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Research Article

On the Laplacian Coefficients and Laplacian-Like Energy of Unicyclic Graphs with *n* Vertices and *m* Pendent Vertices

Xinying Pai[1, 2] and Sanyang Liu[1]

[1] *Department of Mathematics, Xidian University, Shanxi Xi'an 710071, China*
[2] *College of Science, China University of Petroleum, Shandong Qingdao 266580, China*

Correspondence should be addressed to Xinying Pai, paixinying@upc.edu.cn

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Let $\Phi(G,\lambda) = \det(\lambda I_n - L(G)) = \sum_{k=0}^{n}(-1)^k c_k(G)\lambda^{n-k}$ be the characteristic polynomial of the Laplacian matrix of a graph G of order n. In this paper, we give four transforms on graphs that decrease all Laplacian coefficients $c_k(G)$ and investigate a conjecture A. Ilić and M. Ilić (2009) about the Laplacian coefficients of unicyclic graphs with n vertices and m pendent vertices. Finally, we determine the graph with the smallest Laplacian-like energy among all the unicyclic graphs with n vertices and m pendent vertices.

1. Introduction

Let $G = (V,E)$ be a simple undirected graph with n vertices and $|E|$ edges and, let $L(G) = D(G) - A(G)$ be its Laplacian matrix. The Laplacian polynomial of G is the characteristic polynomial of its Laplacian matrix. That is

$$\Phi(G,\lambda) = \det(\lambda I_n - L(G)) = \sum_{k=0}^{n}(-1)^k c_k(G)\lambda^{n-k}. \quad (1.1)$$

The Laplacian matrix $L(G)$ has nonnegative eigenvalues $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq \mu_n = 0$ [1]. From Viette's formulas,

$$c_k(G) = \sigma_k(\mu_1, \mu_2, \ldots, \mu_{n-1}) = \sum_{I\subseteq\{1,2,\ldots,n-1\},|I|=k} \prod_{i\in I}\mu_i \quad (1.2)$$

is a symmetric polynomial of order $n-1$. In particular, we have $c_0(G) = 1, c_1(G) = 2|E(G)|, c_n(G) = 0$ and $c_{n-1}(G) = n\tau(G)$, where $\tau(G)$ is the number of spanning trees of G. If G is a tree, coefficient $c_{n-2}(G)$ is equal to its Wiener index, which is a sum of distance between all pairs of vertices:

$$c_{n-2}(G) = W(G) = \sum_{u,v\in V} d(u,v). \tag{1.3}$$

The Wiener index is considered as one of the most used topological indices with high correlation with many physical and chemical properties of molecular compounds.

A unicyclic graph is a connected graph in which the number of vertices equals the number of edges. Recently, the study on the Laplacian coefficients attracts much attention.

Mohar [2] proved that among all trees of order n, the kth Laplacian coefficients $c_k(G)$ are largest when the tree is a path and are smallest for stars. Stevanović and Ilić [3] showed that among all connected unicyclic graphs of order n, the kth Laplacian coefficients $c_k(G)$ are largest when the graph is a cycle C_n and smallest when the graph is an S_n with an additional edge between two of its pendent vertices, where S_n is a star of order n. He and Shan [4] proved that among all bicyclic graphs of order n, the kth Laplacian coefficients $c_k(G)$ is smallest when the graph is obtained from C_4 by adding one edge connecting two non-adjacent vertices and adding $n-4$ pendent vertices attached to the vertex of degree 3. A. Ilić and M. Ilić [5] verified that among trees on n vertices and m leaves, the balanced starlike tree $S(n,m)$ (see Definition 2.2) has minimal Laplacian coefficients. Some other works on Laplacian coefficients can be found in [6–8].

In this paper, we determine the smallest kth Laplacian coefficients $c_k(G)$ among all unicyclic graphs with n vertices and m pendent vertices. Thus we completely solve a conjecture on the minimal Laplacian coefficients of unicyclic graphs with n vertices and m pendent vertices (see [5]).

Motivated by the results in [3, 4, 9–12] concerning the minimal Laplacian coefficients and Laplacian-like energy of some graphs and the minimal molecular graph energy of unicyclic graphs with n vertices and m pendent vertices, this paper will characterize the unicyclic graphs with n vertices and m pendent vertices, which minimize Laplacian-like energy.

2. Transformations and Lemmas

In this section, we introduce some graphic transformations and lemmas, which can be used to prove our main results. The Laplacian coefficients $c_k(G)$ of a graph G can be expressed in terms of subtree structures of G by the following result of Kelmans and Chelnokov [13]. Let F be a spanning forest of G with components T_i, $i = 1, 2, \ldots, k$ having n_i vertices each, and let $\gamma(F) = \prod_{i=1}^{k} n_i$.

Lemma 2.1 (see [13]). *The Laplacian coefficient $c_{n-k}(G)$ of a graph G is given by*

$$c_{n-k}(G) = \sum_{F\in\mathcal{F}_k} \gamma(F), \tag{2.1}$$

where $\mathcal{F}_k$ is the set of all spanning forests of G with exactly k components.

For a real number x, we use $\lfloor x \rfloor$ to represent the largest integer not greater than x and $\lceil x \rceil$ to represent the smallest integer not less than x.

Definition 2.2 (see [5]). The balanced starlike tree $S(n, m)$, $3 \leq m \leq n-1$, is a tree of order n with just one center vertex v, and each of the m branches of T at v is a path of length $\lfloor (n-1)/m \rfloor$ or $\lceil (n-1)/m \rceil$.

Let P_n be the path with n vertices. A path $P : vv_1v_2 \cdots v_k$ in G is called a pendent path if $d(v_1) = d(v_2) = \cdots = d(v_{k-1}) = 2$ and $d(v_k) = 1$. If $k = 1$, then we say vv_1 is a pendent edge of the graph G. A leaf or pendent vertex is a vertex of degree one. A branching vertex is a vertex of degree greater than two. The k paths $P_{l_1}, P_{l_2}, \ldots, P_{l_k}$ are said to have almost equal lengths if $l_1, l_2, \ldots, l_k$ satisfy $|l_i - l_j| \leq 1$ for $1 \leq i, j \leq k$.

Definition 2.3 (see [5]). The dumbbell $D(n, a, b)$ consists of the path P_{n-a-b} together with a independent vertices adjacent to one leaf of P_{n-a-b} and b independent vertices adjacent to the other leaf.

The union $G = G_1 \bigcup G_2$ of graph G_1 and G_2 with disjoint vertex sets V_1 and V_2 and edge sets E_1 and E_2 is the graph $G = (V, E)$ with $V = V_1 \bigcup V_2$ and $E = E_1 \bigcup E_2$. If G is a union of two paths of lengths a and b, then G is disconnected and has $a + b$ vertices and $a + b - 2$ edges. Let $m_k(G)$ be the number of matchings of G containing exactly k independent edges. Especially, let $m_k(a, b)$ be the number of k matchings in $G = P_a \bigcup P_b$.

Lemma 2.4 (see [5]). *Let v be a vertex of nontrivial connected graph G, and let $G(p, q)$ denote the graph obtained from G by adding pendent paths $P = vv_1v_2 \cdots v_p$ and $Q = vu_1u_2 \cdots u_q$, at vertex v. Assume that both numbers p and q are even. If $p - 2 \geq q + 2 \geq 4$, then for every k we have*

$$m_k(G(p, q)) \leq m_k(G(p-2, q+2)). \tag{2.2}$$

Lemma 2.5 (see [12]). *Let $m_k(a, b)$ be the number of k-matchings in $G = P_a \bigcup P_b$ and $n = 4s + r$ with $0 \leq r \leq 3$. Then the following inequality holds:*

$$m_k(n, 0) \geq m_k(n-2, 2) \geq m_k(n-4, 4) \geq \cdots \geq m_k(2s + r, 2s). \tag{2.3}$$

Lemma 2.6 (see [5]). *Among trees on n vertices and $2 \leq m \leq n-2$ leaves, the balanced starlike tree $S(n, m)$ has minimal Laplacian coefficient $c_k(G)$, for every $k = 0, 1, \ldots, n$.*

Definition 2.7 (see [5]). Let v be a vertex of a tree T of degree $m+1$. Suppose that $P_1, P_2, \ldots, P_m$ are pendent paths incident with v, with lengths $n_i \geq 1, i = 1, 2, \ldots, m$. Let w be the neighbor of v distinct from the starting vertices of paths $v_1, v_2, \ldots, v_m$, respectively. We form a tree $T' = \delta(T, v)$ by removing the edges $vv_1, vv_2, \ldots, vv_{m-1}$ from T and adding $m - 1$ new edges $wv_1, wv_2, \ldots, wv_{m-1}$ incident with w. We say that T' is a δ-transform of T.

Lemma 2.8 (see [5]). *Let T be an arbitrary tree, rooted at the center vertex. Let vertex v be on the deepest level of tree T among all branching vertices with degree at least three. Then for the δ-transformation tree $T' = \delta(T, v)$ and $0 \leq k \leq n$ holds:*

$$c_k(T) \geq c_k(T'). \tag{2.4}$$

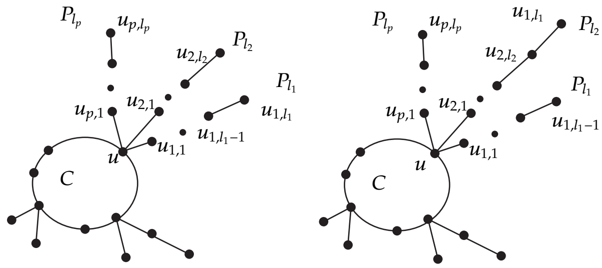

Figure 1: Example of π_1-transformation.

Lemma 2.9 (see [14]). *For every acyclic graph T with n vertices,*

$$c_k(T) = m_k(S(T)), \quad 0 \le k \le n, \tag{2.5}$$

where $S(T)$ means the subdivision graph of T.

3. Main Results

In this section, we present four new graphic transformations that decrease the Laplacian coefficients.

Definition 3.1. Let u be a vertex in the cycle C of a unicyclic graph G, such that u has degree $p+2$ and p pendent paths named $P_{l_1}, P_{l_2}, \ldots, P_{l_p}$, where $P_{l_i}: u_{i,1}, u_{i,2}, \ldots, u_{i,l_i}$, $1 \le i \le p$. If $l_i \ge l_j + 2$, and let

$$G_1 = G - u_{i,l_i-1}u_{i,l_i} + u_{j,l_j}u_{i,l_i} \triangleq \pi_1(G). \tag{3.1}$$

We say that G_1 is a π_1-transformation of G.

It is easy to see that π_1-transformation preserves the size of a cycle of G and the number of pendent vertices.

Theorem 3.2. *Let G be a connected unicyclic graph with n vertices and m pendent vertices, $G_1 = \pi_1(G)$. Then for every $k = 0, 1, \ldots, n$,*

$$c_k(G) \ge c_k(G_1), \tag{3.2}$$

with equality if and only if $k \in \{0, 1, n-1, n\}$.

Proof. It is easy to see that $c_0(G_1) = c_0(G) = 1$, $c_1(G_1) = 2|E(G_1)| = 2|E(G)| = c_1(G), c_n(G_1) = c_n(G) = 0$, $c_{n-1}(G_1) = n\tau(G_1) = n|E(C)| = n\tau(G) = c_{n-1}(G)$.

Now, consider the coefficients c_{n-k} $(k \ne 0, 1, n-1, n)$. Let $\mathcal{F}_k$ and $\mathcal{F}_{k_1}$ be the sets of spanning forests of G and G_1 with exactly k components, respectively.

Without loss of generality, we assume that $l_1 \ge l_2 + 2$. Let $G_1 = \pi_1(G) = G - u_{1,l_1-1}u_{1,l_1} + u_{2,l_2}u_{1,l1}$ (see Figure 1).

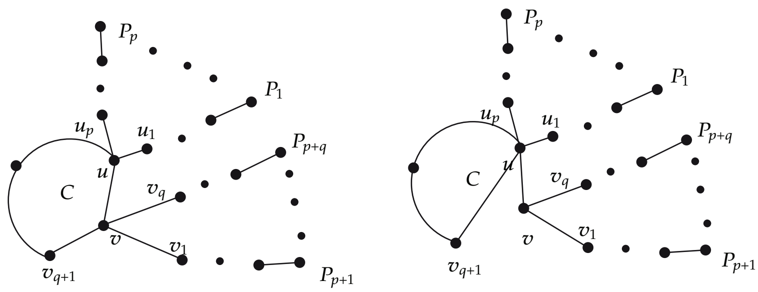

Figure 2: Example of π_2-transformation.

Obviously, by the definition of the spanning forest, the cycle C in the unicyclic graph satisfies that $C \notin F \in \mathcal{F}_k$ and $C \notin F_1 \in \mathcal{F}_{k_1}$, where F and F_1 are the arbitrary forests in $\mathcal{F}_k$ and $\mathcal{F}_{k_1}$, respectively. Without loss of generality, we remove one of the edges in the cycle C, say uv, so we get T and T', respectively. By Lemmas 2.4 and 2.9, we have that for every $k = 0, 1, \dots, n$,

$$c_k(T) \geq c_k(T'), \tag{3.3}$$

with equality if and only if $k \in \{0, 1, n-1, n\}$. If we remove the other edge, say xy, we get S and S', respectively. By Lemmas 2.4 and 2.9, we have that for every $k = 0, 1, \dots, n$,

$$c_k(S) \geq c_k(S'), \tag{3.4}$$

with equality if and only if $k \in \{0, 1, n-1, n\}$.

It is easy to see that $T - xy = S - uv$ and $T' - xy = S' - uv$. We know that the numbers of the same tree of spanning forests of $T - xy$ and $T' - xy$ with exactly k components are equal to the numbers of the same tree of spanning forests of $S - uv$ and $S' - uv$ with exactly k components, respectively.

Applying to Definition 3.1 and Lemma 2.1, we can show that for every $k = 0, 1, \dots, n$,

$$c_k(G) \geq c_k(G_1), \tag{3.5}$$

with equality if and only if $k \in \{0, 1, n-1, n\}$. □

Definition 3.3. Let v be a vertex in a cycle C of a connected unicyclic graph G, where $d(v) \geq 3$. Suppose that u is one of two neighbors adjacent to v in C, such that u has degree $p+2$ and p pendent paths incident with u and v has degree $q+2$ and q pendent paths incident with v. Let

$$G_2 = G - vv_{q+1} + uv_{q+1} \triangleq \pi_2(G), \tag{3.6}$$

where v_{q+1} is one of the other neighbors adjacent to v in C. We say that G_2 is a π_2-transformation of G (see Figure 2).

Obviously, π_2-transformation decreases the size of a cycle of G and preserves the number of pendent vertices.

Theorem 3.4. *Let G be a connected unicyclic graph with n vertices and m pendent vertices, $G_2 = \pi_2(G)$. Then for every $k = 0, 1, \dots, n$,*

$$c_k(G) \geq c_k(G_2), \tag{3.7}$$

with equality if and only if $k \in \{0, 1, n\}$.

Proof. Obviously, $c_0(G_2) = c_0(G) = 1, c_1(G_2) = 2|E(G_2)| = 2|E(G)| = c_1(G), c_n(G_2) = c_n(G) = 0$. For $k = n-1$, the length of a cycle in G is greater than the length of a cycle in G_2. Therefore, $c_{n-1}(G) > c_{n-1}(G_2)$.

Now, consider the coefficients c_{n-k} $(k \neq 0, 1, n-1, n)$. Let $\mathcal{F}_k$ and $\mathcal{F}_{k_2}$ be the sets of spanning forests of G and G_2 with exactly k components, respectively. Let $F_2 \in \mathcal{F}_{k_2}$ and T' be the component of F_2 and $u \in V(T')$. If $v_{q+1} \in V(T')$, we define F with $V(F) = V(G)$ and

$$E(F) = E(F_2) - uv_{q+1} + vv_{q+1}. \tag{3.8}$$

Now, we distinguish F_2 as the following two cases.

Case 1 ($v \in V(T')$). We have trees of equal sizes in both spanning forests thus $\gamma(F) = \gamma(F_2)$.

Case 2 ($v \notin V(T')$). Let vertex v be in the tree S', that is, $v \in V(S')$.

Note the fact that uv is a cut edge of G_2. It is easy to see that F is a spanning forest of G, and the number of components of F is $k-1$ or k. We claim that $F \in \mathcal{F}_k$. Otherwise, u, v belong to one tree of F; then there exists a path P joining v_{q+1} to u in F; then $uPv_{q+1}u$ is a cycle of F_2, which contradicts the fact that F_2 is a forest.

Suppose that $T' - v_{q+1}$ contains $a \geq 1$ vertices in the cycle C (including u) and $b \geq 0$ vertices in the paths $P_1, \dots, P_p$, and $T' - u$ contains $c \geq 1$ vertices in the cycle C. Let S' contain $d \geq 1$ in the paths $P_{p+1}, \dots, P_{p+q}$. Assume the orders of the components of F_2 different from T' and S' are $n_1, n_2, \dots n_{k-2}$. We have

$$\begin{aligned}\gamma(F) - \gamma(F_2) &= [(a+b)(c+d) - (a+b+c)d]\prod_{i=1}^{k-2} n_i \\ &= c(a+b-d)\prod_{i=1}^{k-2} n_i = c(a+b-d)N,\end{aligned} \tag{3.9}$$

where $N = \prod_{i=1}^{k-2} n_i$.

If we sum all differences for such forest, having fixed values a, c and $b + d = M$, we get

$$\begin{aligned}\sum_{F \in \mathcal{F}^*} \gamma(F) - \gamma(F_2) &= \sum_{F \in \mathcal{F}^*} c(a+b-d)N \\ &= cN \sum_{b=0}^{M-1} (a + 2b - M) = (a-1)cNM.\end{aligned} \tag{3.10}$$

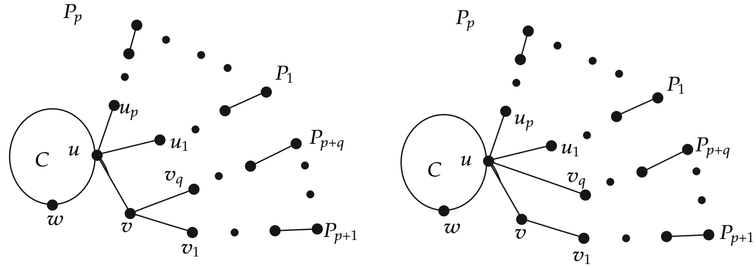

Figure 3: Example of π_3-transformation.

It is easy to see that $a \geq 1$ and $c \geq 1$, so $(a-1)cNM \geq 0$. Since at least one vertex is in $C-u-v_{q+1}$, there exists one forest F_2 such that $a > 1$ and $c \geq 1$, and then $(a-1)cNM > 0$.

If $v_{q+1} \notin V(T')$, thus $\gamma(F) = \gamma(F_2)$.

Therefore, by using Lemma 2.1, we get

$$c_k(G) = \sum_{F \in \mathcal{F}_k} \gamma(F) > \sum_{F_2 \in \mathcal{F}_{k_2}} \gamma(F_2) = c_k(G_2). \tag{3.11}$$

This completes the proof of Theorem 3.4. □

Definition 3.5. Let v (not in the cycle C) be a vertex of degree $q+1$ in a connected unicyclic graph G. Suppose that $P_{p+1}, \ldots, P_{p+q}$ are pendent paths incident with v. Let u be the neighbor of v distinct from the starting vertices of paths $v_1, v_2, \ldots, v_q$, respectively. Let

$$G_3 = \pi_3(G) = G - vv_2 - vv_3 - \cdots - vv_q + uv_2 + uv_3 + \cdots + uv_q. \tag{3.12}$$

We say that G_3 is a π_3-transformation of G (see Figure 3).

It is not difficult to see that π_3-transformation preserves the size of a cycle of G and the number of pendent vertices.

Theorem 3.6. *Let G be a connected unicyclic graph with n vertices and m pendent vertices, $G_3 = \pi_3(G)$. Then for every $k = 0, 1, \ldots, n$,*

$$c_k(G) \geq c_k(G_3), \tag{3.13}$$

with equality if and only if $k \in \{0, 1, n-1, n\}$.

Proof. Obviously, $c_0(G_3) = c_0(G) = 1, c_1(G_3) = 2|E(G_3)| = 2|E(G)| = c_1(G), c_n(G_3) = c_n(G) = 0$, $c_{n-1}(G_3) = n\tau(G_3) = n|E(C)| = n\tau(G) = c_{n-1}(G)$.

Now, consider the coefficients c_{n-k} $(k \neq 0, 1, n-1, n)$. Let $\mathcal{F}_k$ and $\mathcal{F}_{k_3}$ be the sets of spanning forests of G and G_3 with exactly k components, respectively. Obviously, by the definition of the spanning forest, the cycle C in the unicyclic graph satisfies that $C \notin F \in \mathcal{F}_k$ and $C \notin F_3 \in \mathcal{F}_{k_3}$, where F and F_3 are the arbitrary forests in $\mathcal{F}_k$ and $\mathcal{F}_{k_3}$, respectively. Without loss of generality, we remove one of the edges on the cycle, say wu, so we get two trees T and

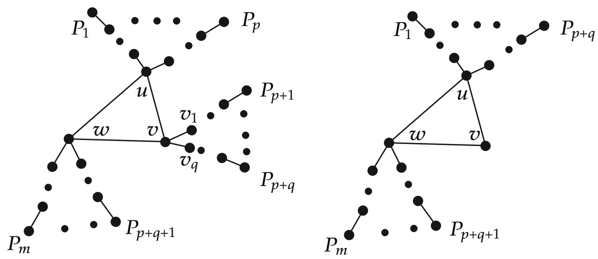

Figure 4: Example of π_4-transformation.

T', respectively. Applying to Definition 2.7, we know that $T' = \delta(T)$. Then using Lemma 2.8, we can get that for every $k = 0, 1, \ldots, n$,

$$c_k(T) \geq c_k(T'), \tag{3.14}$$

with equality if and only if $k \in \{0, 1, n-1, n\}$. If we remove another edge, say xy, we get S and S', respectively. By Definition 2.7, we know that $S' = \delta(S)$. Then applying to Lemma 2.8, we get that for every $k = 0, 1, \ldots, n$,

$$c_k(S) \geq c_k(S'), \tag{3.15}$$

with equality if and only if $k \in \{0, 1, n-1, n\}$.

It is easy to see that $T - xy = S - uv$ and $T' - xy = S' - uv$. We know that the numbers of the same tree of spanning forests of $T - xy$ and $T' - xy$ with exactly k components are equal to the numbers of the same tree of spanning forests of $S - uv$ and $S' - uv$ with exactly k components, respectively.

By Definition 3.5 and Lemma 2.1, we have that for every $k = 0, 1, \ldots, n$,

$$c_k(G) \geq c_k(G_3), \tag{3.16}$$

with equality if and only if $k \in \{0, 1, n-1, n\}$. □

Definition 3.7. Let u, v, and w be three vertices on the triangle in a unicyclic graph G. Suppose that $P_1, \ldots, P_p$ are pendent paths incident with u, $P_{p+1}, \ldots, P_{p+q}$ are pendent paths incident with v, and $P_{p+q+1}, \ldots, P_{p+q+l}$ are pendent paths incident with $w(p + q + l = m)$. Let

$$G_4 = G - vv_1 - \cdots - vv_q + uv_1 + \cdots + uv_q \triangleq \pi_4(G). \tag{3.17}$$

We say that G_4 is a π_4-transformation of G (see Figure 4).

Theorem 3.8. *Let u, v, and w be three vertices on the triangle in a unicyclic graph G, $G_4 = \pi_4(G)$. Then for every $k = 0, 1, \ldots, n$,*

$$c_k(G) \geq c_k(G_4), \tag{3.18}$$

with equality if and only if $k \in \{0, 1, n-1, n\}$.

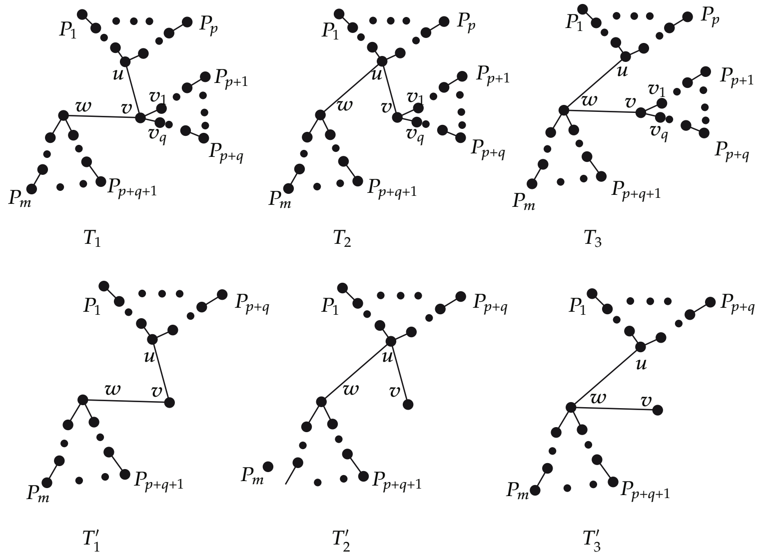

Figure 5: Obtained trees from Figure 4.

Proof. It is obvious to see that $c_0(G_4) = c_0(G) = 1$, $c_1(G_4) = 2|E(G_4)| = 2|E(G)| = c_1(G)$, $c_n(G_4) = c_n(G) = 0$. For $k = n-1$, the length of a cycle in G_4 is equal to the length of a cycle in G. Therefore, $c_{n-1}(G) = c_{n-1}(G_4)$.

Now, consider the coefficient c_{n-k} $(k \neq 0, 1, n-1, n)$. Let $\mathcal{F}_k$ and $\mathcal{F}_{k_4}$ be the sets of spanning forests of G and G_4 with exactly k components, respectively.

Similarly to the proof of Theorem 3.2, we can get 6 trees as shown in Figure 5. Obviously, by Definition 2.7, we know that $T_i' = \delta(T_i)$ $(i = 1, 2, 3)$. And according to Lemma 2.8, we can verify that

$$\begin{aligned} c_k(T_1) &\geq c_k(T_1'), \\ c_k(T_2) &\geq c_k(T_2'), \\ c_k(T_3) &\geq c_k(T_3'). \end{aligned} \tag{3.19}$$

By (3.19), Definition 3.7, and Lemma 2.1, it is easy to see that for every $k = 0, 1, \ldots, n$,

$$c_k(G) \geq c_k(G_4), \tag{3.20}$$

with equality if and only if $k \in \{0, 1, n-1, n\}$. This completes the proof of Theorem 3.8. □

Theorem 3.9. *Let G be a connected unicyclic graph with n vertices and m pendent vertices. Then for $0 \leq k \leq n$,*

$$c_k(G) \geq c_k(S'(n, m)), \tag{3.21}$$

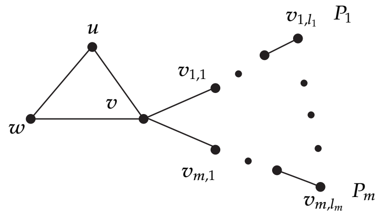

Figure 6: $S'(n,m)$.

with equality if and only if $k \in \{0,1,n\}$, *where* $S'(n,m)$ *is as shown in Figure 6, and each of the* m *branches at* v *is a path of length* $\lfloor (n-3)/m \rfloor$ *or* $\lceil (n-3)/m \rceil$.

Proof. Let $C = w_1 w_2 \cdots w_t w_1$ be a cycle of connected unicyclic graph G, and let T_i be a tree attached at w_i, $i = 1,2,\ldots,t$. We can apply π_3-transformation to T_i, such that the tree contains one branch vertex w_i with pendent path attached to it. Next, we can apply π_2-transformation to decrease the size of the cycle C as long as the length of C is not 3. Then we can apply π_1-transformation at the longest and the shortest path repeatedly, the Laplacian coefficients do not increase while the attached paths become more balanced. Finally, we can apply π_4-transformation as long as it is not $S'(n,m)$.

According to Theorems 3.2, 3.4, 3.6, and 3.8, we know that π_i-transformation ($i = 1,2,3,4$) cannot increase the Laplacian coefficients. So, for an arbitrary unicyclic graph G with n vertices and m pendent vertices, we verify that

$$c_k(G) \geq c_k\left(S'(n,m)\right), \tag{3.22}$$

where $0 \leq k \leq n$ and with equality if and only if $k = 0,1,n$. This completes the proof of Theorem 3.9. □

4. Laplacian-Like Energy of Unicyclic Graphs with *m* Pendent Vertices

Let G be a graph. The Laplacian-like energy of graph G, LEL for short, is defined as follows:

$$\mathrm{LEL}(G) = \sum_{k=1}^{n-1} \sqrt{\mu_k}, \tag{4.1}$$

where $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$ are the Laplacian eigenvalues of G. This concept was introduced by J. Liu and B. Liu [9], where it was demonstrated it has similar feature as molecular graph energy (for more details see [15]). Stevanović in [10] presented a connection between LEL and Laplacian coefficients.

Theorem 4.1. *Let* G *and* H *be two graphs with* n *vertices. If* $c_k(G) \leq c_k(H)$ *for* $k = 1,2,\ldots,n-1$, *then* $\mathrm{LEL}\,(G) \leq \mathrm{LEL}\,(H)$. *Furthermore, if a strict inequality* $c_k(G) < c_k(H)$ *holds for some* $1 \leq k \leq n-1$, *then* $\mathrm{LEL}\,(G) < \mathrm{LEL}\,(H)$.

Using this result, we can conclude the following.

Corollary 4.2. *Let G be a connected unicyclic graph with n vertices and m pendent vertices. Then if* $G \ncong S'(n, m)$

$$\text{LEL}\ (S'(n, m)) < \text{LEL}\ (G), \tag{4.2}$$

where $S'(n, m)$ is shown in Figure 6, and each of the m branches at v is a path of length $\lfloor (n-3)/m \rfloor$ or $\lceil (n-3)/m \rceil$.

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