.

In this case, the solution of (3.3) takes the following form:

$$U(\xi_n) = -\frac{\beta}{2\gamma} - \frac{\alpha_1 m^2 sn(\xi_n, m)cn(\xi_n, m)}{dn(\xi_n, m)} - \frac{\alpha_1 dn(\xi_n, m)}{sn(\xi_n, m)cn(\xi_n, m)}, \tag{3.23}$$

where $\xi_n = dn - (2\gamma\alpha_1^2[m^2 sn^4(d,m) - 1]/sn(d,m)cn(d,m)dn(d,m))t + \xi_0$.

3.2. Example 2. The Discrete Nonlinear Schrodinger Equation

The discrete nonlinear Schrodinger equation (DNSE) is one of the most fundamental nonlinear lattice models [8]. Its arise in nonlinear optics as a model of infinite wave guide arrays [27] and has been recently implemented to describe Bose-Einstein condensates in optical lattices. The class of DNSE model with saturable nonlinearity is also of particular interest in their own right, due to a feature first unveiled in [28]. In this section, we study the DNSE with a saturable nonlinearity [29, 30] form:

$$i\frac{\partial \psi_n}{\partial t} + (\psi_{n+1} + \psi_{n-1} - 2\psi_n) + \frac{\nu|\psi_n|^2}{1+\mu|\psi_n|^2}\psi_n = 0, \tag{3.24}$$

which describes optical pulse propagations in various doped fibers, ψ_n is a complex valued wave function at sites n while ν and μ are real parameters. We make the transformation:

$$\psi_n = \phi(\xi_n)e^{-i(\sigma t+\rho)}, \qquad \xi_n = \alpha n + \beta, \tag{3.25}$$

where σ, ρ, α, and β are arbitrary constants. The transformation (3.25) permits us converting equation (3.24) into the following nonlinear difference equation:

$$(\sigma - 2)\varphi(\xi_n) + \varphi(\xi_n + \alpha) + \varphi(\xi_n - \alpha) + \frac{\nu\varphi^3(\xi_n)}{1+\mu\varphi^2(\xi_n)} = 0. \tag{3.26}$$

We assume that (3.26) has a solution of the form:

$$\phi(\xi_n) = \alpha_1\left(\frac{F'(\xi_n)}{F(\xi_n)}\right) + \beta_1\left(\frac{F(\xi_n)}{F'(\xi_n)}\right) + \alpha_0, \tag{3.27}$$

where α_0, α_1, and β_1 are constants to be determined later and $F(\xi_n)$ satisfies a discrete Jacobi elliptic differential equation (2.6). When we discuss the solutions of (3.26), we have the following types.

Type 1. If $e_0 = 1$, $e_1 = -(1+m^2)$, and $e_2 = m^2$. In this case, the series expansion solution of (3.26) has the form:

$$\phi(\xi_n) = \alpha_1\frac{cn(\xi_n, m)dn(\xi_n, m)}{sn(\xi_n, m)} + \beta_1\frac{sn(\xi_n, m)}{cn(\xi_n, m)dn(\xi_n, m)} + \alpha_0. \tag{3.28}$$

With the help of Maple, we substitute (3.28) and (2.8) into (3.26), cleaning the denominator and collecting all terms with the same order of $sn(\xi_n, m)$, $cn(\xi_n, m)$, $dn(\xi_n, m)$ together, the left-hand side of (3.26) is converted into polynomial in $sn(\xi_n, m)$, $cn(\xi_n, m)$, $dn(\xi_n, m)$. Setting each coefficient of this polynomial to zero, we derive a set of algebraic equations for α_0, α_1, σ, β_1, ρ, α, and β. Solving the set of algebraic equations by using Maple or Mathematica software package, we obtain the following.

Family 1.

$$\alpha_1 = \frac{sn(\alpha,m)cn(\alpha,m)dn(\alpha,m)}{\sqrt{-\mu}\left[m^2sn^4(\alpha,m) - 2sn^2(\alpha,m) + 1\right]}, \qquad \beta_1 = \frac{(m^2-1)sn(\alpha,m)cn(\alpha,m)dn(\alpha,m)}{\sqrt{-\mu}\left[m^2sn^4(\alpha,m) - 2sn^2(\alpha,m) + 1\right]},$$

$$\mu < 0,$$

$$\nu = \frac{-2\mu\left[m^4sn^8(\alpha,m) - 2m^4sn^6(\alpha,m) + 2m^2sn^2(\alpha,m) - 1\right]}{m^4sn^8(\alpha,m) - 4m^2sn^6(\alpha,m) + (2m^2+4)sn^4(\alpha,m) - 4sn^2(\alpha,m) + 1}, \quad \alpha_0 = 0,$$

$$\sigma = \frac{4sn^2(\alpha,m)\left[m^4sn^6(\alpha,m) - (m^4+2m^2)sn^4(\alpha,m) + (m^2+2)sn^2(\alpha,m) + m^2 - 2\right]}{m^4sn^8(\alpha,m) - 4m^2sn^6(\alpha,m) + (2m^2+4)sn^4(\alpha,m) - 4sn^2(\alpha,m) + 1}. \tag{3.29}$$

In this case, the solution of (3.24) takes the following form:

$$\begin{aligned}\psi_n = &\left(\frac{sn(\alpha,m)cn(\alpha,m)dn(\alpha,m)cn(\xi_n,m)dn(\xi_n,m)}{\sqrt{-\mu}\left[m^2sn^4(\alpha,m) - 2sn^2(\alpha,m) + 1\right]sn(\xi_n,m)}\right.\\ &\left.+\frac{(m^2-1)sn(\alpha,m)cn(\alpha,m)dn(\alpha,m)sn(\xi_n,m)}{\sqrt{-\mu}\left[m^2sn^4(\alpha,m) - 2sn^2(\alpha,m) + 1\right]cn(\xi_n,m)dn(\xi_n,m)}\right)\\ &\times\left(\mathrm{Exp}\left\{-i\left[\frac{4tsn^2(\alpha,m)\mathfrak{A}}{m^4sn^8(\alpha,m)-4m^2sn^6(\alpha,m)+(2m^2+4)sn^4(\alpha,m)-4sn^2(\alpha,m)+1}+\rho\right]\right\}\right),\end{aligned} \tag{3.30}$$

where $\mathfrak{A}$ denotes $[m^4sn^6(\alpha,m)-(m^4+2m^2)sn^4(\alpha,m)+(m^2+2)sn^2(\alpha,m)+m^2-2]$ and $\xi_n = \alpha n+\beta$. Figure 2 illustrates the behavior exact solution (3.30).

Family 2.

$$\alpha_1 = \frac{sn(\alpha,m)cn(\alpha,m)dn(\alpha,m)}{\sqrt{-\mu}\left[m^2sn^4(\alpha,m) - 2m^2sn^2(\alpha,m) + 1\right]},$$

$$\beta_1 = -\frac{(m^2-1)sn(\alpha,m)cn(\alpha,m)dn(\alpha,m)}{\sqrt{-\mu}\left[m^2sn^4(\alpha,m) - 2m^2sn^2(\alpha,m) + 1\right]},$$

$$\mu < 0,$$

$$\nu = \frac{-2\mu\left[m^4sn^8(\alpha,m) - 2m^2sn^6(\alpha,m) + 2sn^2(\alpha,m) - 1\right]}{m^4sn^8(\alpha,m) - 4m^4sn^6(\alpha,m) + (2m^2+4m^4)sn^4(\alpha,m) - 4m^2sn^2(\alpha,m) + 1},$$

$$\alpha_0 = 0,$$

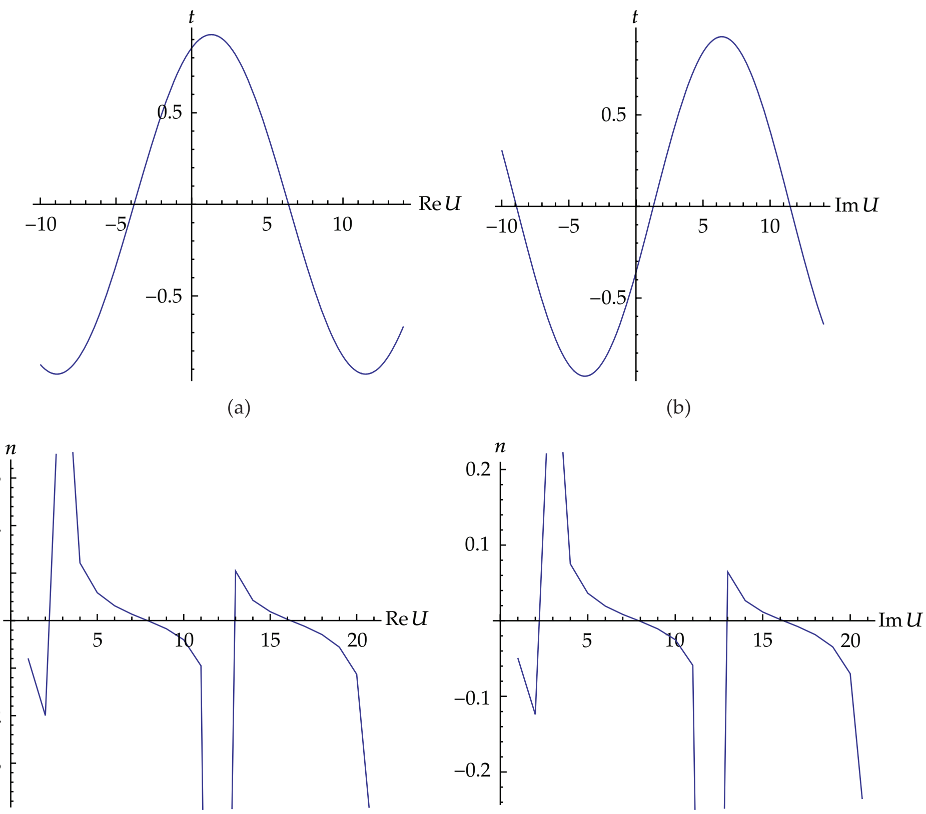

Figure 2: (a) and (b) represent the Jacobi elliptic solution (3.30) when $m = 0.5$, $\beta = 1.5$, $\alpha = 0.2$, $\rho = 0.4$, $n = 2$. (c) and (d) represent the Jacobi elliptic solution (3.30) when $m = 0.5$, $\beta = 1.5$, $\alpha = 0.2$, $\rho = 0.4$, $t = 1.5$.

$$\sigma = \frac{4sn^2(\alpha,m)\left[m^4 sn^6(\alpha,m) - \left(2m^4+m^2\right)sn^4(\alpha,m) + \left(2m^4+m^2\right)sn^2(\alpha,m) - 2m^2+1\right]}{m^4 sn^8(\alpha,m) - 4m^4 sn^6(\alpha,m) + \left(2m^2+4m^4\right)sn^4(\alpha,m) - 4m^2 sn^2(\alpha,m)+1}. \tag{3.31}$$

In this case, the solution of (3.24) takes the following form:

$$\begin{aligned}\varphi_n = {} & \left(\frac{sn(\alpha,m)cn(\alpha,m)dn(\alpha,m)cn(\xi_n,m)dn(\xi_n,m)}{\sqrt{-\mu}\left[m^2 sn^4(\alpha,m) - 2m^2 sn^2(\alpha,m)+1\right]sn(\xi_n,m)}\right.\\ & \left. -\frac{\left(m^2-1\right)sn(\alpha,m)cn(\alpha,m)dn(\alpha,m)sn(\xi_n,m)}{\sqrt{-\mu}\left[m^2 sn^4(\alpha,m) - 2m^2 sn^2(\alpha,m)+1\right]cn(\xi_n,m)dn(\xi_n,m)}\right)\\ & \times\left(\text{Exp}\left\{-i\left[\frac{4t sn^2(\alpha,m)\mathfrak{W}}{m^4 sn^8(\alpha,m) - 4m^4 sn^6(\alpha,m) + \left(2m^2+4m^4\right)sn^4(\alpha,m) - 4m^2 sn^2(\alpha,m)+1}+\rho\right]\right\}\right),\end{aligned} \tag{3.32}$$

where $\mathfrak{W}$ denotes $[m^4 sn^6(\alpha,m) - (2m^4+m^2)sn^4(\alpha,m) + (2m^4+m^2)sn^2(\alpha,m) - 2m^2+1]$ and $\xi_n = \alpha n + \beta$.

Type 2. If $e_0 = 1 - m^2$, $e_1 = 2m^2 - 1$, and $e_2 = -m^2$. In this case, the solution of (3.26) has the form:

$$\phi(\xi_n) = \alpha_0 - \alpha_1 \frac{sn(\xi_n, m)dn(\xi_n, m)}{cn(\xi_n, m)} - \beta_1 \frac{cn(\xi_n, m)}{sn(\xi_n, m)dn(\xi_n, m)}. \tag{3.33}$$

Consequently, using Maple or Mathematica we get the following results.

Family 1.

$$\alpha_1 = \frac{sn(\alpha, m)cn(\alpha, m)dn(\alpha, m)}{\sqrt{-\mu}\left[m^2 sn^4(\alpha, m) - 2sn^2(\alpha, m) + 1\right]}, \qquad \beta_1 = -\frac{sn(\alpha, m)cn(\alpha, m)dn(\alpha, m)}{\sqrt{-\mu}\left[m^2 sn^4(\alpha, m) - 2sn^2(\alpha, m) + 1\right]},$$
$$\mu < 0,$$
$$\nu = \frac{-2\mu\left[m^4 sn^8(\alpha, m) - 2m^4 sn^6(\alpha, m) + 2m^2 sn^2(\alpha, m) - 1\right]}{m^4 sn^8(\alpha, m) - 4m^2 sn^6(\alpha, m) + (2m^2 + 4)sn^4(\alpha, m) - 4sn^2(\alpha, m) + 1}, \qquad \alpha_0 = 0,$$
$$\sigma = \frac{4sn^2(\alpha, m)\left[m^4 sn^6(\alpha, m) - \left(m^4 + 2m^2\right)sn^4(\alpha, m) + \left(m^2 + 2\right)sn^2(\alpha, m) + m^2 - 2\right]}{m^4 sn^8(\alpha, m) - 4m^2 sn^6(\alpha, m) + (2m^2 + 4)sn^4(\alpha, m) - 4sn^2(\alpha, m) + 1}. \tag{3.34}$$

In this case, the solution of (3.24) takes the following form:

$$\begin{aligned}\psi_n = &\left(\frac{-sn(\alpha, m)cn(\alpha, m)dn(\alpha, m)sn(\xi_n, m)dn(\xi_n, m)}{\sqrt{-\mu}\left[m^2 sn^4(\alpha, m) - 2sn^2(\alpha, m) + 1\right]cn(\xi_n, m)}\right.\\ &\left.+ \frac{sn(\alpha, m)cn(\alpha, m)dn(\alpha, m)cn(\xi_n, m)}{\sqrt{-\mu}\left[m^2 sn^4(\alpha, m) - 2sn^2(\alpha, m) + 1\right]sn(\xi_n, m)dn(\xi_n, m)}\right)\\ &\times\left(\operatorname{Exp}\left\{-i\left[\frac{4t sn^2(\alpha, m)\mathfrak{A}}{m^4 sn^8(\alpha, m) - 4m^2 sn^6(\alpha, m) + (2m^2 + 4)sn^4(\alpha, m) - 4sn^2(\alpha, m) + 1} + \rho\right]\right\}\right),\end{aligned} \tag{3.35}$$

where $\xi_n = \alpha n + \beta$.

In the special case, when $m = 0$, the trigonometrical solution (3.35) takes the following form:

$$\psi_n = \frac{\tan(2\alpha)\cot(2\xi_n)}{\sqrt{-\mu}}\left(\operatorname{Exp}\left\{-i\left[-2t\tan^2(2\alpha) + \rho\right]\right\}\right), \quad \text{where } \xi_n = \alpha n + \beta. \tag{3.36}$$

Also if $m = 1$, the hyperbolic solution (3.35) takes the following form:

$$\psi_n = \left(\frac{\sinh(\ 2\alpha\ )}{\sqrt{-\mu}\sinh(\ 2\xi_n\ )}\right)\left(\operatorname{Exp}\left\{i\left[4t\sinh^2(\alpha) - \rho\right]\right\}\right), \quad \text{where } \xi_n = \alpha n + \beta. \tag{3.37}$$

Family 2.

$$\alpha_1 = \frac{sn(\alpha,m)cn(\alpha,m)dn(\alpha,m)}{\sqrt{-\mu}\left[m^2 sn^4(\alpha,m)-1\right]}, \qquad \beta_1 = \frac{sn(\alpha,m)cn(\alpha,m)dn(\alpha,m)}{\sqrt{-\mu}\left[m^2 sn^4(\alpha,m)-1\right]}, \qquad \mu < 0,$$
$$\nu = \frac{2\mu\left[m^4 sn^8(\alpha,m) - 2\left(m^4+m^2\right)sn^6(\alpha,m) + 6m^2 sn^4(\alpha,m) - \left(2+2m^2\right)sn^2(\alpha,m) + 1\right]}{m^4 sn^8(\alpha,m) - 2m^2 sn^4(\alpha,m) + 1},$$
$$\sigma = \frac{4sn^2(\alpha,m)\left[\left(m^4+m^2\right)sn^4(\alpha,m) - 4m^2 sn^2(\alpha,m) + m^2 + 1\right]}{m^4 sn^8(\alpha,m) - 2m^2 sn^4(\alpha,m) + 1}, \qquad \alpha_0 = 0. \tag{3.38}$$

In this case, the solution of (3.24) takes the following form:

$$\begin{aligned}\varphi_n = &\left(\frac{-sn(\alpha,m)cn(\alpha,m)dn(\alpha,m)sn(\xi_n,m)dn(\xi_n,m)}{\sqrt{-\mu}\left[m^2 sn^4(\alpha,m)-1\right]cn(\xi_n,m)}\right.\\ &\left.-\frac{sn(\alpha,m)cn(\alpha,m)dn(\alpha,m)cn(\xi_n,m)}{\sqrt{-\mu}\left[m^2 sn^4(\alpha,m)-1\right]sn(\xi_n,m)dn(\xi_n,m)}\right)\\ &\times\left(\operatorname{Exp}\left\{-i\left[\frac{4tsn^2(\alpha,m)\mathfrak{X}}{m^4 sn^8(\alpha,m) - 2m^2 sn^4(\alpha,m) + 1} + \rho\right]\right\}\right),\end{aligned} \tag{3.39}$$

where $\mathfrak{X}$ denotes $[(m^4+m^2)sn^4(\alpha,m) - 4m^2 sn^2(\alpha,m) + m^2 + 1]$ and $\xi_n = \alpha n + \beta$.

If $m = 0$, the trigonometrical solution (3.39) takes the following form:

$$\varphi_n = \frac{\sin(2\alpha)}{\sqrt{-\mu}\sin(2\xi_n)}\left(\operatorname{Exp}\left\{-i\left[4t\sin^2(\alpha) + \rho\right]\right\}\right), \quad \text{where } \xi_n = \alpha n + \beta. \tag{3.40}$$

Also if $m = 1$, the hyperbolic solution (3.39) takes the following form:

$$\varphi_n = \frac{\tanh(2\alpha)\coth(2\xi_n)}{\sqrt{-\mu}}\left(\operatorname{Exp}\left\{-i\left[2t\tanh^2(2\alpha) + \rho\right]\right\}\right), \quad \text{where } \xi_n = \alpha n + \beta. \tag{3.41}$$

Type 3. if $e_0 = m^2 - 1$, $e_1 = 2 - m^2$, and $e_2 = -1$. In this case, the series expansion solution of (3.26) has the form:

$$\phi(\xi_n) = \alpha_0 - \frac{m^2\alpha_1 sn(\xi_n,m)cn(\xi_n,m)}{dn(\xi_n,m)} - \frac{\beta_1 dn(\xi_n,m)}{m^2 sn(\xi_n,m)cn(\xi_n,m)}. \tag{3.42}$$

Consequently, using Maple or Mathematica we get the following results.

Family 1.

$$\alpha_0 = 0, \qquad \alpha_1 = \frac{sn(\alpha,m)cn(\alpha,m)dn(\alpha,m)}{\sqrt{-\mu}\left[m^2sn^4(\alpha,m) - 2m^2sn^2(\alpha,m) + 1\right]},$$

$$\beta_1 = -\frac{m^2sn(\alpha,m)cn(\alpha,m)dn(\alpha,m)}{\sqrt{-\mu}\left[m^2sn^4(\alpha,m) - 2m^2sn^2(\alpha,m) + 1\right]}, \quad \mu < 0,$$

$$\nu = \frac{-2\mu\left(m^4sn^8(\alpha,m) - 2m^2sn^6(\alpha,m) + 2sn^2(\alpha,m) - 1\right)}{m^4sn^8(\alpha,m) - 4m^4sn^6(\alpha,m) + \left(4m^4 + 2m^2\right)sn^4(\alpha,m) - 4m^2sn^2(\alpha,m) + 1},$$

$$\sigma = \frac{4sn^2(\alpha,m)\left[m^4sn^6(\alpha,m) - \left(2m^4 + m^2\right)sn^4(\alpha,m) + \left(2m^4 + m^2\right)sn^2(\alpha,m) - 2m^2 + 1\right]}{m^4sn^8(\alpha,m) - 4m^4sn^6(\alpha,m) + \left(4m^4 + 2m^2\right)sn^4(\alpha,m) - 4m^2sn^2(\alpha,m) + 1}. \tag{3.43}$$

In this case, the solution of (3.24) takes the following form:

$$\begin{aligned}\varphi_n = &\left(-\frac{m^2sn(\alpha,m)cn(\alpha,m)dn(\alpha,m)sn(\xi_n,m)cn(\xi_n,m)}{\sqrt{-\mu}\left[m^2sn^4(\alpha,m) - 2m^2sn^2(\alpha,m) + 1\right]dn(\xi_n,m)}\right.\\ &\left.+\frac{sn(\alpha,m)cn(\alpha,m)dn(\alpha,m)dn(\xi_n,m)}{\sqrt{-\mu}\left[m^2sn^4(\alpha,m) - 2m^2sn^2(\alpha,m) + 1\right]sn(\xi_n,m)cn(\xi_n,m)}\right)\\ &\times\left(\mathrm{Exp}\left\{-i\left[\frac{4tsn^2(\alpha,m)\mathfrak{W}}{m^4sn^8(\alpha,m)-4m^4sn^6(\alpha,m)+\left(4m^4+2m^2\right)sn^4(\alpha,m)-4m^2sn^2(\alpha,m)+1}+\rho\right]\right\}\right)\end{aligned} \tag{3.44}$$

where $\xi_n = \alpha n + \beta$.

Family 2.

$$\alpha_0 = 0, \qquad \alpha_1 = \frac{sn(\alpha,m)cn(\alpha,m)dn(\alpha,m)}{\sqrt{-\mu}\left[m^2sn^4(\alpha,m) - 1\right]},$$

$$\beta_1 = \frac{m^2sn(\alpha,m)cn(\alpha,m)dn(\alpha,m)}{\sqrt{-\mu}\left[m^2sn^4(\alpha,m) - 1\right]}, \qquad \mu < 0,$$

$$\nu = \frac{2\mu\left[\left(m^4sn^8(\alpha,m) - 2\left(m^4 + m^2\right)sn^6(\alpha,m) + 6m^2sn^4(\alpha,m) - \left(2 + 2m^2\right)sn^2(\alpha,m) + 1\right)\right]}{m^4sn^8(\alpha,m) - 2m^2sn^4(\alpha,m) + 1},$$

$$\sigma = \frac{4sn^2(\alpha,m)\left[\left(m^4 + m^2\right)sn^4(\alpha,m) - 4m^2sn^2(\alpha,m) + m^2 + 1\right]}{m^4sn^8(\alpha,m) - 2m^2sn^4(\alpha,m) + 1}. \tag{3.45}$$

In this case, the solution of (3.24) takes the following form:

$$\varphi_n = \left(-\frac{m^2 sn(\alpha, m) cn(\alpha, m) dn(\alpha, m) sn(\xi_n, m) cn(\xi_n, m)}{\sqrt{-\mu}\left[m^2 sn^4(\alpha, m) - 1\right] dn(\xi_n, m)} - \frac{sn(\alpha, m) cn(\alpha, m) dn(\alpha, m) dn(\xi_n, m)}{\sqrt{-\mu}\left[m^2 sn^4(\alpha, m) - 1\right] sn(\xi_n, m) cn(\xi_n, m)} \right) \times \left(\text{Exp}\left\{ -i\left[\frac{4t sn^2(\alpha, m)\mathfrak{X}}{m^4 sn^8(\alpha, m) - 2m^2 sn^4(\alpha, m) + 1} + \rho \right] \right\} \right), \tag{3.46}$$

where $\xi_n = \alpha n + \beta$.

3.3. Example 3. The Discrete Nonlinear Klein-Gordon Equation

In this section, we consider the following discrete nonlinear Klein-Gordon equation [31]:

$$\frac{d^2 u_n(t)}{dt^2} = g(u_n)(u_{n+1} + u_{n-1} - 2s u_n). \tag{3.47}$$

The nonconstant (in contrast to the standard models of harmonic coupling and linear dispersion [32]) function $g(u_n)$ ensures the presence of nonlinear dispersion, which is critical for the existence of compactly supported solutions and s can take values in the interval $[-1, 1]$. Kevrekidis and Konotop [31] have obtained some exact compaction solutions and claim that this DDE does not have the traveling compact solution.

If we set $g(u_n) = A + C u_n^2$ where A and C are arbitrary constants and take the traveling transformation:

$$u_n(t) = U(\xi_n), \qquad \xi_n = dn + c_1 t + \xi_0, \tag{3.48}$$

where d, c_1, and ξ_0 are constants. The transformation (3.48) permits us converting equation (3.47) into the following form:

$$c_1^2 U''(\xi_n) = \left[A + CU^2(\xi_n)\right]\left[U(\xi_n + d) + U(\xi_n - d) - 2sU(\xi_n)\right], \tag{3.49}$$

where $' = d/d\xi_n$. Considering the homogeneous balance between the highest-order derivative and the nonlinear term in (3.49), we get $K = 1$. Thus, the solutions of (3.49) have the following form:

$$U(\xi_n) = \alpha_0 + \alpha_1\left(\frac{F'(\xi_n)}{F(\xi_n)}\right) + \beta_1\left(\frac{F(\xi_n)}{F'(\xi_n)}\right), \tag{3.50}$$

where α_0, α_1, and β_1 are constants to be determined later and $F(\xi_n)$ satisfies a discrete Jacobi elliptic ordinary differential equation (2.6), we have the following types.

Type 1. If $e_0 = 1$, $e_1 = -(1+m^2)$, and $e_2 = m^2$. In this case, the series expansion solution of (3.49) has the form:

$$U(\xi_n) = \alpha_0 + \frac{\alpha_1 cn(\xi_n, m) dn(\xi_n, m)}{sn(\xi_n, m)} + \beta_1 \frac{sn(\xi_n, m)}{cn(\xi_n, m) dn(\xi_n, m)}. \tag{3.51}$$

With the help of Maple, we substitute (3.51) and (2.8) into (3.49). Cleaning the denominator and collecting all terms with the same degree of $sn(\xi_n, m)$, $cn(\xi_n, m)$, $dn(\xi_n, m)$ together, the left-hand side of (3.49) is converted into polynomial in $sn(\xi_n, m)$, $cn(\xi_n, m)$, $dn(\xi_n, m)$. Setting each coefficient of this polynomial to zero, we derive a set of algebraic equations for α_0, α_1, d, β_1, A, C, s, and c_1. Solving the set of algebraic equations by using Maple or Mathematica software package, we have the following.

Family 1.

$$\beta_1 = \left(m^2 - 1\right) c_1 L_1, \qquad \alpha_0 = 0, \qquad s = \frac{-1}{C L_1^2}, \qquad \alpha_1 = c_1 L_1,$$
$$A = -C L_1^2 c_1^2 \frac{\left[m^4 sn^8(d,m) - 4m^2 sn^6(d,m) + \left(2m^2+4\right) sn^4(d,m) - 4 sn^2(d,m) + 1\right]}{sn^2(d,m) cn^2(d,m) dn^2(d,m)}, \tag{3.52}$$

where d, c_1, and C are arbitrary constants and

$$L_1 = \sqrt{\frac{m^4 sn^4(d,m) cn^4(d,m) + dn^4(d,m)}{C\left[m^4 sn^8(d,m) - 2m^4 sn^6(d,m) + 2m^2 sn^2(d,m) - 1\right]}}. \tag{3.53}$$

In this case, the solution of (3.47) takes the following form:

$$U(\xi_n) = \frac{c_1 L_1 cn(\xi_n, m) dn(\xi_n, m)}{sn(\xi_n, m)} + \frac{(m^2-1) c_1 L_1 sn(\xi_n, m)}{cn(\xi_n, m) dn(\xi_n, m)}, \tag{3.54}$$

where $\xi_n = dn + c_1 t + \xi_0$.

Figure 3 illustrates the behavior of the exact solution (3.54).

Family 2.

$$\beta_1 = -\left(m^2 - 1\right) c_1 L_2, \qquad \alpha_0 = 0, \qquad s = \frac{-1}{C L_2^2}, \qquad \alpha_1 = c_1 L_2,$$
$$A = -C L_2^2 c_1^2 \frac{\left[m^4 sn^8(d,m) - 4m^4 sn^6(d,m) + \left(4m^4 + 2m^2\right) sn^4(d,m) - 4m^2 sn^2(d,m) + 1\right]}{sn^2(d,m) cn^2(d,m) dn^2(d,m)}, \tag{3.55}$$

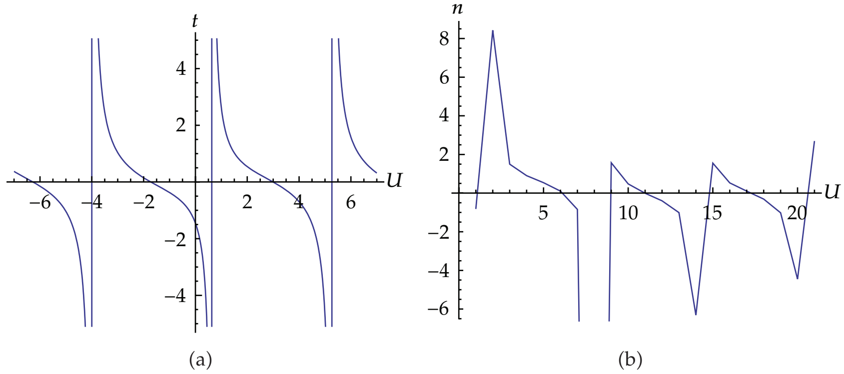

Figure 3: (a) represents the Jacobi elliptic solution (3.54) when $m = 0.5$, $c_1 = 0.4$, $C = -0.6$, $d = 0.3$, $n = 2$. (b) represents the Jacobi elliptic solution (3.54) when $m = 0.5$, $c_1 = 0.4$, $C = -0.6$, $d = 0.3$, $t = 1.5$.

where d, c_1, and C are arbitrary constants and

$$L_2 = \sqrt{\frac{sn^4(d,m)dn^4(d,m) + cn^4(d,m)}{C\left[m^4 sn^8(d,m) - 2m^2 sn^6(d,m) + 2sn^2(d,m) - 1\right]}}. \tag{3.56}$$

In this case, the solution of (3.47) takes the following form:

$$U(\xi_n) = \frac{c_1 L_2 cn(\xi_n, m) dn(\xi_n, m)}{sn(\xi_n, m)} - \frac{(m^2 - 1) c_1 L_2 sn(\xi_n, m)}{cn(\xi_n, m) dn(\xi_n, m)}, \tag{3.57}$$

where $\xi_n = dn + c_1 t + \xi_0$.

Type 2. If $e_0 = 1 - m^2$, $e_1 = 2m^2 - 1$, and $e_2 = -m^2$. In this case, the series expansion solution of (3.49) has the form:

$$U(\xi_n) = \alpha_0 - \alpha_1 \frac{sn(\xi_n, m) dn(\xi_n, m)}{cn(\xi_n, m)} - \beta_1 \frac{cn(\xi_n, m)}{sn(\xi_n, m) dn(\xi_n, m)}. \tag{3.58}$$

Consequently, using Maple or Mathematica we get the following results.

Family 1.

$$\begin{gathered} \beta_1 = -c_1 L_1, \qquad \alpha_0 = 0, \qquad s = \frac{-1}{C L_1^2}, \qquad \alpha_1 = c_1 L_1, \\ A = -C L_1^2 c_1^2 \frac{\left[m^4 sn^8(d,m) - 4m^2 sn^6(d,m) + (2m^2 + 4) sn^4(d,m) - 4sn^2(d,m) + 1\right]}{sn^2(d,m) cn^2(d,m) dn^2(d,m)}, \end{gathered} \tag{3.59}$$

where d, c_1, and C are arbitrary constants.

In this case, the solution of (3.47) takes the following form:

$$U(\xi_n) = \frac{-c_1 L_1 sn(\xi_n, m) dn(\xi_n, m)}{cn(\xi_n, m)} + \frac{c_1 L_1 cn(\xi_n, m)}{sn(\xi_n, m) dn(\xi_n, m)}, \tag{3.60}$$

where $\xi_n = dn + c_1 t + \xi_0$.

In the special case, when $m = 0$, the trigonometrical solution (3.60) takes the following form:

$$U(\xi_n) = \frac{2c_1 \cot(2\xi_n)}{\sqrt{-C}}, \quad \text{where } \xi_n = dn + c_1 t + \xi_0. \tag{3.61}$$

Also when $m = 1$, the hyperbolic solution (3.60) takes the following form:

$$U(\xi_n) = 2c_1 \coth(2\xi_n) \sqrt{\frac{\left[\tanh^4(d) + 1\right]}{C\left[\tanh^4(d) - 1\right]}}, \quad \text{where } \xi_n = dn + c_1 t + \xi_0. \tag{3.62}$$

Family 2.

$$\begin{gathered} \beta_1 = c_1 L_3, \qquad \alpha_0 = 0, \qquad s = \frac{-1}{C L_3^2}, \qquad \alpha_1 = c_1 L_3, \\ A = -C L_3^2 c_1^2 \frac{\left[m^4 sn^8(d, m) - 2m^2 sn^4(d, m) + 1\right]}{sn^2(d, m) cn^2(d, m) dn^2(d, m)}, \end{gathered} \tag{3.63}$$

where d, c_1, and C are arbitrary constants, while

$$L_3 = \sqrt{\frac{m^4 sn^8(d,m) - 2(m^4 + m^2) sn^6(d,m) + 2(m^4 + m^2 + 1) sn^4(d,m) - 2(m^2 + 1) sn^2(d,m) + 1}{-C\left[m^4 sn^8(d,m) - 2(m^4 + m^2) sn^6(d,m) + 6m^2 sn^4(d,m) - 2(m^2 + 1) sn^2(d,m) + 1\right]}}. \tag{3.64}$$

In this case, the solution of (3.47) takes the following form:

$$U(\xi_n) = -\frac{c_1 L_3 sn(\xi_n, m) dn(\xi_n, m)}{cn(\xi_n, m)} - \frac{c_1 L_3 cn(\xi_n, m)}{sn(\xi_n, m) dn(\xi_n, m)}, \tag{3.65}$$

where $\xi_n = dn + c_1 t + \xi_0$.

When $m = 0$, the trigonometrical solution (3.65) takes the following form:

$$U(\xi_n) = \frac{-2c_1}{\sin(2\xi_n)} \sqrt{\frac{-\sin^2(2d) + 2}{-2C \cos(2d)}}, \quad \text{where } \xi_n = dn + c_1 t + \xi_0. \tag{3.66}$$

Also if $m = 1$, the hyperbolic solution (3.65) takes the following form:

$$U(\xi_n) = \frac{-2c_1 \coth(2\xi_n)}{\sqrt{-C}}, \quad \text{where } \xi_n = dn + c_1 t + \xi_0. \tag{3.67}$$

Type 3. If $e_0 = m^2 - 1$, $e_1 = 2 - m^2$, and $e_2 = -1$. In this case, the series expansion solution of (3.49) has the form:

$$U(\xi_n) = \alpha_0 - \frac{m^2 \alpha_1 sn(\xi_n, m) cn(\xi_n, m)}{dn(\xi_n, m)} - \frac{\beta_1 dn(\xi_n, m)}{m^2 sn(\xi_n, m) cn(\xi_n, m)}. \tag{3.68}$$

Consequently, using Maple or Mathematica we get the following results.

Family 1.

$$\beta_1 = -c_1 L_2 m^2, \qquad \alpha_0 = 0, \qquad s = \frac{-1}{CL_2^2}, \qquad \alpha_1 = c_1 L_2,$$
$$A = -CL_2^2 c_1^2 \frac{\left[m^4 sn^8(d,m) - 4m^4 sn^6(d,m) + \left(4m^4 + 2m^2\right) sn^4(d,m) - 4m^2 sn^2(d,m) + 1\right]}{sn^2(d,m) cn^2(d,m) dn^2(d,m)}, \tag{3.69}$$

where d, c_1, and C are arbitrary constants and

$$L_2 = \sqrt{\frac{sn^4(d,m) dn^4(d,m) + cn^4(d,m)}{C\left[m^4 sn^8(d,m) - 2m^2 sn^6(d,m) + 2sn^2(d,m) - 1\right]}}. \tag{3.70}$$

In this case, the solution of (3.47) takes the following form:

$$U(\xi_n) = \frac{-c_1 L_3 m^2 sn(\xi_n, m) cn(\xi_n, m)}{dn(\xi_n, m)} + \frac{c_1 L_3 dn(\xi_n, m)}{sn(\xi_n, m) cn(\xi_n, m)}, \tag{3.71}$$

where $\xi_n = dn + c_1 t + \xi_0$.

Family 2.

$$\beta_1 = m^2 c_1 L_3, \qquad \alpha_0 = 0, \qquad s = \frac{-1}{CL_3^2}, \qquad \alpha_1 = c_1 L_3,$$
$$A = -CL_3^2 c_1^2 \frac{\left[m^4 sn^8(d,m) - 2m^2 sn^4(d,m) + 1\right]}{sn^2(d,m) cn^2(d,m) dn^2(d,m)}, \tag{3.72}$$

where d, c_1, and C are arbitrary constants and

$$L_3 = \sqrt{\frac{m^4 sn^8(d,m) - 2(m^4+m^2) sn^6(d,m) + 2(m^4+m^2+1) sn^4(d,m) - 2(m^2+1) sn^2(d,m) + 1}{-C\left[m^4 sn^8(d,m) - 2(m^4+m^2) sn^6(d,m) + 6m^2 sn^4(d,m) - 2(m^2+1) sn^2(d,m) + 1\right]}}. \tag{3.73}$$

In this case, the solution of (3.47) takes the following form:

$$U(\xi_n) = -\frac{m^2 c_1 L_3 sn(\xi_n, m) cn(\xi_n, m)}{dn(\xi_n, m)} - \frac{c_1 L_3 dn(\xi_n, m)}{sn(\xi_n, m) cn(\xi_n, m)}, \tag{3.74}$$

where $\xi_n = dn + c_1 t + \xi_0$.

3.4. Example 4. The Quintic Discrete Nonlinear Schrodinger Equation

We discuss the quintic discrete nonlinear Schrodinger (QDNLS) equation [33]:

$$\frac{id\psi_n}{dt} + \alpha(\psi_{n+1} - 2\psi_n + \psi_{n-1}) + \beta|\psi_n|^2\psi_n + \gamma|\psi_n|^2(\psi_{n+1} + \psi_{n-1}) + \delta|\psi_n|^4(\psi_{n+1} + \psi_{n-1}) = 0, \tag{3.75}$$

which describes the propagation of discrete self-trapped beams in an array of weakly coupled nonlinear optical waveguides. Equation (3.75) was presented for the first time in [33], together with its localized solutions. We are looking for solutions of the form:

$$\psi_n(t) = \phi_n e^{-i\omega t}, \tag{3.76}$$

where ϕ_n is a real function with respect to the discrete variable n. Substitution of equation (3.76) into (3.75) yields the corresponding symmetric equation as follows:

$$\phi_{n+1} + \phi_{n-1} = \frac{(2\alpha - \omega)\phi_n - \beta\phi_n^3}{\alpha + \gamma\phi_n^2 + \delta\phi_n^4}. \tag{3.77}$$

We take the traveling wave of transformation

$$\phi_n = U(\xi_n), \qquad \xi_n = dn + k, \tag{3.78}$$

where d and k are constants. The transformation (3.78) permits us converting equation (3.77) into the following form:

$$U(\xi_n + d) + U(\xi_n - d) = \frac{(2\alpha - \omega)U(\xi_n) - \beta U^3(\xi_n)}{\alpha + \gamma U^2(\xi_n) + \delta U^4(\xi_n)}. \tag{3.79}$$

We suppose that the solutions of (3.80) have the following form:

$$U(\xi_n) = \alpha_0 + \alpha_1\left(\frac{F'(\xi_n)}{F(\xi_n)}\right) + \beta_1\left(\frac{F(\xi_n)}{F'(\xi_n)}\right), \tag{3.80}$$

where α_0, α_1, and β_1 are constant to be determined later and $F(\xi_n)$ satisfies a discrete Jacobi elliptic ordinary differential equation (2.6), we have the following types.

Type 1. If $e_0 = 1$, $e_1 = -(1+m^2)$, and $e_2 = m^2$. In this case, the series expansion solution of (3.79) has the form:

$$U(\xi_n) = \alpha_0 + \frac{\alpha_1 cn(\xi_n, m) dn(\xi_n, m)}{sn(\xi_n, m)} + \beta_1 \frac{sn(\xi_n, m)}{cn(\xi_n, m) dn(\xi_n, m)}. \tag{3.81}$$

With the help of Maple, we substitute (3.81) and (2.8) into (3.79). Cleaning the denominator and collecting all terms with the same degree of $sn(\xi_n, m)$, $cn(\xi_n, m)$, $dn(\xi_n, m)$ together, the left-hand side of (3.79) is converted into polynomial in $sn(\xi_n, m)$, $cn(\xi_n, m)$, $dn(\xi_n, m)$. Setting each coefficient of this polynomial to be zero, we derive a set of algebraic equations for α_0, α_1, d, β_1, α, ω, β, γ, and δ. Solving the set of algebraic equations by using Maple or Mathematica software package, we have the following.

Family 1.

$$\begin{gathered}
\beta_1 = \alpha_1\left(m^2 - 1\right), \qquad \alpha_0 = 0, \\
\beta = \frac{-2\delta\alpha_1^2\left[m^4 sn^8(d,m) - 2m^4 sn^6(d,m) + 2m^2 sn^2(d,m) - 1\right]}{sn^2(d,m) cn^2(d,m) dn^2(d,m)}, \\
\omega = \frac{4\alpha_1^2\left[H_1\delta\alpha_1^2 + H_2\gamma\right]}{sn^2(d,m) cn^2(d,m) dn^4(d,m)}, \qquad \alpha = \frac{-\alpha_1^2\left[H_3\delta\alpha_1^2 + H_4\gamma\right]}{sn^4(d,m) cn^4(d,m) dn^4(d,m)},
\end{gathered} \tag{3.82}$$

where α_1, δ, and d are arbitrary constants and

$$\begin{aligned}
H_1 &= m^8 sn^{12}(d,m) - 6m^6 sn^{10}(d,m) + \left(14m^4 + m^6\right) sn^8(d,m) - \left(16m^2 + 4m^4\right) sn^6(d,m) \\
&\quad + \left(8 - m^4 + 8m^2\right) sn^4(d,m) + \left(2m^2 - 8\right) sn^2(d,m) - m^2 + 2, \\
H_2 &= m^6 sn^{10}(d,m) - \left(m^6 + 3m^4\right) sn^8(d,m) + \left(4m^2 + 2m^4\right) sn^6(d,m) \\
&\quad + \left(m^4 - 3m^2 - 2\right) sn^4(d,m) + \left(2 - m^2\right) sn^2(d,m), \\
H_3 &= m^8 sn^{16}(d,m) - 8m^6 sn^{14}(d,m) + \left(4m^6 sn^{12} + 24m^4\right) sn^{12}(d,m) \\
&\quad - \left(32m^2 + 24m^4\right) sn^{10}(d,m) + \left(16 + 48m^2 + 6m^4\right) sn^8(d,m) \\
&\quad - \left(24m^2 + 32\right) sn^6(d,m) + \left(24 + 4m^2\right) sn^4(d,m) - 8sn^2(d,m) + 1,
\end{aligned}$$

$$H_4 = m^6 sn^{14}(d,m) - \left(m^6 + 5m^4\right) sn^{12}(d,m) + \left(7m^4 + 8m^2\right) sn^{10}(d,m)$$
$$- \left(14m^2 + 2m^4 + 4\right) sn^8(d,m) + \left(7m^2 + 8\right) sn^6(d,m) - \left(5 + m^2\right) sn^4(d,m) + sn^2(d,m). \tag{3.83}$$

In this case, the solution of (3.75) takes the following form:

$$\psi_n(t) = \frac{\alpha_1 \left[cn^2(\xi_n,m)dn^2(\xi_n,m) + (m^2-1)sn^2(\xi_n,m)\right]}{sn(\xi_n,m)cn(\xi_n,m)dn(\xi_n,m)} \times \operatorname{Exp}\left\{-i\left[\frac{4\alpha_1^2\left[H_1\delta\alpha_1^2 + H_2\gamma\right]}{sn^2(d,m)cn^2(d,m)dn^4(d,m)}\right]t\right\}, \tag{3.84}$$

where $\xi_n = dn + k$.

Figure 4 illustrates the behavior of the exact solution (3.84).

Family 2.

$$\beta_1 = -\alpha_1\left(m^2 - 1\right), \qquad \alpha_0 = 0, \qquad \beta = \frac{-2\delta\alpha_1^2\left[m^4 sn^8(d,m) - 2m^2 sn^6(d,m) + 2sn^2(d,m) - 1\right]}{sn^2(d,m)cn^2(d,m)dn^2(d,m)},$$
$$\omega = \frac{4\alpha_1^2\left[H_5\delta\alpha_1^2 + H_6\gamma\right]}{sn^2(d,m)cn^4(d,m)dn^2(d,m)}, \qquad \alpha = \frac{-\alpha_1^2\left[H_7\delta\alpha_1^2 + H_8\gamma\right]}{sn^4(d,m)cn^4(d,m)dn^4(d,m)}, \tag{3.85}$$

where α_1, δ, and d are arbitrary constants and

$$H_5 = m^6 sn^{12}(d,m) - 6m^6 sn^{10}(d,m) + \left(14m^6 + m^4\right) sn^8(d,m) - \left(16m^6 + 4m^4\right) sn^6(d,m)$$
$$+ \left(8m^4 - m^2 + 8m^6\right) sn^4(d,m) + \left(2m^2 - 8m^4\right) sn^2(d,m) + 2m^2 - 1,$$
$$H_6 = m^4 sn^{10}(d,m) - \left(m^2 + 3m^4\right) sn^8(d,m) + \left(2m^2 + 4m^4\right) sn^6(d,m)$$
$$+ \left(1 - 3m^2 - 2m^4\right) sn^4(d,m) + \left(2m^2 - 1\right) sn^2(d,m),$$
$$H_7 = m^8 sn^{16}(d,m) - 8m^8 sn^{14}(d,m) + \left(24m^8 + 4m^6\right) sn^{12}(d,m) - \left(24m^6 + 32m^8\right) sn^{10}(d,m)$$
$$+ \left(6m^4 + 48m^6 + 16m^8\right) sn^8(d,m) - \left(24m^4 + 32m^6\right) sn^6(d,m) + \left(24m^4 + 4m^2\right) sn^4(d,m)$$
$$- 8m^2 sn^2(d,m) + 1,$$

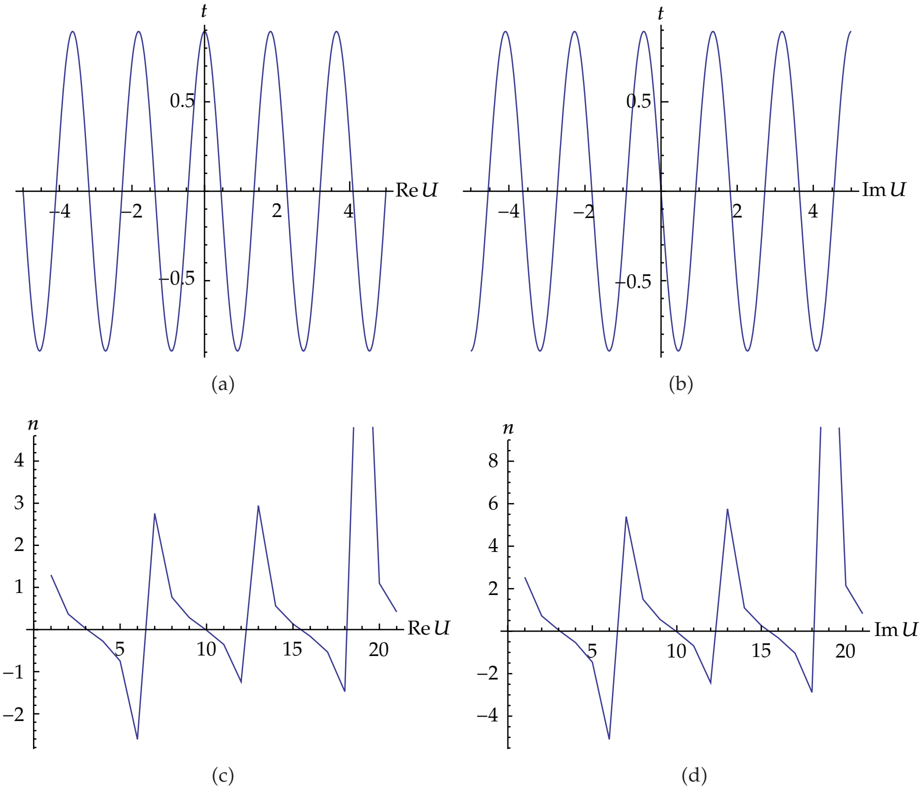

Figure 4: (a) and (b) represent the Jacobi elliptic solution (3.84) when $m = 0.5$, $\alpha_1 = 0.6$, $k = 0.2$, $\delta = 0.4$, $\gamma = 0.1$, $d = 0.3$, $n = 2$. (c) and (d) represent the Jacobi elliptic solution (3.84) when $m = 0.5$, $\alpha_1 = 0.6$, $k = 0.2$, $\delta = 0.4$, $\gamma = 0.1$, $d = 0.3$, $t = 1.5$.

$$\begin{aligned} H_8 = {} & m^6 sn^{14}(d,m) - \left(5m^6 + m^4\right) sn^{12}(d,m) + \left(7m^4 + 8m^6\right) sn^{10}(d,m) \\ & - \left(14m^4 + 2m^2 + 4m^6\right) sn^8(d,m) + \left(8m^4 + 7m^2\right) sn^6(d,m) \\ & - \left(5m^2 + 1\right) sn^4(d,m) + sn^2(d,m). \end{aligned} \tag{3.86}$$

In this case, the solution of (3.75) takes the following form:

$$\begin{aligned} \psi_n(t) = {} & \frac{\alpha_1 \left[cn^2(\xi_n,m)dn^2(\xi_n,m) - (m^2-1)sn^2(\xi_n,m)\right]}{sn(\xi_n,m)cn(\xi_n,m)dn(\xi_n,m)} \\ & \times \operatorname{Exp}\left\{-i\left[\frac{4\alpha_1^2\left[H_5\delta\alpha_1^2 + H_6\gamma\right]}{sn^2(d,m)cn^4(d,m)dn^2(d,m)}\right]t\right\}, \end{aligned} \tag{3.87}$$

where $\xi_n = dn + k$.

Type 2. If $e_0 = 1 - m^2$, $e_1 = 2m^2 - 1$, and $e_2 = -m^2$. In this case, the solution of (3.79) has the form:

$$U(\xi_n) = \alpha_0 - \alpha_1 \frac{sn(\xi_n, m)dn(\xi_n, m)}{cn(\xi_n, m)} - \beta_1 \frac{cn(\xi_n, m)}{sn(\xi_n, m)dn(\xi_n, m)}. \tag{3.88}$$

Consequently, using the Maple or Mathematica we get the following results.

Family 1.

$$\beta_1 = -\alpha_1, \qquad \alpha_0 = 0, \qquad \beta = \frac{-2\delta\alpha_1^2\left[m^4 sn^8(d,m) - 2m^4 sn^6(d,m) + 2m^2 sn^2(d,m) - 1\right]}{sn^2(d,m)cn^2(d,m)dn^2(d,m)},$$
$$\omega = \frac{4\alpha_1^2\left[H_1\delta\alpha_1^2 + H_2\gamma\right]}{sn^2(d,m)cn^2(d,m)dn^4(d,m)}, \quad \alpha = \frac{-\alpha_1^2\left[H_3\delta\alpha_1^2 + H_4\gamma\right]}{sn^4(d,m)cn^4(d,m)dn^4(d,m)}, \tag{3.89}$$

where α_1, δ, and d are arbitrary constants.

In this case, the solution of (3.75) takes the following form:

$$\varphi_n(t) = \frac{\alpha_1\left[cn^2(\xi_n, m) - sn^2(\xi_n, m)dn^2(\xi_n, m)\right]}{sn(\xi_n, m)cn(\xi_n, m)dn(\xi_n, m)} \operatorname{Exp}\left\{-i\left[\frac{4\alpha_1^2\left[H_1\delta\alpha_1^2 + H_2\gamma\right]}{sn^2(d,m)cn^2(d,m)dn^4(d,m)}\right]t\right\}, \tag{3.90}$$

where $\xi_n = dn + k$.

In the special case, when $m = 0$, the trigonometrical solution (3.90) takes the following form:

$$\varphi_n(t) = 2\alpha_1 \cot(2\xi_n) \operatorname{Exp}\left\{-i\left[32\alpha_1^4\delta + \frac{32\alpha_1^4\delta}{\sin^2(2d)} + 8\alpha_1^2\gamma\right]t\right\}, \quad \text{where } \xi_n = dn + k. \tag{3.91}$$

When $m = 1$, the hyperbolic solution (3.90) takes the following form:

$$\varphi_n(t) = \frac{2\alpha_1}{\sinh(2\xi_n)} \operatorname{Exp}\left\{-i\left[\frac{4\alpha_1^2\left[h_1\delta\alpha_1^2 + h_2\gamma\right]}{\tanh^2(d)\operatorname{sech}^6(d)}\right]t\right\}, \tag{3.92}$$

where

$$h_1 = \tanh^{12}(d) - 6\tanh^{10}(d) + 15\tanh^8(d) - 20\tanh^6(d) + 15\tanh^4(d) - 6\tanh^2(d) + 1,$$
$$h_2 = \tanh^{10}(d) - 4\tanh^8(d) + 6\tanh^6(d) - 4\tanh^4(d) + \tanh^2(d), \tag{3.93}$$

and $\xi_n = dn + k$.

Family 2.

$$\beta=\frac{2\delta\alpha_1^2\left[m^4 sn^8(d,m)-2\left(m^4+m^2\right)sn^6(d,m)+6m^2 sn^4(d,m)-2\left(m^2+1\right)sn^2(d,m)+1\right]}{sn^2(d,m)cn^2(d,m)dn^2(d,m)},$$
$$\omega=\frac{-4\alpha_1^2\left[H_9\delta\alpha_1^2+H_{10}\gamma\right]}{sn^2(d,m)cn^4(d,m)dn^4(d,m)},\qquad \alpha=\frac{-\alpha_1^2\left[H_{11}\delta\alpha_1^2+H_{12}\gamma\right]}{sn^4(d,m)cn^4(d,m)dn^4(d,m)},$$
$$\beta_1=\alpha_1,\qquad \alpha_0=0, \tag{3.94}$$

where α_1, δ, and d are arbitrary constants and

$$\begin{aligned}
H_9 &= \left(m^8+m^6\right)sn^{12}(d,m)-4m^6 sn^{10}(d,m)-\left(m^6+m^4\right)sn^8(d,m)+8m^4 sn^6(d,m)\\
&\quad-\left(m^4+m^2\right)sn^4(d,m)-4m^2 sn^2(d,m)+m^2+1,\\
H_{10} &= \left(m^6+m^4\right)sn^{10}(d,m)-\left(m^6+6m^4+m^2\right)sn^8(d,m)+6\left(m^4+m^2\right)sn^6(d,m)\\
&\quad-\left(m^4+6m^2+1\right)sn^4(d,m)+\left(m^2+1\right)sn^2(d,m),\\
H_{11} &= m^8 sn^{16}(d,m)-4m^6 sn^{12}(d,m)+6m^4 sn^8(d,m)-4m^2 sn^4(d,m)+1,\\
H_{12} &= m^6 sn^{14}(d,m)-\left(m^6+m^4\right)sn^{12}(d,m)-m^4 sn^{10}(d,m)+2\left(m^4+m^2\right)sn^8(d,m)\\
&\quad-m^2 sn^6(d,m)-\left(m^2+1\right)sn^4(d,m)+sn^2(d,m).
\end{aligned} \tag{3.95}$$

In this case, the solution of (3.75) takes the following form:

$$\psi_n(t)=\frac{-\alpha_1\left[cn^2(\xi_n,m)+sn^2(\xi_n,m)dn^2(\xi_n,m)\right]}{sn(\xi_n,m)cn(\xi_n,m)dn(\xi_n,m)}\mathrm{Exp}\left\{-i\left[\frac{-4\alpha_1^2\left[H_9\delta\alpha_1^2+H_{10}\gamma\right]}{sn^2(d,m)cn^4(d,m)dn^4(d,m)}\right]t\right\}, \tag{3.96}$$

where $\xi_n=dn+k$.

When $m=0$, the trigonometrical solution (3.96) takes the following form:

$$\psi_n(t)=\frac{-2\alpha_1}{\sin(2\xi_n)}\mathrm{Exp}\left\{-i\left[\frac{-4\alpha_1^2\left[\delta\alpha_1^2+\sin^2(d)\cos^2(d)\gamma\right]}{\sin^2(d)\cos^4(d)}\right]t\right\}, \tag{3.97}$$

where $\xi_n=dn+k$.

When $m=1$, the hyperbolic solution (3.96) takes the following form:

$$\psi_n(t)=-2\alpha_1\coth(2\xi_n)\mathrm{Exp}\left\{-i\left[\frac{-4\alpha_1^2\left[h_9\delta\alpha_1^2+h_{10}\gamma\right]}{\tanh^2(d)\mathrm{sech}^8(d)}\right]t\right\}, \tag{3.98}$$

where

$$h_9 = 2\tanh^{12}(d) - 4\tanh^{10}(d) - 2\tanh^{8}(d) + 8\tanh^{6}(d) - 2\tanh^{4}(d) - 4\tanh^{2}(d) + 2,$$
$$h_{10} = 2\tanh^{10}(d) - 8\tanh^{8}(d) + 12\tanh^{6}(d) - 8\tanh^{4}(d) + 2\tanh^{2}(d), \tag{3.99}$$

and $\xi_n = dn + k$.

Type 3. If $e_0 = m^2 - 1$, $e_1 = 2 - m^2$, and $e_2 = -1$. In this case, the series expansion solution of (3.79) has the form:

$$U(\xi_n) = \alpha_0 - \frac{m^2\alpha_1 sn(\xi_n, m)cn(\xi_n, m)}{dn(\xi_n, m)} - \frac{\beta_1 dn(\xi_n, m)}{m^2 sn(\xi_n, m)cn(\xi_n, m)}. \tag{3.100}$$

Consequently, using the Maple or Mathematica we get the following results:

Family 1.

$$\beta_1 = -\alpha_1 m^2, \qquad \alpha_0 = 0, \qquad \beta = \frac{-2\delta\alpha_1^2\left[m^4 sn^8(d,m) - 2m^2 sn^6(d,m) + 2sn^2(d,m) - 1\right]}{sn^2(d,m)cn^2(d,m)dn^2(d,m)},$$
$$\omega = \frac{4\alpha_1^2\left[H_5\delta\alpha_1^2 + H_6\gamma\right]}{sn^2(d,m)cn^4(d,m)dn^2(d,m)}, \qquad \alpha = \frac{-\alpha_1^2\left[H_7\delta\alpha_1^2 + H_8\gamma\right]}{sn^4(d,m)cn^4(d,m)dn^4(d,m)}, \tag{3.101}$$

where α_1, δ, and d are arbitrary constants.

In this case, the solution of (3.75) takes the following form:

$$\psi_n(t) = \frac{\alpha_1\left[dn^2(\xi_n, m) - m^2 sn^2(\xi_n, m)cn^2(\xi_n, m)\right]}{sn(\xi_n, m)cn(\xi_n, m)dn(\xi_n, m)}\mathrm{Exp}\left\{-i\left[\frac{4\alpha_1^2\left[H_5\delta\alpha_1^2 + H_6\gamma\right]}{sn^2(d,m)cn^4(d,m)dn^2(d,m)}\right]t\right\}, \tag{3.102}$$

where $\xi_n = dn + k$.

Family 2.

$$\beta = \frac{2\delta\alpha_1^2\left[m^4 sn^8(d,m) - 2(m^4 + m^2)sn^6(d,m) + 6m^2 sn^4(d,m) - 2(m^2 + 1)sn^2(d,m) + 1\right]}{sn^2(d,m)cn^2(d,m)dn^2(d,m)},$$
$$\omega = \frac{-4\alpha_1^2\left[H_9\delta\alpha_1^2 + H_{10}\gamma\right]}{sn^2(d,m)cn^4(d,m)dn^4(d,m)}, \qquad \alpha = \frac{-\alpha_1^2\left[H_{11}\delta\alpha_1^2 + H_{12}\gamma\right]}{sn^4(d,m)cn^4(d,m)dn^4(d,m)},$$
$$\beta_1 = m^2\alpha_1, \qquad \alpha_0 = 0, \tag{3.103}$$

where α_1, δ, and d are arbitrary constants.

Table 1: The relation between the Jacobi elliptic functions and the trigonometrical functions, hyperbolic functions.

m	$sn\,(\xi, m)$	$cn\,(\xi, m)$	$dn\,(\xi, m)$	$ns\,(\xi, m)$	$cs\,(\xi, m)$	$ds\,(\xi, m)$
0	sin (ξ)	cos (ξ)	1	csc (ξ)	cot (ξ)	csc (ξ)
1	tanh (ξ)	sech (ξ)	sech (ξ)	coth (ξ)	csch (ξ)	csch (ξ)

In this case, the solution of (3.75) takes the following form:

$$\psi_n(t) = \frac{-\alpha_1\left[m^2 sn^2(\xi_n, m) cn^2(\xi_n, m) + dn^2(\xi_n, m)\right]}{sn(\xi_n, m) cn(\xi_n, m) dn(\xi_n, m)} \times \operatorname{Exp}\left\{-i\left[\frac{-4\alpha_1^2\left[H_9\delta\alpha_1^2 + H_{10}\gamma\right]}{sn^2(d, m) cn^4(d, m) dn^4(d, m)}\right]t\right\}, \tag{3.104}$$

where $\xi_n = dn + k$.

Remark 3.1. Wang and Ma [34] constructed the Jacobi elliptic function solution to some nonlinear DDEs in terms of Jacobi elliptic functions sn or cn or dn. In Zhang [20], an algorithm is devised to derive exact traveling wave solutions of nonlinear DDEs by means of Jacobi elliptic functions. Gepreel and Shehata [22] put a direct method to construct the rational Jacobi elliptic solution for nonlinear DDEs. In this paper, we modified the direct method which was disscussed in [22]. When we are constructing the solutions of nonlinear DDEs we neglected the case $\beta_i = 0$ because if $\beta_i = 0$, we get the same solution which was discussed by [22]. The Jacobi elliptic functions could be generated into the hyperbolic functions when m tends to one in the other hand, they are generated into trigonometrical functions when m tends to zero as shown in Table 1.

4. Conclusion

In this paper, we modified the rational Jacobi elliptic functions method to calculate some new exact solutions for the nonlinear difference differential equations via the lattice equation, the discrete nonlinear Schrodinger equation with a saturable nonlinearity, the discrete nonlinear Klein-Gordon equation, and the quintic discrete nonlinear Schrodinger equation. As a result, many new and more rational Jacobi elliptic solutions are obtained, from which hyperbolic function solutions and trigonometric function solutions are derived when the moduli $m \to 1$ and $m \to 0$.

References

[1] E. Fermi, J. Pasta, and S. Ulam, *Collected Papers of Enrico Fermi II*, University of Chicago Press, Chicago, Ill, USA, 1965.
[2] W. P. Su, J. R. Schrieffer, and A. J. Heeger, "Solitons in polyacetylene," *Physical Review Letters*, vol. 42, no. 25, pp. 1698–1701, 1979.
[3] A. S. Davydov, "The theory of contraction of proteins under their excitation," *Journal of Theoretical Biology*, vol. 38, no. 3, pp. 559–569, 1973.
[4] P. Marquié, J. M. Bilbault, and M. Remoissenet, "Observation of nonlinear localized modes in an electrical lattice," *Physical Review E*, vol. 51, no. 6, pp. 6127–6133, 1995.
[5] M. Toda, *Theory of Nonlinear Lattices*, Springer, Berlin, Germany, 1989.

[6] M. Wadati, "Transformation theories for nonlinear discrete systems," *Progress of Theoretical Physics*, vol. 59, pp. 36–63, 1976.

[7] Y. Ohta and R. Hirota, "A discrete KdV equation and its Casorati determinant solution," *Journal of the Physical Society of Japan*, vol. 60, no. 6, p. 2095, 1991.

[8] M. J. Ablowitz and J. Ladik, "Nonlinear differential-difference equations," *Journal of Mathematical Physics*, vol. 16, pp. 598–603, 1975.

[9] X. B. Hu and W. X. Ma, "Application of Hirota's bilinear formalism to the Toeplitz lattice—some special soliton-like solutions," *Physics Letters A*, vol. 293, no. 3-4, pp. 161–165, 2002.

[10] D. Baldwin, U. Goktas, and W. Hereman, "Symbolic computation of hyperbolic tangent solutions for nonlinear differential-difference equations," *Computer Physics Communications*, vol. 162, no. 3, pp. 203–217, 2004.

[11] S. K. Liu, Z. T. Fu, Z. G. Wang, and S. D. Liu, "Periodic solutions for a class of nonlinear differential-difference equations," *Communications in Theoretical Physics*, vol. 49, no. 5, pp. 1155–1158, 2008.

[12] C. Qiong and L. Bin, "Applications of Jacobi elliptic function expansion method for nonlinear differential-difference equations," *Communications in Theoretical Physics*, vol. 43, no. 3, pp. 385–388, 2005.

[13] F. Xie, M. Jia, and H. Zhao, "Some solutions of discrete sine-Gordon equation," *Chaos, Solitons & Fractals*, vol. 33, no. 5, pp. 1791–1795, 2007.

[14] S. D. Zhu, "Exp-function method for the Hybrid-Lattice system," *International Journal of Nonlinear Sciences and Numerical Simulation*, vol. 8, no. 3, pp. 461–464, 2007.

[15] I. Aslan, "A discrete generalization of the extended simplest equation method," *Communications in Nonlinear Science and Numerical Simulation*, vol. 15, no. 8, pp. 1967–1973, 2010.

[16] P. Yang, Y. Chen, and Z. B. Li, "ADM-Padé technique for the nonlinear lattice equations," *Applied Mathematics and Computation*, vol. 210, no. 2, pp. 362–375, 2009.

[17] S. D. Zhu, Y. M. Chu, and S. L. Qiu, "The homotopy perturbation method for discontinued problems arising in nanotechnology," *Computers & Mathematics with Applications*, vol. 58, no. 11-12, pp. 2398–2401, 2009.

[18] S. Zhang, L. Dong, J. Ba, and Y. Sun, "The $(\frac{G'}{G})$-expansion method for nonlinear differential-difference equations," *Physics Letters A*, vol. 373, no. 10, pp. 905–910, 2009.

[19] I. Aslan, "The Ablowitz-Ladik lattice system by means of the extended (G'/G)-expansion method," *Applied Mathematics and Computation*, vol. 216, no. 9, pp. 2778–2782, 2010.

[20] S. Zhang, "Discrete Jacobi elliptic function expansion method for nonlinear difference equation," *Physica Scripta*, vol. 80, pp. 045002–045010, 2009.

[21] A. Khaled Gepreel, "Rational Jacobi elliptic solutions for nonlinear difference differential equations," *Nonlinear Science Letters A*, vol. 2, pp. 151–158, 2011.

[22] K. A. Gepreel and A. R. Shehata, "Rational Jacobi elliptic solutions for nonlinear differential difference lattice equations," *Applied Mathematics Letters*, vol. 25, no. 9, pp. 1173–1178, 2012.

[23] G. Wu and T. Xia, "A new method for constructing soliton solutions to differential-difference equation with symbolic computation," *Chaos, Solitons & Fractals*, vol. 39, no. 5, pp. 2245–2248, 2009.

[24] F. Xie and J. Wang, "A new method for solving nonlinear differential-difference equation," *Chaos, Solitons & Fractals*, vol. 27, no. 4, pp. 1067–1071, 2006.

[25] C. Liu, "Exponential function rational expansion method for nonlinear differential-difference equations," *Chaos, Solitons & Fractals*, vol. 40, no. 2, pp. 708–716, 2009.

[26] Q. Wang and Y. Yu, "New rational formal solutions for $(1+1)$-dimensional Toda equation and another Toda equation," *Chaos, Solitons & Fractals*, vol. 29, no. 4, pp. 904–915, 2006.

[27] A. Trombettoni and A. Smerzi, "Discrete solitons and breathers with dilute Bose-Einstein condensates," *Physical Review Letters*, vol. 86, no. 11, pp. 2353–2356, 2001.

[28] L. Hadžievski, A. Maluckov, M. Stepić, and D. Kip, "Power controlled soliton stability and steering in lattices with saturable nonlinearity," *Physical Review Letters*, vol. 93, no. 3, Article ID 033901, 4 pages, 2004.

[29] S. Gate and J. Herrmann, "Soliton propagation in materials with saturable nonlinearity," *Journal of the Optical Society of America B*, vol. 9, pp. 2296–2302, 1991.

[30] S. Gate and J. Herrmann, "Soliton propagation and soliton collision in double doped fibers with a non- Kerr- like nonlinear refractive- index change," *Optics Letters*, vol. 17, pp. 484–486, 1992.

[31] P. Kevrekidis and V. Konotop, "Bright compact breathers," *Physical Review E*, vol. 65, no. 6, pp. 066614–066619, 2002.

[32] P. Kevrekidis, V. Konotop, and A. Bishop, "Discrete compactons: some exact results," *Journal of Physics A*, vol. 35, no. 45, pp. L641–L652, 2002.

[33] K. Maruno, Y. Ohta, and N. Joshi, "Exact localized solutions of quintic discrete nonlinear Schrödinger equation," *Physics Letters A*, vol. 311, no. 2-3, pp. 214–220, 2003.

[34] Z. Wang and W.-X. Ma, "Discrete Jacobi sub-equation method for nonlinear differential-difference equations," *Mathematical Methods in the Applied Sciences*, vol. 33, no. 12, pp. 1463–1472, 2010.