Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 420890, 13 pages
doi:10.1155/2012/420890

*Research Article*

# Warped Product Pseudo-Slant Submanifolds of a Nearly Cosymplectic Manifold

**Siraj Uddin, Bernardine R. Wong, and A. A. Mustafa**

*Institute of Mathematical Sciences, Faculty of Science, University of Malaya, 50603 Kuala Lumpur, Malaysia*

Correspondence should be addressed to Siraj Uddin, siraj.ch@gmail.com

Received 17 March 2012; Accepted 23 June 2012

Academic Editor: Liming Ge



We study warped product pseudo-slant submanifolds of a nearly cosymplectic manifold. We obtain some characterization results on the existence or nonexistence of warped product pseudo-slant submanifolds of a nearly cosymplectic manifold in terms of the canonical structures $P$ and $F$.

## 1. Introduction

To study the manifolds with negative curvature, Bishop and O'Neill [1] introduced the notion of warped product manifolds by homothetically warping the product metric of a product manifold $N_1 \times N_2$ onto the fibers $p \times N_2$ for each $p \in N_1$. Later on, the geometrical aspect of these manifolds has been studied by many researchers (cf. [2–4]). Pseudo-slant submanifolds were introduced by Carriazo [5] as a special case of bislant submanifolds.

Almost contact manifolds with Killing structure tensors were defined in [6] as nearly cosymplectic manifolds, and it was shown that the normal nearly cosymplectic manifolds are cosymplectic (see also [7]). Later on, Blair and Showers [8] studied nearly cosymplectic structure $(\phi, \xi, \eta, g)$ on a Riemannian manifold $\overline{M}$ with $\eta$ closed from the topological viewpoint.

Recently, Sahin [9] studied the warped product hemislant (pseudo-slant) submanifolds of Kaehler manifolds. He proved that the warped product submanifolds of the type $M = N_\perp \times_f N_\theta$ of a Kaehler manifold $\overline{M}$ do not exist and obtained some characterization results on the existence of warped product submanifold $M = N_\theta \times_f N_\perp$, where $N_\perp$ and $N_\theta$ are totally real and proper slant submanifolds of a Kaehler manifold $\overline{M}$, respectively. After that, we have extended this study to the more general setting of nearly Kaehler manifolds [4]. The warped product semi-invariant submanifolds of a nearly cosymplectic manifold had been studied in [10].

In this paper, we study warped product pseudo-slant submanifolds of a nearly cosymplectic manifold. We obtain some characterization results of warped product submanifolds of the types $N_{\perp}\times_f N_{\theta}$ and $N_{\theta}\times_f N_{\perp}$ in terms of the canonical structures $P$ and $F$, where $N_{\perp}$ and $N_{\theta}$ are anti-invariant and proper slant submanifolds of a nearly cosymplectic manifold $\overline{M}$, respectively.

## 2. Preliminaries

A $(2n+1)$-dimensional $C^{\infty}$ manifold $\overline{M}$ is said to have an *almost contact structure* if there exist on $\overline{M}$ a tensor field $\phi$ of type $(1,1)$, a vector field $\xi$, and a 1-form $\eta$ satisfying [8]

$$\phi^2 = -I + \eta \otimes \xi, \qquad \phi\xi = 0, \qquad \eta \circ \phi = 0, \qquad \eta(\xi) = 1. \tag{2.1}$$

There always exists a Riemannian metric $g$ on an almost contact manifold $\overline{M}$ satisfying the following compatibility condition:

$$\eta(X) = g(X, \xi), \qquad g\left(\phi X, \phi Y\right) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

where $X$ and $Y$ are vector fields on $\overline{M}$ [8].

An almost contact structure $(\phi, \xi, \eta)$ is said to be *normal* if the almost complex structure $J$ on the product manifold $\overline{M} \times \mathbb{R}$ given by

$$J\left(X, f\frac{d}{dt}\right) = \left(\phi X - f\xi, \eta(X)\frac{d}{dt}\right), \tag{2.3}$$

where $f$ is a $C^{\infty}$-function on $\overline{M} \times \mathbb{R}$ has no torsion, that is, $J$ is integrable, the condition for normality in terms of $\phi, \xi$ and $\eta$ is $[\phi, \phi] + 2d\eta \otimes \xi = 0$ on $\overline{M}$, where $[\phi, \phi]$ is the Nijenhuis tensor of $\phi$. Finally the *fundamental 2-form* $\Phi$ is defined by $\Phi(X, Y) = g(X, \phi Y)$.

An almost contact metric structure $(\phi, \xi, \eta, g)$ is said to be *cosymplectic*, if it is normal and both $\Phi$ and $\eta$ are closed [8]. The structure is said to be *nearly cosymplectic* if $\phi$ is Killing, that is, if

$$\left(\overline{\nabla}_X\phi\right)Y + \left(\overline{\nabla}_Y\phi\right)X = 0, \tag{2.4}$$

for any $X, Y \in T\overline{M}$, where $T\overline{M}$ is the tangent bundle of $\overline{M}$ and $\overline{\nabla}$ denotes the Riemannian connection of the metric $g$. Equation (2.4) is equivalent to $(\overline{\nabla}_X\phi)X = 0$, for each $X \in T\overline{M}$. The structure is said to be *closely cosymplectic* if $\phi$ is Killing and $\eta$ is closed. It is well known that an almost contact metric manifold is *cosymplectic* if and only if $\overline{\nabla}\phi$ vanishes identically, that is, $(\overline{\nabla}_X\phi)Y = 0$ and $\overline{\nabla}_X\xi = 0$.

**Proposition 2.1** (see [8]). *On a nearly cosymplectic manifold the vector field $\xi$ is Killing.*

From the above proposition, one has $\overline{\nabla}_X\xi = 0$, for any vector field $X$ tangent to $\overline{M}$, where $\overline{M}$ is a nearly cosymplectic manifold.

Let $M$ be submanifold of an almost contact metric manifold $\overline{M}$ with induced metric $g$ and if $\nabla$ and $\nabla^{\perp}$ are the induced connections on the tangent bundle $TM$ and the normal bundle $T^{\perp}M$ of $M$, respectively, then Gauss and Weingarten formulae are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.5}$$

$$\overline{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \tag{2.6}$$

for each $X, Y \in TM$ and $N \in T^{\perp}M$, where $h$ and $A_N$ are the second fundamental form and the shape operator (corresponding to the normal vector field $N$), respectively, for the immersion of $M$ into $\overline{M}$. They are related as

$$g(h(X, Y), N) = g(A_N X, Y), \tag{2.7}$$

where $g$ denotes the Riemannian metric on $\overline{M}$ as well as induced on $M$.

For any $X \in TM$, one writes

$$\phi X = PX + FX, \tag{2.8}$$

where $PX$ is the tangential component and $FX$ is the normal component of $\phi X$.

Similarly for any $N \in T^{\perp}M$, one writes

$$\phi N = BN + CN, \tag{2.9}$$

where $BN$ is the tangential component and $CN$ is the normal component of $\phi N$.

Now, denote by $\mathcal{P}_X Y$ and $\mathcal{Q}_X Y$ the tangential and normal parts of $(\overline{\nabla}_X \phi)Y$, that is,

$$\left(\overline{\nabla}_X \phi\right) Y = \mathcal{P}_X Y + \mathcal{Q}_X Y \tag{2.10}$$

for all $X, Y \in TM$. Making use of (2.8), (2.10), and the Gauss and Weingarten formulae, the following equations may easily be obtained:

$$\begin{aligned} \mathcal{P}_X Y &= \left(\overline{\nabla}_X P\right) Y - A_{FY} X - Bh(X, Y), \\ \mathcal{Q}_X Y &= \left(\overline{\nabla}_X F\right) Y + h(X, PY) - Ch(X, Y). \end{aligned} \tag{2.11}$$

Similarly, for any $N \in T^{\perp}M$, denoting tangential and normal parts of $(\overline{\nabla}_X \phi)N$ by $\mathcal{P}_X N$ and $\mathcal{Q}_X N$, respectively, one obtains

$$\begin{aligned} \mathcal{P}_X N &= \left(\overline{\nabla}_X B\right) N + PA_N X - A_{CN} X, \\ \mathcal{Q}_X N &= \left(\overline{\nabla}_X C\right) N + h(BN, X) + FA_N X, \end{aligned} \tag{2.12}$$

where the covariant derivatives of $P, F, B$, and $C$ are defined by

$$\left(\overline{\nabla}_X P\right)Y = \nabla_X PY - P\nabla_X Y, \tag{2.13}$$

$$\left(\overline{\nabla}_X F\right)Y = \nabla_X^{\perp} FY - F\nabla_X Y, \tag{2.14}$$

$$\left(\overline{\nabla}_X B\right)N = \nabla_X BN - B\nabla_X^{\perp} N, \tag{2.15}$$

$$\left(\overline{\nabla}_X C\right)N = \nabla_X^{\perp} CN - C\nabla_X^{\perp} N, \tag{2.16}$$

for all $X, Y \in TM$ and $N \in T^{\perp}M$.

It is straightforward to verify the following properties of $\mathcal{P}$ and $\mathcal{Q}$, which one enlists here for later use

$(p_1)$ (i) $\mathcal{P}_{X+Y}W = \mathcal{P}_X W + \mathcal{P}_Y W$, (ii) $\mathcal{Q}_{X+Y}W = \mathcal{Q}_X W + \mathcal{Q}_Y W$,

$(p_2)$ (i) $\mathcal{P}_X(Y+W) = \mathcal{P}_X Y + \mathcal{P}_X W$, (ii) $\mathcal{Q}_X(Y+W) = \mathcal{Q}_X Y + \mathcal{Q}_X W$,

$(p_3)$ (i) $g(\mathcal{P}_X Y, W) = -g(Y, \mathcal{P}_X W)$, (ii) $g(\mathcal{Q}_X Y, N) = -g(Y, \mathcal{P}_X N)$,

$(p_4)$ $\mathcal{P}_X \phi Y + \mathcal{Q}_X \phi Y = -\phi(\mathcal{P}_X Y + \mathcal{Q}_X Y)$,

for all $X, Y, W \in TM$ and $N \in T^{\perp}M$.

On a submanifold $M$ of a nearly cosymplectic manifold, by (2.4) and (2.10), one has

$$\text{(a) } \mathcal{P}_X Y + \mathcal{P}_Y X = 0, \qquad \text{(b) } \mathcal{Q}_X Y + \mathcal{Q}_Y X = 0, \tag{2.17}$$

for any $X, Y \in TM$.

The submanifold $M$ is said to be *invariant* if $F$ is identically zero, that is, $\phi X \in TM$ for any $X \in TM$. On the other hand, $M$ is said to be *anti-invariant* if $P$ is identically zero, that is, $\phi X \in T^{\perp}M$, for any $X \in TM$.

One will always consider $\xi$ to be tangent to the submanifold $M$. There is another class of submanifolds that is called the *slant submanifold*. For each nonzero vector $X$ tangent to $M$ at any $x \in M$, such that $X$ is not proportional to $\xi_x$, one denotes by $0 \le \theta(X) \le \pi/2$, the angle between $\phi X$ and $T_x M$ is called the slant angle. If the slant angle $\theta(X)$ is constant for all $X \in T_x M - \langle \xi_x \rangle$ and $x \in M$, then M is said to be a slant submanifold [11]. Obviously, if $\theta = 0$, then $M$ is an invariant submanifold and if $\theta = \pi/2$, then $M$ is an anti-invariant submanifold. A slant submanifold is said to be *proper slant* if it is neither invariant nor anti-invariant.

One recalls the following result for a slant submanifold.

**Theorem 2.2** (see [11]). *Let M be a submanifold of an almost contact metric manifold $\overline{M}$, such that $\xi \in TM$. Then M is slant if and only if there exists a constant $\lambda \in [0,1]$ such that*

$$P^2 = \lambda\left(-I + \eta \otimes \xi\right). \tag{2.18}$$

*Furthermore, if $\theta$ is slant angle, then $\lambda = \cos^2\theta$.*

The following relations are straightforward consequence of (2.18):

$$g(PX, PY) = \cos^2\theta\big(g(X,Y) - \eta(Y)\eta(X)\big), \tag{2.19}$$

$$g(FX, FY) = \sin^2\theta\big(g(X,Y) - \eta(Y)\eta(X)\big), \tag{2.20}$$

for all $X, Y \in TM$.

A submanifold $M$ of an almost contact manifold $\overline{M}$ is said to be a *pseudo-slant submanifold* if there exist two orthogonal complementary distributions $D_1$ and $D_2$ satisfying:

(i) $TM = D_1 \oplus D_2 \oplus \langle\xi\rangle$,

(ii) $D_1$ is a slant distribution with slant angle $\theta \neq \pi/2$,

(iii) $D_2$ is anti-invariant that is, $\phi D_2 \subseteq T^\perp M$.

A pseudo-slant submanifold $M$ of an almost contact manifold $\overline{M}$ is *mixed geodesic* if

$$h(X, Z) = 0, \tag{2.21}$$

for any $X \in D_1$ and $Z \in D_2$.

If $\mu$ is the invariant subspace of the normal bundle $T^\perp M$, then in the case of pseudo-slant submanifold, the normal bundle $T^\perp M$ can be decomposed as follows:

$$T^\perp M = FD_1 \oplus FD_2 \oplus \mu. \tag{2.22}$$

## 3. Warped Product Pseudo-Slant Submanifolds

Bishop and O'Neill [1] introduced the notion of warped product manifolds. These manifolds are the natural generalizations of Riemannian product manifolds. They defined these manifolds as follows Let $(N_1, g_1)$ and $(N_2, g_2)$ be two Riemannian manifolds and $f$, a positive differentiable function on $N_1$. The warped product of $N_1$ and $N_2$ is the Riemannian manifold $N_1 \times_f N_2 = (N_1 \times N_2, g)$, where

$$g = g_1 + f^2 g_2. \tag{3.1}$$

A warped product manifold $N_1 \times_f N_2$ is said to be *trivial* if the warping function $f$ is constant. We recall the following general formula on a warped product manifold [1]:

$$\nabla_X Z = \nabla_Z X = \big(X \ln f\big)Z, \tag{3.2}$$

where $X$ is tangential to $N_1$ and $Z$ is tangential to $N_2$.

Let $M = N_1 \times_f N_2$ be a warped product manifold. This means that $N_1$ is totally geodesic and $N_2$ is a totally umbilical submanifold of $M$, respectively [1].

Throughout this section, we consider the warped product pseudo-slant submanifolds which are either in the form $N_\perp \times_f N_\theta$ or $N_\theta \times_f N_\perp$ in a nearly cosymplectic manifold $\overline{M}$, where $N_\theta$ and $N_\perp$ are proper slant and anti-invariant submanifolds of a nearly cosymplectic manifold $\overline{M}$, respectively. On a warped product submanifold $M = N_1 \times_f N_2$ of a nearly cosymplectic manifold $\overline{M}$, we have the following result.

**Theorem 3.1** (see [10]). *A warped product submanifold $M = N_1 \times_f N_2$ of a nearly cosymplectic manifold $\overline{M}$ is an usual Riemannian product if the structure vector field $\xi$ is tangential to $M_2$, where $M_1$ and $M_2$ are the Riemannian submanifolds of $\overline{M}$.*

Now, one considers the warped product pseudo-slant submanifolds in the form $M = N_\perp \times_f N_\theta$ of a nearly cosymplectic manifold $\overline{M}$. If one considers the structure vector field $\xi \in TN_\theta$ then by Theorem 3.1, the warping function $f$ is constant and hence one will considers $\xi \in TN_\perp$.

**Proposition 3.2.** *Let $M = N_\perp \times_f N_\theta$ be a warped product pseudo-slant submanifold of a nearly cosymplectic manifold $\overline{M}$. Then,*

$$g\left(\nabla^\perp_{PX} FPX - \nabla^\perp_X FX, FZ\right) = (Z \ln f) \sin^2\theta \|X\|^2 + \left(1 + \cos^2\theta\right) g(h(X, PX), FZ), \tag{3.3}$$

*for any $X \in TN_\theta$ and $Z \in TN_\perp$, where $N_\theta$ and $N_\perp$ are proper slant and anti-invariant submanifolds of $\overline{M}$, respectively.*

*Proof.* For any $X \in TN_\theta$ and $Z \in TN_\perp$, by (2.8), we have

$$g\left(\overline{\nabla}_X \phi X, FZ\right) = g\left(\overline{\nabla}_X PX, FZ\right) + g\left(\overline{\nabla}_X FX, FZ\right). \tag{3.4}$$

Using (2.5), (2.6), and the covariant derivative property of $\phi$, we obtain

$$g\left(\left(\overline{\nabla}_X \phi\right) X, FZ\right) + g\left(\phi \overline{\nabla}_X X, \phi Z\right) = g(h(X, PX), FZ) + g\left(\nabla^\perp_X FX, FZ\right). \tag{3.5}$$

Then from (2.2), (2.4), and the fact that $\xi$ is a Killing vector field on $\overline{M}$, thus we obtain

$$g\left(\overline{\nabla}_X X, Z\right) = g(h(X, PX), FZ) + g\left(\nabla^\perp_X FX, FZ\right). \tag{3.6}$$

Using the property of $\overline{\nabla}$, we get

$$-g\left(X, \overline{\nabla}_X Z\right) = g(h(X, PX), FZ) + g\left(\nabla^\perp_X FX, FZ\right). \tag{3.7}$$

Then by (2.5) and (3.2), we derive

$$-(Z \ln f) \|X\|^2 = g(h(PX, X), FZ) + g\left(\nabla^\perp_X FX, FZ\right). \tag{3.8}$$

Interchanging $X$ by $PX$ in (3.8) and using (2.18), (2.19), and the fact that $\xi \in TN_{\perp}$, we obtain

$$-(Z\ln f)\cos^2\theta\|X\|^2 = -\cos^2\theta g(h(X,PX),FZ) + g\left(\nabla^{\perp}_{PX}FPX,FZ\right). \tag{3.9}$$

Thus, the result follows from (3.8) and (3.9). □

**Proposition 3.3.** *Let $M = N_{\perp}\times_f N_{\theta}$ be a warped product pseudo-slant submanifold of a nearly cosymplectic manifold $\overline{M}$. Then,*

$$g\left(\left(\overline{\nabla}_X F\right)X,FZ\right) = sec^2\theta g\left(\left(\overline{\nabla}_{PX} F\right)PX,FZ\right) \tag{3.10}$$

*for any $X \in TN_{\theta}$ and $Z \in TN_{\perp}$, where $N_{\theta}$ and $N_{\perp}$ are proper slant and anti-invariant submanifolds of $\overline{M}$, respectively.*

*Proof.* For any $X \in TN_{\theta}$ and $Z \in TN_{\perp}$ by (2.14), we have

$$g\left(\nabla^{\perp}_X FX,FZ\right) = g\left(\left(\overline{\nabla}_X F\right)X,FZ\right) + g(F\nabla_X X,FZ). \tag{3.11}$$

Using (2.20), (2.5), and the fact that $\xi$ is killing vector field, we obtain

$$g\left(\nabla^{\perp}_X FX,FZ\right) = g\left(\left(\overline{\nabla}_X F\right)X,FZ\right) - \sin^2\theta g(X,\nabla_X Z). \tag{3.12}$$

Then from (3.2), we derive

$$g\left(\nabla^{\perp}_X FX,FZ\right) = g\left(\left(\overline{\nabla}_X F\right)X,FZ\right) - (Z\ln f)\sin^2\theta\|X\|^2. \tag{3.13}$$

Now, from (3.8) and (3.13), we obtain

$$g\left(\left(\overline{\nabla}_X F\right)X,FZ\right) = -(Z\ln f)\cos^2\theta\|X\|^2 - g(h(X,PX),FZ). \tag{3.14}$$

Interchanging $X$ by $PX$ in (3.14) and then using (2.18), (2.19), and the fact that $\xi \in TN_{\perp}$, we get

$$g\left(\left(\overline{\nabla}_{PX} F\right)PX,FZ\right) = -(Z\ln f)\cos^4\theta\|X\|^2 - \cos^2\theta g(h(X,PX),FZ). \tag{3.15}$$

From (3.14) and (3.15), we arrive at

$$g\left(\left(\overline{\nabla}_X F\right)X,FZ\right) = \sec^2\theta g\left(\left(\overline{\nabla}_{PX} F\right)PX,FZ\right). \tag{3.16}$$

Hence, the result is proved. □

**Lemma 3.4.** *Let $M = N_\perp \times_f N_\theta$ be a warped product pseudo-slant submanifold of a nearly cosymplectic manifold $\overline{M}$. Then,*

$$g(\wp_X PX, Z) = g(h(X,Z), FPX) - g(h(PX,Z), FX) \tag{3.17}$$

*for any $X \in TN_\theta$ and $Z \in TN_\perp$, where $N_\theta$ and $N_\perp$ are proper slant and anti-invariant submanifolds of $\overline{M}$, respectively.*

*Proof.* For any $X \in TN_\theta$ and $Z \in TN_\perp$ by (2.5), we have

$$g(h(PX,Z), FX) = g\left(\overline{\nabla}_Z PX, FX\right) = -g\left(PX, \overline{\nabla}_Z FX\right). \tag{3.18}$$

Then from (2.8), we derive

$$g(h(PX,Z), FX) = g\left(PX, \overline{\nabla}_Z PX\right) - g\left(PX, \overline{\nabla}_Z \phi X\right). \tag{3.19}$$

From the covariant derivative property of $\phi$ and (2.5), we obtain

$$g(h(PX,Z), FX) = g(PX, \nabla_Z PX) - g\left(PX, \left(\overline{\nabla}_Z \phi\right) X\right) - g\left(PX, \phi \overline{\nabla}_Z X\right). \tag{3.20}$$

By (2.2), (2.10), and (3.2), we derive

$$g(h(PX,Z), FX) = (Z \ln f) g(PX, PX) - g(PX, \wp_Z X) + g\left(\phi PX, \overline{\nabla}_Z X\right). \tag{3.21}$$

Using (2.5), (2.8), (2.17)(a), (2.19) and the fact that $\xi \in TN_\perp$, we get

$$\begin{aligned} g(h(PX,Z), FX) = {} & (Z \ln f) \cos^2\theta \|X\|^2 + g(PX, \wp_X Z) \\ & + g\left(P^2 X, \nabla_Z X\right) + g(h(X,Z), FPX). \end{aligned} \tag{3.22}$$

Thus, by property $(p_3)$(i), (2.18), and (3.2) and the fact that $\xi \in TN_\perp$, we obtain

$$\begin{aligned} g(h(PX,Z), FX) = {} & (Z \ln f) \cos^2\theta \|X\|^2 - g(\wp_X PX, Z) \\ & - (Z \ln f) \cos^2\theta \|X\|^2 + g(h(X,Z), FPX). \end{aligned} \tag{3.23}$$

Hence, the above equation takes the form

$$g(\wp_X PX, Z) = g(h(X,Z), FPX) - g(h(PX,Z), FX), \tag{3.24}$$

which proves our assertion. □

**Theorem 3.5.** *Let $M = N_\perp \times_f N_\theta$ be a warped product submanifold of a nearly cosymplectic manifold $\overline{M}$. Then $M$ is Riemannian product of $N_\perp$ and $N_\theta$ if and only if $\mathcal{P}_X TX \in TN_\theta$, for any $X \in TN_\theta$, where $N_\theta$ and $N_\perp$ are proper slant and anti-invariant submanifolds of $\overline{M}$, respectively.*

*Proof.* If the structure vector field $\xi \in TN_\theta$, then, by Theorem 3.1, $M$ is Riemannian product of $N_\perp$ and $N_\theta$. Now, we consider $\xi \in TN_\perp$, then for any $X \in TN_\theta$ and $Z \in TN_\perp$ from (2.5), we have

$$g(h(X,PX),FZ) = g\left(\overline{\nabla}_{PX}X, \phi Z\right). \tag{3.25}$$

Then by (2.2), we get

$$g(h(X,PX),FZ) = -g\left(\phi\overline{\nabla}_{PX}X, Z\right). \tag{3.26}$$

Using the covariant derivative formula of $\phi$, we derive

$$g(h(X,PX),FZ) = g\left(\left(\overline{\nabla}_{PX}\phi\right)X, Z\right) - g\left(\overline{\nabla}_{PX}\phi X, Z\right). \tag{3.27}$$

Then from (2.10) and the property of $\overline{\nabla}$, we obtain

$$g(h(X,PX),FZ) = g(\mathcal{P}_{PX}X, Z) + g\left(\phi X, \overline{\nabla}_{PX}Z\right). \tag{3.28}$$

Thus by (2.5), (2.8), and (2.17)(a), we arrive at

$$g(h(X,PX),FZ) = -g(\mathcal{P}_X PX, Z) + g(PX, \nabla_{PX}Z) + g(h(PX,Z),FX). \tag{3.29}$$

Using (3.2) and then (2.19) and the fact that $\xi \in TN_\perp$, we get

$$\begin{aligned} g(h(X,PX),FZ) = &-g(\mathcal{P}_X PX, Z) + (Z\ln f)\cos^2\theta\|X\|^2 \\ &+ g(h(PX,Z),FX). \end{aligned} \tag{3.30}$$

By property $(p_3)(i)$, we derive

$$\begin{aligned} g(h(X,PX),FZ) = &\ g(PX, \mathcal{P}_X Z) + (Z\ln f)\cos^2\theta\|X\|^2 \\ &+ g(h(PX,Z),FX). \end{aligned} \tag{3.31}$$

Interchanging $X$ by $PX$ in (3.30) and then using (2.18), (2.19), and the fact that $\xi \in TN_\perp$, we obtain

$$\begin{aligned} -\cos^2\theta g(h(X,PX),FZ) = &-\cos^2\theta g(X, \mathcal{P}_{PX}Z) + (Z\ln f)\cos^4\theta\|X\|^2 \\ &-\cos^2\theta g(h(X,Z),FPX). \end{aligned} \tag{3.32}$$

Using the property $(p_3)$(i) and then (2.17)(a), we arrive at

$$-g(h(X,PX),FZ) = -g(\mathcal{P}_X PX,Z) + (Z\ln f)\cos^2\theta\|X\|^2 - g(h(X,Z),FPX). \tag{3.33}$$

Then from (3.30) and (3.33), we obtain

$$2(Z\ \ln f)\cos^2\theta\|X\|^2 = 2g(\mathcal{P}_X PX,Z) + g(h(X,Z),FPX) - g(h(PX,Z),FX). \tag{3.34}$$

Thus, by Lemma 3.4, we conclude that

$$(Z\ \ln f)\cos^2\theta\|X\|^2 = \frac{3}{2}g(\mathcal{P}_X PX,Z). \tag{3.35}$$

Since $N_\theta$ is proper slant, thus we get $(Z\ln f) = 0$, if and only if $\mathcal{P}_X PX$ lies in $TN_\theta$ for all $X \in TN_\theta$ and $Z \in TN_\perp$. This proves the theorem completely. □

Now, we discuss the other case, that is, the warped product submanifold $M = N_\theta \times_f N_\perp$ of a nearly cosymplectic manifold $\overline{M}$. In this case also, if the structure vector filed $\xi \in TN_\perp$ then the warping function $f$ is constant (by Theorem 3.1), thus we consider $\xi \in TN_\theta$.

**Proposition 3.6.** *Let $M = N_\theta \times_f N_\perp$ be a warped product pseudo-slant submanifold of a nearly cosymplectic manifold $\overline{M}$. Then,*

$$g\left(\left(\overline{\nabla}_X F\right)Z,FX\right)+g\left(\left(\overline{\nabla}_{PX} F\right)Z,FPX\right) = \sin^2\theta g(h(X,PX),FZ) + \left(1+\cos^2\theta\right)g(\mathcal{P}_X Z,PX)-\cos^2\theta\eta(X)g(\mathcal{P}_\xi Z,PX) - g(Q_Z X,FX) - g(Q_Z PX,FPX) \tag{3.36}$$

*for any $X \in TN_\theta$ and $Z \in TN_\perp$, where $N_\theta$ and $N_\perp$ are proper slant and anti-invariant submanifolds of $\overline{M}$, respectively.*

*Proof.* For any $X \in TN_\theta$ and $Z \in TN_\perp$, by (2.2) we have

$$g\left(\phi\overline{\nabla}_X Z,\phi X\right) = g\left(\overline{\nabla}_X Z,X\right) - \eta(X)g\left(\overline{\nabla}_X Z,\xi\right). \tag{3.37}$$

Using the property of the connection $\overline{\nabla}$ and the fact that $\xi$ is a Killing vector field, then, from (2.5), we obtain

$$g\left(\phi\overline{\nabla}_X Z,\phi X\right) = g(\nabla_X Z,X). \tag{3.38}$$

Thus by (3.2) and the covariant derivative formula of $\phi$, we derive

$$g\left(\overline{\nabla}_X \phi Z, \phi X\right) - g\left(\left(\overline{\nabla}_X \phi\right) Z, \phi X\right) = (X \ln f) g(Z, X). \tag{3.39}$$

Then form (2.6), (2.8), (2.10), and by the orthogonality of two distributions, we get

$$-g(A_{FZ} X, PX) + g\left(\nabla^{\perp}_X FZ, FX\right) - g(\mathcal{P}_X Z, PX) - g(\mathcal{Q}_X Z, FX) = 0. \tag{3.40}$$

Thus, on using (2.7) and (2.17)(b), the above equation takes the form

$$g\left(\nabla^{\perp}_X FZ, FX\right) = g(h(X, PX), FZ) + g(\mathcal{P}_X Z, PX) - g(\mathcal{Q}_Z X, FX). \tag{3.41}$$

Now, for any $X \in TN_\theta$ and $Z \in TN_\perp$ from (2.14), we have

$$g\left(\nabla^{\perp}_X FZ, FX\right) = g\left(\left(\overline{\nabla}_X F\right) Z, FX\right) + g(F \nabla_X Z, FX). \tag{3.42}$$

Using (3.2), we obtain

$$g\left(\nabla^{\perp}_X FZ, FX\right) = g\left(\left(\overline{\nabla}_X F\right) Z, FX\right) + (X \ln f) g(FZ, FX). \tag{3.43}$$

By orthogonality of two normal distributions, we get

$$g\left(\nabla^{\perp}_X FZ, FX\right) = g\left(\left(\overline{\nabla}_X F\right) Z, FX\right). \tag{3.44}$$

Then, from (3.41) and (3.44), we obtain

$$g\left(\left(\overline{\nabla}_X F\right) Z, FX\right) = g(h(X, PX), FZ) + g(\mathcal{P}_X Z, PX) - g(\mathcal{Q}_Z X, FX). \tag{3.45}$$

Interchanging $X$ by $PX$ in (3.45) and using (2.18) and the fact that $h(X, \xi) = 0$, for any $X$ on a nearly cosymplectic manifold $\overline{M}$, hence we get

$$\begin{aligned} g\left(\left(\overline{\nabla}_{PX} F\right) Z, FPX\right) = &-\cos^2\theta g(h(X, PX), FZ) - \cos^2\theta g(\mathcal{P}_{PX} Z, X) \\ &+ \cos^2\theta \eta(X) g(\mathcal{P}_{PX} Z, \xi) - g(\mathcal{Q}_Z PX, FPX). \end{aligned} \tag{3.46}$$

Using property $(p_3)$(i) and (2.17), we derive

$$\begin{aligned} g\left(\left(\overline{\nabla}_{PX} F\right) Z, FPX\right) = &-\cos^2\theta g(h(X, PX), FZ) - \cos^2\theta g(\mathcal{P}_X PX, Z) \\ &+ \cos^2\theta \eta(X) g(\mathcal{P}_\xi PX, Z) - g(\mathcal{Q}_Z PX, FPX). \end{aligned} \tag{3.47}$$

Again, by property $(p_3)$(i), we obtain

$$g\left(\left(\overline{\nabla}_{PX}F\right)Z, FPX\right) = -\cos^2\theta g(h(X, PX), FZ) + \cos^2\theta g(\mathcal{P}_X Z, PX) - \cos^2\theta\eta(X)g(\mathcal{P}_\xi Z, PX) - g(\mathcal{Q}_Z PX, FPX). \tag{3.48}$$

Thus, the result follows from (3.45) and (3.48). □

**Theorem 3.7.** *Let $M = N_\theta \times_f N_\perp$ be a warped product submanifold of a nearly cosymplectic manifold $\overline{M}$. Then $M$ is Riemannian product of $N_\theta$ and $N_\perp$ if and only if*

$$g(h(X, Z), FZ) = g(h(Z, Z), FX), \tag{3.49}$$

*for any $X \in TN_\theta$ and $Z \in TN_\perp$, where $N_\theta$ and $N_\perp$ are proper slant and anti-invariant submanifolds of $\overline{M}$, respectively.*

*Proof.* If $\xi \in TN_\perp$, then by Theorem 3.1, $f$ is constant on $M$. Now, we consider $\xi \in TN_\theta$. In this case, for any $X \in TN_\theta$ and $Z \in TN_\perp$ by (2.5), we have

$$g(h(PX, Z), FZ) = g\left(\overline{\nabla}_Z PX, \phi Z\right). \tag{3.50}$$

Using (2.2), we get

$$g(h(PX, Z), FZ) = -g\left(\phi\overline{\nabla}_Z PX, Z\right). \tag{3.51}$$

Thus, on using the covariant derivative property of $\phi$, we obtain

$$g(h(PX, Z), FZ) = g\left(\left(\overline{\nabla}_Z\phi\right)PX, Z\right) - g\left(\overline{\nabla}_Z\phi PX, Z\right). \tag{3.52}$$

Then from (2.8) and (2.10), we get

$$g(h(PX, Z), FZ) = g(\mathcal{P}_Z PX, Z) - g\left(\overline{\nabla}_Z P^2X, Z\right) - g\left(\overline{\nabla}_Z FPX, Z\right). \tag{3.53}$$

Using property $(p_3)$(i) and the property of the connection $\overline{\nabla}$, we derive

$$g(h(PX, Z), FZ) = -g(\mathcal{P}_Z Z, PX) + g\left(P^2X, \nabla_Z Z\right) + g\left(FPX\overline{\nabla}_Z Z\right). \tag{3.54}$$

As we have $\mathcal{P}_Z Z = 0$ from (2.4) and (2.10), then by (2.18) the above equation reduced to

$$g(h(PX, Z), FZ) = -\cos^2\theta g\left(X, \overline{\nabla}_Z Z\right) + \cos^2\theta\eta(X)g\left(\xi, \overline{\nabla}_Z Z\right) + g(h(Z, Z), FPX). \tag{3.55}$$

Since $\xi$ is a Killing vector field on $\overline{M}$, then by (2.5), (3.2), and the property of the connection $\overline{\nabla}$, the above equation takes the form

$$g(h(PX, Z), FZ) = (X \ln f)\cos^2\theta\|Z\|^2 + g(h(Z, Z), FPX). \tag{3.56}$$

Interchanging $X$ by $PX$ in (3.56) and using (2.18), we obtain

$$\begin{aligned}\cos^2\theta g(h(X, Z), FZ) + \cos^2\theta\eta(X)g(h(Z, \xi), FZ) \\ = -(PX \ln f)\cos^2\theta\|Z\|^2 + \cos^2\theta g(h(Z, Z), FX).\end{aligned} \tag{3.57}$$

Since $h(Z, \xi) = 0$, for nearly cosymplectic, then the above equation reduces to

$$(PX \ln f)\|Z\|^2 = g(h(Z, Z), FX) - g(h(X, Z), FZ). \tag{3.58}$$

Thus, from (3.58), we obtain $(PX \ln f) = 0$ if and only if $g(h(Z, Z), FX) = g(h(X, Z), FZ)$. This proves the theorem completely. □

## References


[1] R. L. Bishop and B. O'Neill, "Manifolds of negative curvature," *Transactions of the American Mathematical Society*, vol. 145, pp. 1–49, 1969.

[2] B.-Y. Chen, "Geometry of warped product CR-submanifolds in Kaehler manifolds," *Monatshefte für Mathematik*, vol. 133, no. 3, pp. 177–195, 2001.

[3] I. Hasegawa and I. Mihai, "Contact CR-warped product submanifolds in Sasakian manifolds," *Geometriae Dedicata*, vol. 102, pp. 143–150, 2003.

[4] S. Uddin and A. Y. M. Chi, "Warped product pseudo-slant submanifolds of nearly Kaehler manifolds," *Mathematical Journal of the Ovidius University of Constantza*, vol. 19, no. 3, pp. 195–204, 2011.

[5] A. Carriazo, *New Developments in Slant Submanifolds Theory*, Narosa Publishing House, New Delhi, India, 2002.

[6] D. E. Blair, "Almost contact manifolds with Killing structure tensors," *Pacific Journal of Mathematics*, vol. 39, pp. 285–292, 1971.

[7] D. E. Blair and K. Yano, "Affine almost contact manifolds and $f$-manifolds with affine Killing structure tensors," *Kōdai Mathematical Seminar Reports*, vol. 23, pp. 473–479, 1971.

[8] D. E. Blair and D. K. Showers, "Almost contact manifolds with Killing structure tensors. II," *Journal of Differential Geometry*, vol. 9, pp. 577–582, 1974.

[9] B. Sahin, "Warped product submanifolds of Kaehler manifolds with a slant factor," *Annales Polonici Mathematici*, vol. 95, no. 3, pp. 207–226, 2009.

[10] S. Uddin, S. H. Kon, M. A. Khan, and K. Singh, "Warped product semi-invariant submanifolds of nearly cosymplectic manifolds," *Mathematical Problems in Engineering*, vol. 2011, Article ID 230374, 12 pages, 2011.

[11] J. L. Cabrerizo, A. Carriazo, L. M. Fernández, and M. Fernández, "Slant submanifolds in Sasakian manifolds," *Glasgow Mathematical Journal*, vol. 42, no. 1, pp. 125–138, 2000.