Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2012, Article ID 405978, 10 pages
doi:10.1155/2012/405978

Research Article

Lie Group Classification of a Generalized Lane-Emden Type System in Two Dimensions

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Received 18 September 2012; Accepted 13 October 2012

Academic Editor: Fazal M. Mahomed

The aim of this work is to perform a complete Lie symmetry classification of a generalized Lane-Emden type system in two dimensions which models many physical phenomena in biological and physical sciences. The classical approach of group classification is employed for classification. We show that several cases arise in classifying the arbitrary parameters, the forms of which include amongst others the power law nonlinearity, and exponential and quadratic forms.

1. Introduction

For many years extensive studies using various approaches have been done on the Lane-Emden type equation

$$\Delta y + F(y) = 0, \tag{1.1}$$

which has applications in astrophysics. Recently, the Lane-Emden type systems have attracted a lot of attention in modelling physical phenomena in biological and physical sciences [1, 2]. In many of these investigations the nonlinearity in the system is often assumed. However, the symmetry-based approach provides a systematic way to specify the nonlinearities in the models of physical importance and mathematical interest.

The bidimensional Lane-Emden system [3]:

$$
\begin{aligned}
u_{xx} + u_{yy} + v^{q} &= 0, \\
v_{xx} + v_{yy} + u^{p} &= 0,
\end{aligned}
\tag{1.2}
$$

is studied from the view point of both Lie and Nöether point symmetry classification where $p, q \notin \{0, 1\}$ are arbitrary constants. In particular, the Lie point symmetry classification is obtained for the cases $p \neq q$ $(p = 1/q)$ and $p = q$ where $q \neq 0, \pm 1$ including the special case $p = q = -1$.

In the current study we generalize the last system by considering the bidimensional Lane-Emden system of the form

$$
\begin{aligned}
u_{xx} + u_{yy} + f(v) &= 0, \\
v_{xx} + v_{yy} + g(u) &= 0,
\end{aligned}
\tag{1.3}
$$

where $f(v)$ and $g(u)$ are nonzero arbitrary functions of their respective arguments. The underlying system (1.3) is a two-dimensional Euler-Lagrange model system in elastostatics [4].

Recently, in [5] Nöether point symmetry classification of system (1.3) is performed and various forms of the arbitrary functions are obtained which include linear, power, exponential, and logarithmic types.

The plan of this work is organized as follows. In Section 2 we generate the classifying relations (determining equations for the arbitrary elements). The computation of the equivalence transformations is presented in Section 3. In Section 4 the Lie group classification of the underlying system is performed. Finally, we summarize our investigations in Section 5.

2. Generator of Symmetry Group and Classifying Relations

According to the Lie algorithm we seek the generator of Lie point symmetries for system (1.3) of the form

$$
X = \xi^{1}(x, y, u, v)\partial_x + \xi^{2}(x, y, u, v)\partial_y + \eta^{1}(x, y, u, v)\partial_u + \eta^{2}(x, y, u, v)\partial_v. \tag{2.1}
$$

The application of the second prolongation of (2.1) on the underlying system yields the determining equations which are solved for ξ^1, ξ^2, η^1, and η^2, see for details [4, 6, 7]. The manual generation and manipulation of determining equations is a tiring task. Fortunately, nowadays the Lie algorithm has been implemented using the computer software packages for symbolic computation such as the `YaLie` package [8] which is used in this work. Therefore, with the help of the `YaLie` package written in Mathematica the following determining equations are generated:

$$\begin{gathered}
\xi^1_u = \xi^1_v = 0, \qquad \xi^2_u = \xi^2_v = 0, \qquad \eta^1_{uu} = \eta^1_{vv} = 0, \qquad \eta^2_{uu} = \eta^2_{vv} = 0, \\
\eta^1_{uv} = \eta^1_{yv} = \eta^1_{xv} = 0, \qquad \eta^2_{uv} = \eta^2_{yu} = \eta^2_{xu} = 0, \\
\xi^2_y - \xi^1_x = 0, \qquad \xi^1_y + \xi^2_x = 0, \\
\xi^1_{xx} + \xi^1_{yy} - 2\eta^1_{xu} = 0, \qquad 2\eta^1_{yu} - \xi^2_{yy} - \xi^2_{xx} = 0, \\
\xi^1_{xx} + \xi^1_{yy} - 2\eta^2_{xv} = 0, \qquad 2\eta^2_{yv} - \xi^2_{yy} - \xi^2_{xx} = 0, \\
\eta^2 f'(v) - \eta^1_v g(u) + \left(2\xi^1_x - \eta^1_u\right) f(v) + \eta^1_{xx} + \eta^1_{yy} = 0, \\
\eta^1 g'(u) + \left(2\xi^2_y - \eta^2_v\right) g(u) - \eta^2_u f(v) + \eta^2_{xx} + \eta^2_{yy} = 0,
\end{gathered} \tag{2.2}$$

where subscripts denote partial differentiation with respect to the indicated variables and "prime" indicates total derivative with respect to the given argument.

The manipulation of (2.2) yields the general generator of symmetry group for system (1.3) in the form

$$X = a(x,y)\partial_x + b(x,y)\partial_y + [C_1 u + c(x,y)]\partial_u + [C_2 v + d(x,y)]\partial_v, \tag{2.3}$$

where C_1, C_2 are arbitrary constants and a, b, c, d are the arbitrary smooth functions which satisfy

$$a_x - b_y = 0, \qquad a_y + b_x = 0, \qquad a_{xx} + a_{yy} = 0, \qquad b_{xx} + b_{yy} = 0, \tag{2.4}$$

$$(2a_x - C_1) f(v) + (C_2 v + d) f'(v) + c_{xx} + c_{yy} = 0, \tag{2.5}$$

$$\left(2b_y - C_2\right) g(u) + (C_1 u + c) g'(u) + d_{xx} + d_{yy} = 0. \tag{2.6}$$

The determining equations (2.5) and (2.6) are known as the classifying relations/equations.

Since the variables x and y do not appear explicitly in the underlying system (1.3), the principal symmetry Lie algebra admitted by this system is spanned by at least two operators, namely, $X_1 = \partial_x$ and $X_2 = \partial_y$ (to be established in Section 3).

3. Equivalence Transformations

Following the infinitesimal approach [9] we consider the generator of equivalence group of the form

$$Y = \xi^1\partial_x + \xi^2\partial_y + \eta^1\partial_u + \eta^2\partial_v + \mu^1\partial_f + \mu^2\partial_g, \tag{3.1}$$

where $\xi^j = \xi^j(x,y,u,v)$, $\eta^j = \eta^j(x,y,u,v)$, and $\mu^j = \mu^j(x,y,u,v,f,g)$ for $j = 1,2$.

The operator (3.1) is the generator of equivalence group for system (1.3) provided it is admitted by the extended system

$$\begin{gathered} u_{xx} + u_{yy} + f = 0, \qquad v_{xx} + v_{yy} + g = 0, \\ f_x = f_y = f_u = 0, \qquad g_x = g_y = g_v = 0. \end{gathered} \tag{3.2}$$

We require the prolonged operator for the extended system (3.2) having the form

$$\widetilde{Y} = Y^{[2]} + \omega_1^1 \partial_{f_x} + \omega_2^1 \partial_{f_y} + \omega_3^1 \partial_{f_u} + \omega_4^1 \partial_{f_v} + \omega_1^2 \partial_{g_x} + \omega_2^2 \partial_{g_y} + \omega_3^2 \partial_{g_u} + \omega_4^2 \partial_{g_v}, \tag{3.3}$$

where $Y^{[2]}$ is the second prolongation of (3.1) given by

$$\begin{aligned} Y^{[2]} = {} & \xi^1 \partial_x + \xi^2 \partial_y + \eta^1 \partial_u + \eta^2 \partial_v + \mu^1 \partial_f + \mu^2 \partial_g \\ & + \zeta_{11}^1 \partial_{u_{xx}} + \zeta_{11}^2 \partial_{v_{xx}} + \zeta_{22}^1 \partial_{u_{yy}} + \zeta_{22}^2 \partial_{v_{yy}}. \end{aligned} \tag{3.4}$$

The variables ζ_j^i and ω_k^i are given by the prolongation formulae

$$\begin{aligned}
\zeta_{xx}^1 &= D_x\left(D_x\left(\eta^1\right) - u_x D_x\left(\xi^1\right) - u_y D_x\left(\xi^2\right)\right) - u_{xx} D_x\left(\xi^1\right) - u_{xy} D_x\left(\xi^2\right), \\
\zeta_{yy}^1 &= D_y\left(D_y\left(\eta^1\right) - u_x D_y\left(\xi^1\right) - u_y D_y\left(\xi^2\right)\right) - u_{xy} D_y\left(\xi^1\right) - u_{yy} D_y\left(\xi^2\right), \\
\zeta_{xx}^2 &= D_x\left(D_x\left(\eta^2\right) - u_x D_x\left(\xi^1\right) - u_y D_x\left(\xi^2\right)\right) - u_{xx} D_x\left(\xi^1\right) - u_{xy} D_x\left(\xi^2\right), \\
\zeta_{yy}^2 &= D_y\left(D_y\left(\eta^2\right) - u_x D_y\left(\xi^1\right) - u_y D_y\left(\xi^2\right)\right) - u_{xy} D_y\left(\xi^1\right) - u_{yy} D_y\left(\xi^2\right), \\
\omega_1^1 &= \widetilde{D}_x\left(\mu^1\right) - f_x \widetilde{D}_x\left(\xi^1\right) - f_y \widetilde{D}_x\left(\xi^2\right) - f_u \widetilde{D}_x\left(\eta^1\right) - f_v \widetilde{D}_x\left(\eta^2\right), \\
\omega_2^1 &= \widetilde{D}_y\left(\mu^1\right) - f_x \widetilde{D}_y\left(\xi^1\right) - f_y \widetilde{D}_y\left(\xi^2\right) - f_u \widetilde{D}_y\left(\eta^1\right) - f_v \widetilde{D}_y\left(\eta^2\right), \\
\omega_3^1 &= \widetilde{D}_u\left(\mu^1\right) - f_x \widetilde{D}_u\left(\xi^1\right) - f_y \widetilde{D}_u\left(\xi^2\right) - f_u \widetilde{D}_u\left(\eta^1\right) - f_v \widetilde{D}_u\left(\eta^2\right), \\
\omega_1^2 &= \widetilde{D}_x\left(\mu^2\right) - g_x \widetilde{D}_x\left(\xi^1\right) - g_y \widetilde{D}_x\left(\xi^2\right) - g_u \widetilde{D}_x\left(\eta^1\right) - g_v \widetilde{D}_x\left(\eta^2\right), \\
\omega_2^2 &= \widetilde{D}_y\left(\mu^2\right) - g_x \widetilde{D}_y\left(\xi^1\right) - g_y \widetilde{D}_y\left(\xi^2\right) - g_u \widetilde{D}_y\left(\eta^1\right) - g_v \widetilde{D}_y\left(\eta^2\right), \\
\omega_4^2 &= \widetilde{D}_v\left(\mu^2\right) - g_x \widetilde{D}_u\left(\xi^1\right) - g_y \widetilde{D}_u\left(\xi^2\right) - g_u \widetilde{D}_u\left(\eta^1\right) - g_v \widetilde{D}_u\left(\eta^2\right),
\end{aligned} \tag{3.5}$$

respectively, where

$$D_x = \partial_x + u_x \partial_u + v_x \partial_v + \cdots, \qquad D_y = \partial_y + u_y \partial_u + v_y \partial_v + \cdots \tag{3.6}$$

are the total derivative operators and the total derivative operators for the extended system are given by

$$\begin{aligned}
\widetilde{D}_x &= \partial_x + f_x\partial_f + g_x\partial_g + \cdots, \\
\widetilde{D}_y &= \partial_y + f_y\partial_f + g_y\partial_g + \cdots, \\
\widetilde{D}_u &= \partial_u + f_u\partial_f + g_u\partial_g + \cdots, \\
\widetilde{D}_v &= \partial_v + f_v\partial_f + g_v\partial_g + \cdots.
\end{aligned} \tag{3.7}$$

Upon application of the prolongation (3.3), the invariance conditions of system (3.2) read

$$\begin{gathered}
\mu^1 + \zeta^1_{11} + \zeta^1_{22} = 0, \qquad \mu^2 + \zeta^2_{11} + \zeta^2_{22} = 0, \\
\omega^1_1 = \omega^1_2 = \omega^1_3 = 0, \qquad \omega^2_1 = \omega^2_2 = \omega^2_4 = 0.
\end{gathered} \tag{3.8}$$

The solution of system (3.8) is given by

$$\begin{gathered}
\xi^1 = c_5x + c_6y + c_7, \qquad \xi^2 = c_5y - c_6x + c_8, \qquad \eta^1 = c_1u + c_2, \\
\eta^2 = c_3v + c_4, \qquad \mu^1 = (c_1 - 2c_5)f, \qquad \mu^2 = (c_3 - 2c_5)g,
\end{gathered} \tag{3.9}$$

where $c_1, \ldots, c_8$ are arbitrary constants.

Therefore, system (1.3) has 8-dimensional equivalence Lie algebra spanned by the operators

$$\begin{gathered}
Y_1 = \partial_x, \qquad Y_2 = \partial_y, \qquad Y_3 = y\partial_x - x\partial_y, \qquad Y_4 = \partial_u, \qquad Y_5 = \partial_v, \\
Y_6 = x\partial_x + y\partial_y - 2f\partial_f - 2g\partial_g, \qquad Y_7 = u\partial_u + f\partial_f, \qquad Y_8 = v\partial_v + g\partial_g.
\end{gathered} \tag{3.10}$$

The composition of the one-parameter group of transformations for each Y_i yields the equivalence group for system (1.3) given by the transformations

$$\begin{gathered}
\overline{x} = \alpha_1 x + \alpha_2 y + \beta_1, \qquad \overline{y} = -\alpha_2 x + \alpha_1 y + \beta_2, \\
\overline{u} = \alpha_3 u + \beta_3, \qquad \overline{v} = \alpha_4 v + \beta_4, \qquad \overline{f} = \alpha_5 f, \qquad \overline{g} = \alpha_6 g,
\end{gathered} \tag{3.11}$$

where $\alpha_1, \ldots, \alpha_6 \neq 0$ and $\beta_1, \ldots, \beta_4$ are arbitrary constants.

Next we use the theorem on projections of equivalence Lie algebras [9] to establish the principal Lie algebra for system (1.3). The projections of the equivalence generator (3.1) are given by

$$X = \mathrm{pr}_{(x,y,u,v)}(Y) \equiv \xi^1\partial_x + \xi^2\partial_y + \eta^1\partial_u + \eta^2\partial_v, \tag{3.12}$$

$$Z = \mathrm{pr}_{(u,v,f,g)}(Y) \equiv \eta^1\partial_u + \eta^2\partial_v + \mu^1\partial_f + \mu^2\partial_g, \tag{3.13}$$

where $\mathrm{pr}_{(x,y,u,v)}$ denotes projection onto the space (x, y, u, v) and $\mathrm{pr}_{(u,v,f,g)}$ onto the space (u, v, f, g).

An operator X spans the principal Lie algebra provided the following condition holds:

$$Z = \mathrm{pr}_{(u,v,f,g)}(Y) = 0. \tag{3.14}$$

In view of (3.1) taking into account (3.9) and (3.13), equation (3.14) is recast as

$$(c_1 u + c_2)\partial_u + (c_3 v + c_4)\partial_v + (c_1 - 2c_5) f \partial_f + (c_3 - 2c_5) g \partial_g = 0. \tag{3.15}$$

Thus, we have

$$c_1 = 0, \qquad c_2 = 0, \qquad c_3 = 0, \qquad c_4 = 0, \qquad c_5 = 0. \tag{3.16}$$

Now the generator of equivalence group (3.1) reduces to

$$Y = (c_6 y + c_7)\partial_x + (c_8 - c_6 x)\partial_y, \tag{3.17}$$

and therefore the principal Lie algebra for system (1.3) is three-dimensional and it is spanned by the operators

$$X_1 = \partial_x, \qquad X_2 = \partial_y, \qquad X_3 = y\partial_x - x\partial_y. \tag{3.18}$$

Remark 3.1. The principal Lie algebra (3.18) can be achieved alternatively by solving the resulting equations obtained from splitting the determining equations (2.5) and (2.6) with respect to the arbitrary elements and their derivatives.

Our goal in Section 4 is to extend the principal Lie algebra, that is, we obtain the functional forms of the arbitrary elements $f(v)$ and $g(u)$ which provide additional operator(s).

4. Group Classification

Case 1. Following the classical approach of group classification [7], the classifying relations (2.5) and (2.6) become

$$(h + kv) f'(v) + \ell f(v) + m = 0, \tag{4.1}$$

$$(n + pu) g'(u) + q g(u) + r = 0, \tag{4.2}$$

where h, k, ℓ, m, n, p, q, and r are arbitrary constants.

It is noted that the analysis of (4.1) and (4.2) is similar.

Upon the use of equivalence transformations (3.11), the classifying equations (4.1) and (4.2) take the form

$$\left(h+\frac{\alpha_5}{\alpha_4}\beta_4 k+\alpha_5 k v\right)f'(v)+\alpha_5\ell f(v)+m=0,$$
$$\left(n+\frac{\alpha_6}{\alpha_3}\beta_3 p+\alpha_6 p u\right)g'(u)+\alpha_6 q g(u)+r=0, \tag{4.3}$$

provided

$$\overline{h}=h+\frac{\alpha_5}{\alpha_4}\beta_4 k,\qquad \overline{k}=\alpha_5 k,\qquad \overline{\ell}=\alpha_5\ell,\qquad \overline{m}=m;$$
$$\overline{n}=n+\frac{\alpha_6}{\alpha_3}\beta_3 p,\qquad \overline{p}=\alpha_6 p,\qquad \overline{q}=\alpha_6 q,\qquad \overline{r}=r. \tag{4.4}$$

As an illustration in the analysis of (4.3) we consider the case $k\neq 0$ $(\overline{h}=0)$ and $p\neq 0$ $(\overline{n}=0)$. Thus, we obtain

$$vf'(v)+\alpha f(v)+\beta=0;\quad \alpha,\beta=0,\pm1,$$
$$ug'(u)+\gamma g(u)+\delta=0;\quad \gamma,\delta=0,\pm1, \tag{4.5}$$

where $\ell=\alpha k$, $m=\alpha_5 k\beta$, $q=\gamma p$, and $r=\alpha_6 p\delta$.

Consider also the case $k=0$ $(\overline{h}\neq 0)$ and $p=0$ $(\overline{n}\neq 0)$, then we have

$$f'(v)+\lambda f(v)+\mu=0;\quad \lambda,\mu=0,\pm1,$$
$$g'(u)+\nu g(u)+\rho=0;\quad \nu,\rho=0,\pm1, \tag{4.6}$$

where $h\lambda=\alpha_5 m$, $m=\mu h$, $n\nu=\alpha_6 q$, and $r=\rho n$.

From (4.5)-(4.6) we obtain the functional forms of the arbitrary parameters $f(v)$ and $g(u)$ together with their corresponding extra operator(s) given by

$$f=f_0 v^{-\alpha},\qquad g=g_0 u^{-\gamma};$$
$$X_4=(\alpha\gamma-1)x\partial_x+(\alpha\gamma-1)y\partial_y+2(\alpha-1)u\partial_u+2(\gamma-1)v\partial_v, \tag{4.7}$$

$$f=\tilde{f}_0 e^{-\lambda v},\qquad g=\tilde{g}_0 e^{-\nu u},\qquad \lambda,\nu\neq 0;$$
$$X_4=\lambda\nu(y-x)\partial_x-\lambda\nu(x+y)\partial_y-2\lambda\partial_u-2\nu\partial_v,$$
$$X_5=\lambda\nu\left(x^2-2xy-y^2\right)\partial_x+\lambda\nu\left(x^2+2xy-y^2\right)\partial_y$$
$$+4\lambda(x-y)\partial_u+4\nu(x-y)\partial_v, \tag{4.8}$$

where f_0, g_0, $\tilde{f}_0$, and $\tilde{g}_0$ are nonzero arbitrary constants.

The cases $k\neq 0$ $(p=0)$ and $k=0$ $(p\neq 0)$ yield the classification results given in Table 1.

Table 1: Classification results: $k \neq 0$ $(p = 0)$ and $k = 0$ $(p \neq 0)$.

f	g	Condition on const.	Extra operator(s)
$\tilde{f}_0 e^{-\lambda v}$	$g_0 u^{-\gamma}$	$\lambda \neq 0$	$X_4 = \lambda\gamma x\partial_x + \lambda\gamma y\partial_y + 2\lambda u\partial_u + 2(\gamma-1)\partial_v$
	g_0	$\lambda \neq 0$	$X_4 = \lambda u\partial_u - \partial_v$, $X_c = c(x,y)\partial_u$
$f_0 v^{-1}$	$\delta \ln u$	$\delta = \pm 1$	$X_4 = x\partial_x + y\partial_y + 2v\partial_v$
	$g_0 u^{-\gamma}$	$\mu = \pm 1$	$X_4 = (1+\gamma)x\partial_x + (1+\gamma)y\partial_y + 4u\partial_u + 2(1-\gamma)v\partial_v$
μv	$\tilde{g}_0 e^{-\nu u}$	$\nu \neq 0$	$X_4 = \nu x\partial_x + \nu y\partial_y + 4u\partial_u - 2\nu v\partial_v$
	ρu	$\rho = \pm 1$	$X_4 = u\partial_u + v\partial_v$, $X_d = d(x,y)\partial_v$
$\beta \ln v$	g_0	$\beta = \pm 1$	$X_4 = x\partial_x + y\partial_y + 2u\partial_u + 2v\partial_v$, $X_c = c(x,y)\partial_u$
$f(v)$	$g_0 u^{-1}$		$X_4 = x\partial_x + y\partial_y + 2u\partial_u$
	g_0		$X_c = c(x,y)\partial_u$

Case 2. Suppose that $f(v)$ and $g(u)$ are nonlinear functions. Differentiation of (2.5) and (2.6) twice with respect to v and u, respectively, leads to

$$\begin{aligned}
&(2a_x - C_1 + 2C_2)f'' + (d + C_2 v)f''' = 0, \\
&(2b_y - 2C_1 + C_2)g'' + (c + C_1 v)g''' = 0.
\end{aligned} \tag{4.9}$$

Thus, the last equations prompt consideration of the following set of cases:

$$f'''(v) = 0, \qquad g'''(u) = 0, \tag{4.10}$$

$$f'''(v) = 0, \qquad g'''(u) \neq 0, \tag{4.11}$$

$$f'''(v) \neq 0, \qquad g'''(u) = 0, \tag{4.12}$$

$$f'''(v) \neq 0, \qquad g'''(u) \neq 0. \tag{4.13}$$

Consider case (4.10) for illustration. We obtain $f = Av^2 + Bv + C$, $g = \overline{A}u^2 + \overline{B}u + \overline{C}$, respectively, where A, $\overline{A}$, B, $\overline{B}$, C and $\overline{C}$ are arbitrary constants of integration. We make use of the equivalence transformations (3.11) in order to have simplified forms of f and g.

Firstly consider f, then

$$\begin{aligned}
\overline{f} &= \alpha_5 f = \alpha_5 A\left(\frac{\overline{v} - \beta_4}{\alpha_4}\right)^2 + \alpha_5 B\left(\frac{\overline{v} - \beta_4}{\alpha_4}\right) + \alpha_5 C \\
&\approx \alpha_5 \overline{v}^2 + \frac{\alpha_5}{\sqrt{A}} B\overline{v} + \alpha_5 C.
\end{aligned} \tag{4.14}$$

Therefore, when we drop the bars and set $\alpha_5 = 1$, $\beta_4 = 0$ the equivalence relation for f is given by $v^2 + \sigma v + \tau$, where $B = \sigma\sqrt{A}$ and $\tau = C$. Likewise, $g = u^2 + \phi u + \omega$ for arbitrary constants ϕ and ω. Substitution of the forms of f and g into (2.4)–(2.6) yields

$$\begin{gathered} C_1 = C_2, \qquad \tilde{a}_y + \tilde{b}_x = 0, \qquad \tilde{a}_{yy} = 0, \qquad \tilde{b}_{xx} = 0, \\ (\sigma + 2v)d - C_2(2\tau + \sigma v) + c_{xx} + c_{yy} = 0, \\ (\phi + 2u)c - C_1(2\omega + \phi u) + d_{xx} + d_{yy} = 0, \end{gathered} \tag{4.15}$$

where

$$\begin{aligned} a(x,y) &= \left(\frac{C_1}{2} - C_2\right)x + \tilde{a}(y), \\ b(x,y) &= \left(\frac{C_2}{2} - C_1\right)y + \tilde{b}(x), \end{aligned} \tag{4.16}$$

according to (4.9)

The solution of (4.15) leads to the four-dimensional symmetry Lie algebra spanned by the generators

$$X_1 = \partial_x, \qquad X_2 = \partial_y, \qquad X_3 = y\partial_x - x\partial_y, \qquad X_4 = x\partial_x + y\partial_y - 2u\partial_u - 2v\partial_v, \tag{4.17}$$

provided f and g are quadratic in v and u, respectively, that is,

$$f = v^2, \qquad g = u^2. \tag{4.18}$$

However, the last result (4.18) is included in (4.7) for $\alpha = \gamma = -2$.

Next, in considering case (4.11), we obtain the classification result that if $f = v^2$ and $g = e^u$, then the principal Lie algebra is extended by the operator

$$X_4 = x\partial_x + y\partial_y - 3\partial_u - v\partial_v. \tag{4.19}$$

The classification results of case (4.11) can be mapped into those of (4.12) by the use of the equivalence transformations $u \mapsto v$ and $f \mapsto g$. The last case (4.13) does not yield the forms of $f(v)$ and $g(u)$ such that the principal Lie algebra is extended.

Note. It should be noted that not included in the preceding classification results are the cases for which the functional forms of the arbitrary elements do not extend the principal Lie algebra, this includes amongst others the case for which both functions are of logarithmic forms. Moreover, the cases which are the same under the equivalence transformations $u \mapsto v$ and $f \mapsto g$ are also excluded. The constant coefficient case is also excluded.

5. Conclusion

In this work we performed the Lie symmetry classification of a generalized bidimensional Lane-Emden type system. The functional forms of the arbitrary parameters were specified via the classical method of group classification, and these include the combination of power law nonlinearity, exponential, logarithmic, quadratic, linear, and constant forms. Many cases yielded four symmetries apart from the five-dimensional symmetry Lie algebra obtained in the case for which both parameters are of exponential forms. The other cases possess infinite dimensional symmetry Lie algebra.

Acknowledgments

M. Molati and C. M. Khalique would like to thank the Organizing Committee of "Symmetries, Differential Equations and Applications": Galois Bicentenary (SDEA2012) Conference for their kind hospitality during the conference. M. Molati also acknowledges the financial support from the North-West University, Mafikeng Campus, through the postdoctoral fellowship.

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