Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2012, Article ID 398085, 14 pages
doi:10.1155/2012/398085

Research Article

On the Hermitian R-Conjugate Solution of a System of Matrix Equations

Chang-Zhou Dong,[1] Qing-Wen Wang,[2] and Yu-Ping Zhang[3]

[1] *School of Mathematics and Science, Shijiazhuang University of Economics, Shijiazhuang, Hebei 050031, China*
[2] *Department of Mathematics, Shanghai University, Shanghai, Shanghai 200444, China*
[3] *Department of Mathematics, Ordnance Engineering College, Shijiazhuang, Hebei 050003, China*

Correspondence should be addressed to Qing-Wen Wang, wqw858@yahoo.com.cn

Received 3 October 2012; Accepted 26 November 2012

Academic Editor: Yang Zhang

Let R be an n by n nontrivial real symmetric involution matrix, that is, $R = R^{-1} = R^T \neq I_n$. An $n \times n$ complex matrix A is termed R-conjugate if $\overline{A} = RAR$, where $\overline{A}$ denotes the conjugate of A. We give necessary and sufficient conditions for the existence of the Hermitian R-conjugate solution to the system of complex matrix equations $AX = C$ and $XB = D$ and present an expression of the Hermitian R-conjugate solution to this system when the solvability conditions are satisfied. In addition, the solution to an optimal approximation problem is obtained. Furthermore, the least squares Hermitian R-conjugate solution with the least norm to this system mentioned above is considered. The representation of such solution is also derived. Finally, an algorithm and numerical examples are given.

1. Introduction

Throughout, we denote the complex $m \times n$ matrix space by $\mathbb{C}^{m\times n}$, the real $m \times n$ matrix space by $\mathbb{R}^{m\times n}$, and the set of all matrices in $\mathbb{R}^{m\times n}$ with rank r by $\mathbb{R}_r^{m\times n}$. The symbols $I, \overline{A}, A^T, A^*, A^\dagger$, and $\|A\|$ stand for the identity matrix with the appropriate size, the conjugate, the transpose, the conjugate transpose, the Moore-Penrose generalized inverse, and the Frobenius norm of $A \in \mathbb{C}^{m\times n}$, respectively. We use V_n to denote the $n \times n$ backward matrix having the elements 1 along the southwest diagonal and with the remaining elements being zeros.

Recall that an $n \times n$ complex matrix A is centrohermitian if $\overline{A} = V_n A V_n$. Centrohermitian matrices and related matrices, such as k-Hermitian matrices, Hermitian Toeplitz matrices, and generalized centrohermitian matrices, appear in digital signal processing and others areas (see, [1–4]). As a generalization of a centrohermitian matrix and related matrices, Trench [5] gave the definition of R-conjugate matrix. A matrix $A \in \mathbb{C}^{n\times n}$ is R-conjugate if $\overline{A} = RAR$, where R is a nontrivial real symmetric involution matrix, that is, $R = R^{-1} = R^T$ and $R \neq I_n$. At the same time, Trench studied the linear equation $Az = w$ for R-conjugate matrices in [5], where z, w are known column vectors.

Investigating the matrix equation

$$AX = B \tag{1.1}$$

with the unknown matrix X being symmetric, reflexive, Hermitian-generalized Hamiltonian, and repositive definite is a very active research topic [6–14]. As a generalization of (1.1), the classical system of matrix equations

$$AX = C, \qquad XB = D \tag{1.2}$$

has attracted many author's attention. For instance, [15] gave the necessary and sufficient conditions for the consistency of (1.2), [16, 17] derived an expression for the general solution by using singular value decomposition of a matrix and generalized inverses of matrices, respectively. Moreover, many results have been obtained about the system (1.2) with various constraints, such as bisymmetric, Hermitian, positive semidefinite, reflexive, and generalized reflexive solutions (see, [18–28]). To our knowledge, so far there has been little investigation of the Hermitian R-conjugate solution to (1.2).

Motivated by the work mentioned above, we investigate Hermitian R-conjugate solutions to (1.2). We also consider the optimal approximation problem

$$\left\|\widehat{X} - E\right\| = \min_{X\in S_X}\|X - E\|, \tag{1.3}$$

where E is a given matrix in $\mathbb{C}^{n\times n}$ and S_X the set of all Hermitian R-conjugate solutions to (1.2). In many cases the system (1.2) has not Hermitian R-conjugate solution. Hence, we need to further study its least squares solution, which can be described as follows: Let $RH\mathbb{C}^{n\times n}$ denote the set of all Hermitian R-conjugate matrices in $\mathbb{C}^{n\times n}$:

$$S_L = \left\{X \mid \min_{X\in RH\mathbb{C}^{n\times n}}\left(\|AX - C\|^2 + \|XB - D\|^2\right)\right\}. \tag{1.4}$$

Find $\widetilde{X} \in \mathbb{C}^{n\times n}$ such that

$$\left\|\widetilde{X}\right\| = \min_{X\in S_L}\|X\|. \tag{1.5}$$

In Section 2, we present necessary and sufficient conditions for the existence of the Hermitian R-conjugate solution to (1.2) and give an expression of this solution when the solvability conditions are met. In Section 3, we derive an optimal approximation solution to (1.3). In Section 4, we provide the least squares Hermitian R-conjugate solution to (1.5). In Section 5, we give an algorithm and a numerical example to illustrate our results.

2. *R*-Conjugate Hermitian Solution to (1.2)

In this section, we establish the solvability conditions and the general expression for the Hermitian R-conjugate solution to (1.2).

We denote $R\mathbb{C}^{n\times n}$ and $RH\mathbb{C}^{n\times n}$ the set of all R-conjugate matrices and Hermitian R-conjugate matrices, respectively, that is,

$$R\mathbb{C}^{n\times n} = \left\{A \mid \overline{A} = RAR\right\},$$
$$HR\mathbb{C}^{n\times n} = \left\{A \mid \overline{A} = RAR, A = A^*\right\}, \tag{2.1}$$

where R is $n \times n$ nontrivial real symmetric involution matrix.

Chang et al. in [29] mentioned that for nontrivial symmetric involution matrix $R \in \mathbb{R}^{n\times n}$, there exist positive integer r and $n \times n$ real orthogonal matrix $[P, Q]$ such that

$$R = [P,\ Q]\begin{bmatrix} I_r & 0 \\ 0 & -I_{n-r}\end{bmatrix}\begin{bmatrix} P^T \\ Q^T\end{bmatrix}, \tag{2.2}$$

where $P \in \mathbb{R}^{n\times r}$, $Q \in \mathbb{R}^{n\times(n-r)}$. By (2.2),

$$RP = P, \qquad RQ = -Q, \qquad P^TP = I_r, \qquad Q^TQ = I_{n-r}, \qquad P^TQ = 0, \qquad Q^TP = 0. \tag{2.3}$$

Throughout this paper, we always assume that the nontrivial symmetric involution matrix R is fixed which is given by (2.2) and (2.3). Now, we give a criterion of judging a matrix is R-conjugate Hermitian matrix.

Theorem 2.1. *A matrix $K \in HR\mathbb{C}^{n\times n}$ if and only if there exists a symmetric matrix $H \in \mathbb{R}^{n\times n}$ such that $K = \Gamma H \Gamma^*$, where*

$$\Gamma = [P,\ iQ], \tag{2.4}$$

with P, Q being the same as (2.2).

Proof. If $K \in HR\mathbb{C}^{n\times n}$, then $\overline{K} = RKR$. By (2.2),

$$\overline{K} = RKR = [P,\ Q]\begin{bmatrix} I_r & 0 \\ 0 & -I_{n-r}\end{bmatrix}\begin{bmatrix} P^T \\ Q^T\end{bmatrix}K[P,\ Q]\begin{bmatrix} I_r & 0 \\ 0 & -I_{n-r}\end{bmatrix}\begin{bmatrix} P^T \\ Q^T\end{bmatrix}, \tag{2.5}$$

which is equivalent to

$$\begin{bmatrix} P^T \\ Q^T \end{bmatrix} \overline{K}[P,\ Q] = \begin{bmatrix} I_r & 0 \\ 0 & -I_{n-r} \end{bmatrix} \begin{bmatrix} P^T \\ Q^T \end{bmatrix} K[P,\ Q] \begin{bmatrix} I_r & 0 \\ 0 & -I_{n-r} \end{bmatrix}. \tag{2.6}$$

Suppose that

$$\begin{bmatrix} P^T \\ Q^T \end{bmatrix} K[P,\ Q] = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}. \tag{2.7}$$

Substituting (2.7) into (2.6), we obtain

$$\begin{bmatrix} \overline{K_{11}} & \overline{K_{12}} \\ \overline{K_{21}} & \overline{K_{22}} \end{bmatrix} = \begin{bmatrix} I_r & 0 \\ 0 & -I_{n-r} \end{bmatrix} \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} I_r & 0 \\ 0 & -I_{n-r} \end{bmatrix} = \begin{bmatrix} K_{11} & -K_{12} \\ -K_{21} & K_{22} \end{bmatrix}. \tag{2.8}$$

Hence, $\overline{K_{11}} = K_{11}$, $\overline{K_{12}} = -K_{12}$, $\overline{K_{21}} = -K_{21}$, $\overline{K_{22}} = K_{22}$, that is, K_{11}, iK_{12}, iK_{21}, K_{22} are real matrices. If we denote $M = iK_{12}$, $N = -iK_{21}$, then by (2.7)

$$K = [P,\ Q] \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} P^T \\ Q^T \end{bmatrix} = [P,\ iQ] \begin{bmatrix} K_{11} & M \\ N & K_{22} \end{bmatrix} \begin{bmatrix} P^T \\ -iQ^T \end{bmatrix}. \tag{2.9}$$

Let $\Gamma = [P, iQ]$, and

$$H = \begin{bmatrix} K_{11} & M \\ N & K_{22} \end{bmatrix}. \tag{2.10}$$

Then, K can be expressed as $\Gamma H \Gamma^*$, where Γ is unitary matrix and H is a real matrix. By $K = K^*$

$$\Gamma H^T \Gamma^* = K^* = K = \Gamma H \Gamma^*, \tag{2.11}$$

we obtain $H = H^T$.

Conversely, if there exists a symmetric matrix $H \in \mathbb{R}^{n\times n}$ such that $K = \Gamma H \Gamma^*$, then it follows from (2.3) that

$$RKR = R\Gamma H\Gamma^* R = R[P,\ iQ]H\begin{bmatrix} P^T \\ -iQ^T \end{bmatrix} R = [P,\ -iQ]H\begin{bmatrix} P^T \\ iQ^T \end{bmatrix} = \overline{\Gamma} H \overline{\Gamma^*} = \overline{K}, \\ K^* = \Gamma H^T \Gamma^* = \Gamma H \Gamma^* = K, \tag{2.12}$$

that is, $K \in HR\mathbb{C}^{n\times n}$. □

Theorem 2.1 implies that an arbitrary complex Hermitian R-conjugate matrix is equivalent to a real symmetric matrix.

Lemma 2.2. *For any matrix $A \in \mathbb{C}^{m\times n}$, $A = A_1 + iA_2$, where*

$$A_1 = \frac{A + \overline{A}}{2}, \qquad A_2 = \frac{A - \overline{A}}{2i}. \tag{2.13}$$

Proof. For any matrix $A \in \mathbb{C}^{m\times n}$, it is obvious that $A = A_1 + iA_2$, where A_1, A_2 are defined as (2.13). Now, we prove that the decomposition $A = A_1 + iA_2$ is unique. If there exist B_1, B_2 such that $A = B_1 + iB_2$, then

$$A_1 - B_1 + i(B_2 - A_2) = 0. \tag{2.14}$$

It follows from A_1, A_2, B_1, and B_2 are real matrix that

$$A_1 = B_1, \qquad A_2 = B_2. \tag{2.15}$$

Hence, $A = A_1 + iA_2$ holds, where A_1, A_2 are defined as (2.13). □

By Theorem 2.1, for $X \in HR\mathbb{C}^{n\times n}$, we may assume that

$$X = \Gamma Y \Gamma^*, \tag{2.16}$$

where Γ is defined as (2.4) and $Y \in \mathbb{R}^{n\times n}$ is a symmetric matrix.

Suppose that $A\Gamma = A_1 + iA_2 \in \mathbb{C}^{m\times n}$, $C\Gamma = C_1 + iC_2 \in \mathbb{C}^{m\times n}$, $\Gamma^* B = B_1 + iB_2 \in \mathbb{C}^{n\times l}$, and $\Gamma^* D = D_1 + iD_2 \in \mathbb{C}^{n\times l}$, where

$$A_1 = \frac{A\Gamma + \overline{A\Gamma}}{2}, \quad A_2 = \frac{A\Gamma - \overline{A\Gamma}}{2i}, \quad C_1 = \frac{C\Gamma + \overline{C\Gamma}}{2}, \quad C_2 = \frac{C\Gamma - \overline{C\Gamma}}{2i}, \\ B_1 = \frac{\Gamma^* B + \overline{\Gamma^* B}}{2}, \quad B_2 = \frac{\Gamma^* B - \overline{\Gamma^* B}}{2i}, \quad D_1 = \frac{\Gamma^* D + \overline{\Gamma^* D}}{2}, \quad D_2 = \frac{\Gamma^* D - \overline{\Gamma^* D}}{2i}. \tag{2.17}$$

Then, system (1.2) can be reduced into

$$(A_1 + iA_2)Y = C_1 + iC_2, \qquad Y(B_1 + iB_2) = D_1 + iD_2, \tag{2.18}$$

which implies that

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} Y = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}, \qquad Y\left[B_1,\ B_2\right] = \left[D_1,\ D_2\right]. \tag{2.19}$$

Let

$$\begin{gathered} F = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}, \qquad G = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}, \qquad K = \left[B_1,\ B_2\right], \\ L = \left[D_1,\ D_2\right], \qquad M = \begin{bmatrix} F \\ K^T \end{bmatrix}, \qquad N = \begin{bmatrix} G \\ L^T \end{bmatrix}. \end{gathered} \tag{2.20}$$

Then, system (1.2) has a solution X in $HR\mathbb{C}^{n\times n}$ if and only if the real system

$$MY = N \tag{2.21}$$

has a symmetric solution Y in $\mathbb{R}^{n\times n}$.

Lemma 2.3 (Theorem 1 in [7]). *Let $A \in \mathbb{R}^{m\times n}$. The SVD of matrix A is as follows*

$$A = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T, \tag{2.22}$$

where $U = [U_1, U_2] \in \mathbb{R}^{m\times m}$ and $V = [V_1, V_2] \in \mathbb{R}^{n\times n}$ are orthogonal matrices, $\Sigma = \operatorname{diag}(\sigma_1, \ldots, \sigma_r)$, $\sigma_i > 0$ $(i = 1, \ldots, r)$, $r = \operatorname{rank}(A)$, $U_1 \in \mathbb{R}^{m\times r}$, $V_1 \in \mathbb{R}^{n\times r}$. Then, (1.1) *has a symmetric solution if and only if*

$$AB^T = BA^T, \qquad U_2^T B = 0. \tag{2.23}$$

In that case, it has the general solution

$$X = V_1\Sigma^{-1}U_1^T B + V_2V_2^T B^T U_1 \Sigma^{-1} V_1^T + V_2 G V_2^T, \tag{2.24}$$

where G is an arbitrary $(n - r) \times (n - r)$ symmetric matrix.

By Lemma 2.3, we have the following theorem.

Theorem 2.4. *Given $A \in \mathbb{C}^{m\times n}$, $C \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times l}$, and $D \in \mathbb{C}^{n\times l}$. Let A_1, A_2, C_1, C_2, B_1, B_2, D_1, D_2, F, G, K, L, M, and N be defined in* (2.17), (2.20), *respectively. Assume that the SVD of $M \in \mathbb{R}^{(2m+2l)\times n}$ is as follows*

$$M = U\begin{bmatrix} M_1 & 0 \\ 0 & 0 \end{bmatrix}V^T, \tag{2.25}$$

where $U = [U_1, U_2] \in \mathbb{R}^{(2m+2l)\times(2m+2l)}$ and $V = [V_1, V_2] \in \mathbb{R}^{n\times n}$ are orthogonal matrices, $M_1 = \operatorname{diag}(\sigma_1, \ldots, \sigma_r)$, $\sigma_i > 0$ $(i = 1, \ldots, r)$, $r = \operatorname{rank}(M)$, $U_1 \in \mathbb{R}^{(2m+2l)\times r}$, $V_1 \in \mathbb{R}^{n\times r}$. Then, system (1.2) *has a solution in $HR\mathbb{C}^{n\times n}$ if and only if*

$$MN^T = NM^T, \qquad U_2^T N = 0. \tag{2.26}$$

In that case, it has the general solution

$$X = \Gamma\left(V_1 M_1^{-1} U_1^T N + V_2 V_2^T N^T U_1 M_1^{-1} V_1^T + V_2 G V_2^T\right)\Gamma^*, \tag{2.27}$$

where G is an arbitrary $(n-r)\times(n-r)$ symmetric matrix.

3. The Solution of Optimal Approximation Problem (1.3)

When the set S_X of all Hermitian R-conjugate solution to (1.2) is nonempty, it is easy to verify S_X is a closed set. Therefore, the optimal approximation problem (1.3) has a unique solution by [30].

Theorem 3.1. *Given $A \in \mathbb{C}^{m\times n}$, $C \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times l}$, $D \in \mathbb{C}^{n\times l}$, $E \in \mathbb{C}^{n\times n}$, and $E_1 = (1/2)(\Gamma^* E\Gamma + \overline{\Gamma^* E\Gamma})$. Assume S_X is nonempty, then the optimal approximation problem* (1.3) *has a unique solution $\widehat{X}$ and*

$$\widehat{X} = \Gamma\left(V_1 M_1^{-1} U_1^T N + V_2 V_2^T N^T U_1 M_1^{-1} V_1^T + V_2 V_2^T E_1 V_2 V_2^T\right)\Gamma^*. \tag{3.1}$$

Proof. Since S_X is nonempty, $X \in S_X$ has the form of (2.27). By Lemma 2.2, $\Gamma^* E\Gamma$ can be written as

$$\Gamma^* E\Gamma = E_1 + iE_2, \tag{3.2}$$

where

$$E_1 = \frac{1}{2}\left(\Gamma^* E\Gamma + \overline{\Gamma^* E\Gamma}\right), \qquad E_2 = \frac{1}{2i}\left(\Gamma^* E\Gamma - \overline{\Gamma^* E\Gamma}\right). \tag{3.3}$$

According to (3.2) and the unitary invariance of Frobenius norm

$$\begin{aligned}\|X - E\| &= \left\|\Gamma\left(V_1 M^{-1} U_1^T N + V_2 V_2^T N^T U_1 M^{-1} V_1^T + V_2 G V_2^T\right)\Gamma^* - E\right\| \\ &= \left\|\left(E_1 - V_1 M^{-1} U_1^T N - V_2 V_2^T N^T U_1 M^{-1} V_1^T - V_2 G V_2^T\right) + iE_2\right\|.\end{aligned} \tag{3.4}$$

We get

$$\|X - E\|^2 = \left\|E_1 - V_1 M^{-1} U_1^T N - V_2 V_2^T N^T U_1 M^{-1} V_1^T - V_2 G V_2^T\right\|^2 + \|E_2\|^2. \tag{3.5}$$

Then, $\min_{X\in S_X}\|X - E\|$ is consistent if and only if there exists $G \in \mathbb{R}^{(n-r)\times(n-r)}$ such that

$$\min\left\|E_1 - V_1 M^{-1} U_1^T N - V_2 V_2^T N^T U_1 M^{-1} V_1^T - V_2 G V_2^T\right\|. \tag{3.6}$$

For the orthogonal matrix V

$$\begin{aligned}&\left\|E_1 - V_1 M^{-1} U_1^T N - V_2 V_2^T N^T U_1 M^{-1} V_1^T - V_2 G V_2^T\right\|^2 \\ &\quad= \left\|V^T(E_1 - V_1 M^{-1} U_1^T N - V_2 V_2^T N^T U_1 M^{-1} V_1^T - V_2 G V_2^T)V\right\|^2 \\ &\quad= \left\|V_1^T\left(E_1 - V_1 M^{-1} U_1^T N\right)V_1\right\|^2 + \left\|V_1^T(E_1 - V_1 M^{-1} U_1^T N)V_2\right\|^2 \\ &\qquad+ \left\|V_2^T\left(E_1 - V_2 V_2^T N^T U_1 M^{-1} V_1^T\right)V_1\right\|^2 + \left\|V_2^T\left(E_1 - V_2 G V_2^T\right)V_2\right\|.\end{aligned} \tag{3.7}$$

Therefore,

$$\min\left\|E_1 - V_1 M^{-1} U_1^T N - V_2 V_2^T N^T U_1 M^{-1} V_1^T - V_2 G V_2^T\right\| \tag{3.8}$$

is equivalent to

$$G = V_2^T E_1 V_2. \tag{3.9}$$

Substituting (3.9) into (2.27), we obtain (3.1). □

4. The Solution of Problem (1.5)

In this section, we give the explicit expression of the solution to (1.5).

Theorem 4.1. *Given $A \in \mathbb{C}^{m\times n}$, $C \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times l}$, and $D \in \mathbb{C}^{n\times l}$. Let A_1, A_2, C_1, C_2, B_1, B_2, D_1, D_2, F, G, K, L, M, and N be defined in* (2.17), (2.20), *respectively. Assume that the*

SVD of $M \in \mathbb{R}^{(2m+2l)\times n}$ *is as* (2.25) *and system* (1.2) *has not a solution in* $HR\mathbb{C}^{n\times n}$. *Then,* $X \in S_L$ *can be expressed as*

$$X = \Gamma V \begin{bmatrix} M_1^{-1} U_1^T N V_1 & M_1^{-1} U_1^T N V_2 \\ V_2^T N^T U_1 M_1^{-1} & Y_{22} \end{bmatrix} V^T \Gamma^*, \tag{4.1}$$

where $Y_{22} \in \mathbb{R}^{(n-r)\times(n-r)}$ *is an arbitrary symmetric matrix.*

Proof. It yields from (2.17)–(2.21) and (2.25) that

$$\begin{aligned} \|AX - C\|^2 + \|XB - D\|^2 &= \|MY - N\|^2 \\ &= \left\| U \begin{bmatrix} M_1 & 0 \\ 0 & 0 \end{bmatrix} V^T Y - N \right\|^2 \\ &= \left\| \begin{bmatrix} M_1 & 0 \\ 0 & 0 \end{bmatrix} V^T Y V - U^T N V \right\|^2. \end{aligned} \tag{4.2}$$

Assume that

$$V^T Y V = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix}, \quad Y_{11} \in \mathbb{R}^{r\times r},\ Y_{22} \in \mathbb{R}^{(n-r)\times(n-r)}. \tag{4.3}$$

Then, we have

$$\begin{aligned} &\|AX - C\|^2 + \|XB - D\|^2 \\ &\quad = \left\| M_1 Y_{11} - U_1^T N V_1 \right\|^2 + \left\| M_1 Y_{12} - U_1^T N V_2 \right\|^2 \\ &\qquad + \left\| U_2^T N V_1 \right\|^2 + \left\| U_2^T N V_2 \right\|^2. \end{aligned} \tag{4.4}$$

Hence,

$$\min\left(\|AX - C\|^2 + \|XB - D\|^2 \right) \tag{4.5}$$

is solvable if and only if there exist Y_{11}, Y_{12} such that

$$\begin{aligned} \left\| M_1 Y_{11} - U_1^T N V_1 \right\|^2 &= \min, \\ \left\| M_1 Y_{12} - U_1^T N V_2 \right\|^2 &= \min. \end{aligned} \tag{4.6}$$

It follows from (4.6) that

$$\begin{aligned} Y_{11} &= M_1^{-1} U_1^T N V_1, \\ Y_{12} &= M_1^{-1} U_1^T N V_2. \end{aligned} \tag{4.7}$$

Substituting (4.7) into (4.3) and then into (2.16), we can get that the form of elements in S_L is (4.1). □

Theorem 4.2. *Assume that the notations and conditions are the same as Theorem 4.1. Then,*

$$\left\| \widetilde{X} \right\| = \min_{X \in S_L} \|X\| \tag{4.8}$$

if and only if

$$\widetilde{X} = \Gamma V \begin{bmatrix} M_1^{-1} U_1^T N V_1 & M_1^{-1} U_1^T N V_2 \\ V_2^T N^T U_1 M_1^{-1} & 0 \end{bmatrix} V^T \Gamma^*. \tag{4.9}$$

Proof. In Theorem 4.1, it implies from (4.1) that $\min_{X \in S_L} \|X\|$ is equivalent to X has the expression (4.1) with $Y_{22} = 0$. Hence, (4.9) holds. □

5. An Algorithm and Numerical Example

Base on the main results of this paper, we in this section propose an algorithm for finding the solution of the approximation problem (1.3) and the least squares problem with least norm (1.5). All the tests are performed by MATLAB 6.5 which has a machine precision of around 10^{-16}.

Algorithm 5.1. (1) Input $A \in \mathbb{C}^{m \times n}$, $C \in \mathbb{C}^{m \times n}$, $B \in \mathbb{C}^{n \times l}$, $D \in \mathbb{C}^{n \times l}$.

(2) Compute $A_1, A_2, C_1, C_2, B_1, B_2, D_1, D_2, F, G, K, L, M$, and N by (2.17) and (2.20).

(3) Compute the singular value decomposition of M with the form of (2.25).

(4) If (2.26) holds, then input $E \in \mathbb{C}^{n \times n}$ and compute the solution $\widehat{X}$ of problem (1.3) according (3.1), else compute the solution $\widetilde{X}$ to problem (1.5) by (4.9).

To show our algorithm is feasible, we give two numerical example. Let an nontrivial symmetric involution be

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \tag{5.1}$$

We obtain $[P, Q]$ in (2.2) by using the spectral decomposition of R, then by (2.4)

$$\Gamma = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & i \end{bmatrix}. \tag{5.2}$$

Example 5.2. Suppose $A \in \mathbb{C}^{2\times 4}$, $C \in \mathbb{C}^{2\times 4}$, $B \in \mathbb{C}^{4\times 3}$, $D \in \mathbb{C}^{4\times 3}$, and

$$A = \begin{bmatrix} 3.33 - 5.987i & 45i & 7.21 & -i \\ 0 & -0.66i & 7.694 & 1.123i \end{bmatrix},$$
$$C = \begin{bmatrix} 0.2679 - 0.0934i & 0.0012 + 4.0762i & -0.0777 - 0.1718i & -1.2801i \\ 0.2207 & -0.1197i & 0.0877 & 0.7058i \end{bmatrix},$$
$$B = \begin{bmatrix} 4 + 12i & 2.369i & 4.256 - 5.111i \\ 4i & 4.66i & 8.21 - 5i \\ 0 & 4.83i & 56 + i \\ 2.22i & -4.666 & 7i \end{bmatrix}, \tag{5.3}$$
$$D = \begin{bmatrix} 0.0616 + 0.1872i & -0.0009 + 0.1756i & 1.6746 - 0.0494i \\ 0.0024 + 0.2704i & 0.1775 + 0.4194i & 0.7359 - 0.6189i \\ -0.0548 + 0.3444i & 0.0093 - 0.3075i & -0.4731 - 0.1636i \\ 0.0337i & 0.1209 - 0.1864i & -0.2484 - 3.8817i \end{bmatrix}.$$

We can verify that (2.26) holds. Hence, system (1.2) has an Hermitian R-conjugate solution. Given

$$E = \begin{bmatrix} 7.35i & 8.389i & 99.256 - 6.51i & -4.6i \\ 1.55 & 4.56i & 7.71 - 7.5i & i \\ 5i & 0 & -4.556i & -7.99 \\ 4.22i & 0 & 5.1i & 0 \end{bmatrix}. \tag{5.4}$$

Applying Algorithm 5.1, we obtain the following:

$$\widehat{X} = \begin{bmatrix} 1.5597 & 0.0194i & 2.8705 & 0.0002i \\ -0.0194i & 9.0001 & 0.2005i & -3.9997 \\ 2.8705 & -0.2005i & -0.0452 & 7.9993i \\ -0.0002i & -3.9997 & -7.9993i & 5.6846 \end{bmatrix}. \tag{5.5}$$

Example 5.2 illustrates that we can solve the optimal approximation problem with Algorithm 5.1 when system (1.2) have Hermitian R-conjugate solutions.

Example 5.3. Let A, B, and C be the same as Example 5.2, and let D in Example 5.2 be changed into

$$D = \begin{bmatrix} 0.0616 + 0.1872i & -0.0009 + 0.1756i & 1.6746 + 0.0494i \\ 0.0024 + 0.2704i & 0.1775 + 0.4194i & 0.7359 - 0.6189i \\ -0.0548 + 0.3444i & 0.0093 - 0.3075i & -0.4731 - 0.1636i \\ 0.0337i & 0.1209 - 0.1864i & -0.2484 - 3.8817i \end{bmatrix}. \tag{5.6}$$

We can verify that (2.26) does not hold. By Algorithm 5.1, we get

$$\widetilde{X} = \begin{bmatrix} 0.52 & 2.2417i & 0.4914 & 0.3991i \\ -2.2417i & 8.6634 & 0.1921i & -2.8232 \\ 0.4914 & -0.1921i & 0.1406 & 1.3154i \\ -0.3991i & -2.8232 & -1.3154i & 6.3974 \end{bmatrix}. \tag{5.7}$$

Example 5.3 demonstrates that we can get the least squares solution with Algorithm 5.1 when system (1.2) has not Hermitian R-conjugate solutions.

Acknowledgments

This research was supported by the Grants from the Key Project of Scientific Research Innovation Foundation of Shanghai Municipal Education Commission (13ZZ080), the National Natural Science Foundation of China (11171205), the Natural Science Foundation of Shanghai (11ZR1412500), the Ph.D. Programs Foundation of Ministry of Education of China (20093108110001), the Discipline Project at the corresponding level of Shanghai (A. 13-0101-12-005), Shanghai Leading Academic Discipline Project (J50101), the Natural Science Foundation of Hebei province (A2012403013), and the Natural Science Foundation of Hebei province (A2012205028). The authors are grateful to the anonymous referees for their helpful comments and constructive suggestions.

References

[1] R. D. Hill, R. G. Bates, and S. R. Waters, “On centro-Hermitian matrices,” *SIAM Journal on Matrix Analysis and Applications*, vol. 11, no. 1, pp. 128–133, 1990.

[2] R. Kouassi, P. Gouton, and M. Paindavoine, “Approximation of the Karhunen-Loeve tranformation and Its application to colour images,” *Signal Processing: Image Communication*, vol. 16, pp. 541–551, 2001.

[3] A. Lee, “Centro-Hermitian and skew-centro-Hermitian matrices,” *Linear Algebra and Its Applications*, vol. 29, pp. 205–210, 1980.

[4] Z.-Y. Liu, H.-D. Cao, and H.-J. Chen, “A note on computing matrix-vector products with generalized centrosymmetric (centrohermitian) matrices,” *Applied Mathematics and Computation*, vol. 169, no. 2, pp. 1332–1345, 2005.

[5] W. F. Trench, “Characterization and properties of matrices with generalized symmetry or skew symmetry,” *Linear Algebra and Its Applications*, vol. 377, pp. 207–218, 2004.

[6] K.-W. E. Chu, “Symmetric solutions of linear matrix equations by matrix decompositions,” *Linear Algebra and Its Applications*, vol. 119, pp. 35–50, 1989.

[7] H. Dai, “On the symmetric solutions of linear matrix equations,” *Linear Algebra and Its Applications*, vol. 131, pp. 1–7, 1990.

[8] F. J. H. Don, “On the symmetric solutions of a linear matrix equation,” *Linear Algebra and Its Applications*, vol. 93, pp. 1–7, 1987.

[9] I. Kyrchei, “Explicit representation formulas for the minimum norm least squares solutions of some quaternion matrix equations,” *Linear Algebra and Its Applications*, vol. 438, no. 1, pp. 136–152, 2013.

[10] Y. Li, Y. Gao, and W. Guo, “A Hermitian least squares solution of the matrix equation $AXB = C$ subject to inequality restrictions,” *Computers & Mathematics with Applications*, vol. 64, no. 6, pp. 1752–1760, 2012.

[11] Z.-Y. Peng and X.-Y. Hu, “The reflexive and anti-reflexive solutions of the matrix equation $AX = B$,” *Linear Algebra and Its Applications*, vol. 375, pp. 147–155, 2003.

[12] L. Wu, “The re-positive definite solutions to the matrix inverse problem $AX = B$,” *Linear Algebra and Its Applications*, vol. 174, pp. 145–151, 1992.

[13] S. F. Yuan, Q. W. Wang, and X. Zhang, “Least-squares problem for the quaternion matrix equation $AXB + CYD = E$ over different constrained matrices,” Article ID 722626, *International Journal of Computer Mathematics*. In press.

[14] Z.-Z. Zhang, X.-Y. Hu, and L. Zhang, “On the Hermitian-generalized Hamiltonian solutions of linear matrix equations,” *SIAM Journal on Matrix Analysis and Applications*, vol. 27, no. 1, pp. 294–303, 2005.

[15] F. Cecioni, “Sopra operazioni algebriche,” *Annali della Scuola Normale Superiore di Pisa*, vol. 11, pp. 17–20, 1910.

[16] K.-W. E. Chu, “Singular value and generalized singular value decompositions and the solution of linear matrix equations,” *Linear Algebra and Its Applications*, vol. 88-89, pp. 83–98, 1987.

[17] S. K. Mitra, “A pair of simultaneous linear matrix equations $A_1XB_1 = C_1, A_2XB_2 = C_2$ and a matrix programming problem,” *Linear Algebra and Its Applications*, vol. 131, pp. 107–123, 1990.

[18] H.-X. Chang and Q.-W. Wang, “Reflexive solution to a system of matrix equations,” *Journal of Shanghai University*, vol. 11, no. 4, pp. 355–358, 2007.

[19] A. Dajić and J. J. Koliha, “Equations $ax = c$ and $xb = d$ in rings and rings with involution with applications to Hilbert space operators,” *Linear Algebra and Its Applications*, vol. 429, no. 7, pp. 1779–1809, 2008.

[20] A. Dajić and J. J. Koliha, “Positive solutions to the equations $AX = C$ and $XB = D$ for Hilbert space operators,” *Journal of Mathematical Analysis and Applications*, vol. 333, no. 2, pp. 567–576, 2007.

[21] C.-Z. Dong, Q.-W. Wang, and Y.-P. Zhang, “The common positive solution to adjointable operator equations with an application,” *Journal of Mathematical Analysis and Applications*, vol. 396, no. 2, pp. 670–679, 2012.

[22] X. Fang, J. Yu, and H. Yao, “Solutions to operator equations on Hilbert C^*-modules,” *Linear Algebra and Its Applications*, vol. 431, no. 11, pp. 2142–2153, 2009.

[23] C. G. Khatri and S. K. Mitra, “Hermitian and nonnegative definite solutions of linear matrix equations,” *SIAM Journal on Applied Mathematics*, vol. 31, no. 4, pp. 579–585, 1976.

[24] I. Kyrchei, “Analogs of Cramer’s rule for the minimum norm least squares solutions of some matrix equations,” *Applied Mathematics and Computation*, vol. 218, no. 11, pp. 6375–6384, 2012.

[25] F. L. Li, X. Y. Hu, and L. Zhang, “The generalized reflexive solution for a class of matrix equations $(AX = C, XB = D)$,” *Acta Mathematica Scientia*, vol. 28B, no. 1, pp. 185–193, 2008.

[26] Q.-W. Wang and C.-Z. Dong, “Positive solutions to a system of adjointable operator equations over Hilbert C^*-modules,” *Linear Algebra and Its Applications*, vol. 433, no. 7, pp. 1481–1489, 2010.

[27] Q. Xu, “Common Hermitian and positive solutions to the adjointable operator equations $AX = C$, $XB = D$,” *Linear Algebra and Its Applications*, vol. 429, no. 1, pp. 1–11, 2008.

[28] F. Zhang, Y. Li, and J. Zhao, “Common Hermitian least squares solutions of matrix equations $A_1XA_1^* = B_1$ and $A_2XA_2^* = B_2$ subject to inequality restrictions,” *Computers & Mathematics with Applications* , vol. 62, no. 6, pp. 2424–2433, 2011.

[29] H.-X. Chang, Q.-W. Wang, and G.-J. Song, “(R, S)-conjugate solution to a pair of linear matrix equations,” *Applied Mathematics and Computation*, vol. 217, no. 1, pp. 73–82, 2010.

[30] E. W. Cheney, *Introduction to Approximation Theory*, McGraw-Hill, New York, NY, USA, 1966.